Fisher-KPP propagation in the presence of a line: further effects

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Abstract

This paper is a continuation of [4] where a new model of biological invasions in the plane directed by a line was introduced. Here we include new features such as transport and reaction terms on the line. Their interaction with the pure diffusivity in the plane is quantified in terms of enhancement of the propagation speed. We establish conditions that determine whether the spreading speed exceeds the standard Fisher KPP invasion speed. These conditions involve the ratio of the diffusivities on the line and in the field, the transport term and the reactions. We derive the asymptotic behaviour for large diffusions or large transports. We also discuss the biological interpretation of these findings.

Keywords: KPP equations, reaction-diffusion system, fast diffusion on a line, asymptotic speed of propagation.

MSC: 35K57, 92D25, 35B40, 35K40, 35B53.

1 Introduction

This paper is a sequel of [4] in which we introduced a new model to describe biological invasions in the plane when a strong diffusion takes place on a line. The purpose of this model is to understand the effect of a line interacting with a homogeneous environment. This type of questions arises for instance in studying the propagation of diseases directed by roads [10] or the movements of animal populations in the presence of pathways allowing for a more rapid movement. An example of the latter is provided by recent observations of movements of wolves along seismic lines in Western Canada [9]. In [4] we derived the asymptotic speed of spreading in the direction of the line. There we showed that for low diffusion the line has no effect, whereas, past a threshold, the line enhances global diffusion in the plane in the direction of the line. Moreover, the propagation velocity on the line increases indefinitely as the diffusivity on the line grows to infinity.

The goal of the present paper is to include new features in this model such as transport and reaction or mortality on the road and to understand the resulting new effects. Taking into account these new elements is important when discussing propagation directed by roads or along water stream networks. These require new developments that turn out to make more transparent the case considered in [4] as well.

Consider the line $\{(x,0) : x \in \mathbb{R}\}$ in the plane $\mathbb{R}^2$; we will refer to the plane as "the field" and the line as "the road". For a single species, we consider a system that combines the density of this population in the field $v(x,y,t)$ and the density on the line $u(x,t)$. The main questions that we want to understand are the effects of a transport term $q\partial_x u$ as well as a decay rate $\rho \geq 0$ on the road. An invasive species that can be carried by streams of water is an example of a situation where such additional terms are required. The transport and the decay rate are considered here to be uniform, that is, q and ρ are constant. Due to the symmetry of the problem, we can restrict our analysis to the upper half-plane $\Omega := \{(x,y) \ : \ x \in \mathbb{R},\ y > 0\}$. The equations for u and v then read:

$$\begin{cases} \partial_t u - D\partial_{xx}u + q\partial_x u + \rho u = \nu v(x,0,t) - \mu u & x \in \mathbb{R},\ t > 0 \\ \partial_t v - d\Delta v = f(v) & (x,y) \in \Omega,\ t > 0 \\ -d\partial_y v(x,0,t) = \mu u(x,t) - \nu v(x,0,t) & x \in \mathbb{R},\ t > 0. \end{cases} \tag{1.1}$$

Here, $d, D, \mu, \nu > 0$, $\rho \geq 0$, $q \in \mathbb{R}$ and $f \in C^1([0,+\infty))$ satisfies the usual KPP type assumptions:

$$f(0) = f(1) = 0, \quad f > 0 \text{ in } (0,1), \quad f < 0 \text{ in } (1,+\infty), \quad f(s) \leq f'(0)s \text{ for } s > 0. \tag{1.2}$$

Note that transport and pure decay only occur on the road and the question is to know how these interact with the diffusivity and growth in the field. We combine the system with the initial condition

$$\begin{cases} u|_{t=0} = u_0 & \text{in } \mathbb{R} \\ v|_{t=0} = v_0 & \text{in } \Omega, \end{cases}$$

where u_0, v_0 are always assumed to be nonnegative and continuous.

Let c_K denote the classical KPP invasion speed [8] in the field:

$$c_K = 2\sqrt{df'(0)}.$$

This is the asymptotic speed at which the population would spread in the open space - i.e., when no line is present (see [2]).

We say that (1.1) admits the *asymptotic speeds of spreading* $w_*^\pm$ (in the directions $\pm e_1 = \pm(1,0)$) if for any solution (u,v) starting from a compactly supported initial datum $(u_0, v_0) \not\equiv (0,0)$, and for all $\varepsilon > 0$, the following hold true:

$$\lim_{t\to+\infty} \sup_{\substack{x<-(w_*^- +\varepsilon)t \\ y\geq 0}} |(u(x,t), v(x,y,t))| = 0, \qquad \lim_{t\to+\infty} \sup_{\substack{x>(w_*^+ +\varepsilon)t \\ y\geq 0}} |(u(x,t), v(x,y,t))| = 0,$$

$$\forall a > 0, \qquad \lim_{t\to+\infty} \sup_{\substack{-(w_*^- -\varepsilon)t<x<(w_*^+ -\varepsilon)t \\ 0\leq y<a}} |(u(x,t), v(x,y,t)) - (U, V(y))| = 0,$$

where $(U, V) = (U, V(y))$ is the unique positive, bounded, stationary solution of (1.1).

Thus, the first step for proving the existence of the asymptotic speeds of spreading consists in deriving a uniqueness result for the stationary system. We call it a Liouville-type result. The existence of the asymptotic speeds of spreading implies in particular that

$$\forall c \notin [-w_*^-, w_*^+], \qquad \lim_{t\to+\infty} (u(x+ct,t), v(x+ct,y,t)) = (0,0),$$

$$\forall c \in (-w_*^-, w_*^+), \qquad \lim_{t\to+\infty} (u(x+ct,t), v(x+ct,y,t)) = (U, V(y)),$$

locally uniformly in $(x, y) \in \overline{\Omega}$. This kind of weaker formulation is sometimes used in the literature as the definition of the asymptotic speed.

1.1 Statement of the main results

In [4], we proved that, in the case $q, \rho = 0$, (1.1) admits asymptotic speeds of spreading $w_*^\pm$. We further identified a threshold situation: $D/d = 2$ below which $w_*^\pm = c_K$ and above which $w_*^+ = w_*^- > c_K$. The first question we address in the present paper is how this threshold is modified by the presence of the transport and decay terms q and ρ. This question is solved by the following

Theorem 1.1. *Under the assumption* (1.2)*, problem* (1.1) *admits asymptotic speeds of spreading $w_*^\pm$ (in the directions $\pm e_1$). Moreover, if* $\dfrac{D}{d} \leq 2 + \dfrac{\rho}{f'(0)} \mp \dfrac{q}{\sqrt{df'(0)}}$, *then $w_*^\pm = c_K$, else $w_*^\pm > c_K$.*

In the case $q = \rho = 0$, we recover the result of [4]. Theorem 1.1 shows that a mortality term $\rho > 0$ always rises the threshold for D/d after which the effect of the road is felt. The threshold for the enhancement of the speed towards right, w_*^+, is decreased if the transport term q is positive and increased if it is negative.

We will actually carry out the whole study of system (1.1) in the case where the mortality term $-\rho u$ is replaced by a more general reaction term $g(u)$. This term can be used to model situations where reproduction occurs on the road as well. The general system reads

$$\begin{cases} \partial_t u - D\partial_{xx} u + q\partial_x u = \nu v(x,0,t) - \mu u + g(u) & x \in \mathbb{R},\ t > 0 \\ \partial_t v - d\Delta v = f(v) & (x,y) \in \Omega,\ t > 0 \\ -d\partial_y v(x,0,t) = \mu u(x,t) - \nu v(x,0,t) & x \in \mathbb{R},\ t > 0. \end{cases} \tag{1.3}$$

The hypotheses are: $f, g \in C^1([0,+\infty))$ and satisfy

$$f(0) = f(1) = 0, \qquad f > 0 \text{ in } (0,1), \qquad f < 0 \text{ in } (1,+\infty), \tag{1.4}$$

$$g(0) = 0, \qquad \exists S > 0,\ g(S) \leq 0, \tag{1.5}$$

$$s \mapsto f(s)/s, \quad s \mapsto g(s)/s \quad \text{are nonincreasing.} \tag{1.6}$$

Condition (1.6) holds if f and g are concave. It implies that f and g are of KPP type: $f(s) \leq f'(0)s$ and $g(s) \leq g'(0)s$.

Theorem 1.2. *Assume that* (1.4)-(1.6) *hold. Then:*

(i) (Liouville-type result). *Problem* (1.3) *admits a unique, positive, bounded, stationary solution* (U, V). *Moreover,* $U \equiv$ *constant and* $V \equiv V(y)$.

(ii) (Spreading). *Problem* (1.3) *admits asymptotic speeds of spreading* $w_*^{\pm}$.

(iii) (Spreading velocity). *If* $\dfrac{D}{d} \leq 2 - \dfrac{g'(0)}{f'(0)} \mp \dfrac{q}{\sqrt{df'(0)}}$, *then* $w_*^{\pm} = c_K$. *Otherwise* $w_*^{\pm} > c_K$.

Notice that, in Theorem 1.1, $s \mapsto f(s)/s$ is not assumed to be nonincreasing. This hypothesis is only required in the proof of statement (i) of Theorem 1.2. An alternative hypothesis is given in Proposition 3.4 below. It is to be noted that the Liouville-type result may fail if $s \mapsto f(s)/s$ is not nonincreasing. In this case, we still get a convergence result, but for initial data satisfying $u_0 \leq \nu/\mu$, $v_0 \leq 1$, and the convergence holds to the minimal positive, stationary solution (see Remark 3.5). Furthermore, the spreading speeds are still well defined and statements (ii) and (iii) of Theorem 1.2 hold true.

When there is no transport on the road, w_*^- and w_*^+ coincide and the threshold condition given by Theorem 1.1 part (iii) becomes

$$\frac{D}{d} \leq 2 - \frac{g'(0)}{f'(0)}.$$

This allows us to understand the - somewhat mysterious - factor 2 in the threshold condition of [4]. Indeed, when $g \equiv 0$, we recover the condition $D \leq 2d$ of [4]. We further note that, if $f'(0) = g'(0)$, the threshold condition becomes $D = d$ - which is what one would have expected. Thus, when the same reaction occurs on the road and in the field, the effect of the road is felt as soon as the diffusivity there is larger than in the field. The factor 2 of [4] is therefore explained by the absence of reaction on the road.

In the last part of the paper, we investigate the limits of the asymptotic speeds of spreading $w_*^{\pm}$ as the diffusion and the transport on the road tend to $+\infty$. We find the following asymptotic behaviours:

Theorem 1.3. *As functions of the variables D and q respectively, $w_*^\pm$ satisfy*

$$\lim_{D\to\infty}\frac{w_*^\pm}{\sqrt{D}}=h,\qquad \lim_{q\to\pm\infty}\frac{w_*^\pm}{|q|}=\begin{cases}k & \text{if } g'(0)<\mu\\ 1 & \text{if } g'(0)\geq\mu,\end{cases}$$

with $h>0$ independent of q and $0<k<1$ independent of D.

The expression of k is explicitly given in the proof of the theorem in Section 5.2.

1.2 Biological interpretation

What we are aiming at understanding here is how transport, diffusivity and reaction on the road combine to yield a spreading speed larger than c_K, the KPP invasion speed in the open field (without the road). For definiteness, let us consider propagation to the right (that is, in the direction e_1).

In the case when the reaction on the road consists in a pure mortality term $-\rho u$, Theorem 1.1 asserts that the threshold for D/d after which the effect of the road is felt grows linearly in ρ. This means that the larger the ρ, the more likely the road has no effect on the overall propagation. When $\rho=0$, the threshold condition reads

$$\frac{D}{d}>2-\frac{q}{\sqrt{df'(0)}}=2\left(1-\frac{q}{c_K}\right).$$

The above formula can also be written as

$$q>c_K\left(1-\frac{D}{2d}\right).$$

A consequence of this formula is that a transport q larger than c_K is sufficient to enhance the overall propagation, no matter what the diffusivity ratio is. This explains for instance how a river may help the invasion of a parasite [7]. It may seem at first sight a natural effect that a drift term larger than c_K on the road will enhance the overal invasion speed and bring it above c_k. However, it should be noted that in the situation we are looking at, on the road alone - without the exchange terms with the field - there would be extinction of the population even with a large q. The global spreading speed results from a rather delicate interaction between the field and the road. Therefore, it is remarkable that the threshold is precisely $q=c_K$. If $q\leq c_K(1-D/2d)$ then the spreading is with the usual KPP velocity. This yields an interesting interpretation when considering the propagation against the direction of the transport, that is, when $q<0$. For $-q$ large enough, the asymptotic speed of spreading in the direction e_1 in the field is the KPP invasion speed. For instance, asymptotic propagation upstream against a river flow in the neighbouring field is unaffected by the river, but downstream propagation, in the direction of the flow, can be significantly enhanced.

Theorem 1.3 asserts that the spreading speed approaches a portion of the speed q of the flow when the latter is very large. This portion is equal to 1 only if there is a sufficiently large reaction g on the road. It is worthwhile to note that the condition on g only involves the rate μ at which the individuals leave the line.

Another phenomenon for which our model is relevant is the dynamics of a population that favours disturbed habitats[1]. It is known that certain invasive plants, some weeds in particular, thrive in disturbed habitats, such as cultivated fields or along roads [1]. It has been observed that the presence of disturbances increases their speed of spreading [5]. A model has been proposed in [2] to describe this type of phenomena and in particular the propagation of the "unscented chamomile" in North America. This is an invasive weed widely distributed in croplands, pastures and infrastructure (road edges). In such cases, the disturbance, represented by the road, provides better environmental conditions than the rest of the territory (more diffuse light, bare ground that permit seedling establishment), rather than a greater diffusivity. This results in having $d = D$ and $g > f$ in our model. If one actually has $g'(0) > f'(0)$ then Theorem 1.2 implies that there is enhancement of the invasion speed, which is in agreement with the observations. Notice that if $g'(0) \geq 2f'(0)$ then the enhancement always occurs, whatever the diffusivity ratio is. We plan to study the effect due to propagation through a road crossing a globally unfavourable area in a separate note.

1.3 Organization of the paper

The paper is organized as follows. In the next section, we start with recalling the results of [4] concerning the well-posedness of the Cauchy problems and the comparison principles. Then, we show that the large time limits of solutions emerging from non-trivial, bounded initial data are trapped between two positive, bounded, stationary solutions which do not depend on x. In Section 3, we derive the Liouville-type results for such class of solutions, first for (1.1) and then for (1.3). The large-time behaviour of solutions is thereby characterized, at least locally uniformly in space. In particular, Theorem 1.2 part (i) follows.

In section 4, we investigate the spreading property of solutions, that is, the set where they converge to 0 or to the positive, stationary solution. We give the proof of Theorem 1.2 parts (ii) and (iii) insisting on the main differences with [4].

Section 5 is dedicated to the proof of Theorem 1.3.

2 Long time behaviour

Throughout this section, we assume that (1.4), (1.5) hold. The unique solvability for the Cauchy problem associated with (1.3) in the class of nonnegative, bounded functions follows from the same arguments as in [4], Proposition 3.1. There, the case $g \equiv 0$ is treated, but a slight modification of the arguments allows one to handle the presence of the nonlinear term g. The existence result is obtained by first solving the Cauchy problem for u with $v \equiv 0$ and then the one for v with the so obtained u. Repeating this procedure, starting with the new v, one constructs an increasing,

[1]The authors thank Mark Lewis for having brought this issue to their attention.

bounded sequence (u_n, v_n) which eventually converges to a solution of (1.3). The uniqueness result is a consequence of the comparison principle between sub and supersolutions (i.e., pairs of functions satisfying the system with the "=" replaced by "$\leq$" and "$\geq$" respectively). We recall two versions of the comparison principle that will be used several time in the sequel:

Proposition 2.1 ([4])**.** *Let $(\underline{u}, \underline{v})$ and $(\overline{u}, \overline{v})$ be respectively a subsolution bounded from above and a supersolution bounded from below of (1.3) satisfying $\underline{u} \leq \overline{u}$ and $\underline{v} \leq \overline{v}$ at $t = 0$. Then, either $\underline{u} < \overline{u}$ and $\underline{v} < \overline{v}$ for all t, or there exists $T > 0$ such that $(\underline{u}, \underline{v}) = (\overline{u}, \overline{v})$ for $t \leq T$.*

The second version deals with subsolutions in a generalised sense (see [3] for a related notion).

Proposition 2.2 ([4])**.** *Let $E \subset \mathbb{R}^N \times (0, +\infty)$ and $F \subset \Omega \times (0, +\infty)$ be two open sets and let (u_1, v_1), (u_2, v_2) be two subsolutions of (1.3) bounded from above and satisfying*

$$u_1 \leq u_2 \quad \text{on } (\partial E) \cap (\mathbb{R}^N \times (0, +\infty)), \qquad v_1 \leq v_2 \text{ on } (\partial F) \cap (\Omega \times (0, +\infty)).$$

If the functions $\underline{u}$, $\underline{v}$ defined by

$$\underline{u}(x,t) := \begin{cases} \max(u_1(x,t), u_2(x,t)) & \text{if } (x,t) \in \overline{E} \\ u_2(x,t) & \text{otherwise,} \end{cases}$$

$$\underline{v}(x,y,t) := \begin{cases} \max(v_1(x,y,t), v_2(x,y,t)) & \text{if } (x,y,t) \in \overline{F} \\ v_2(x,y,t) & \text{otherwise,} \end{cases}$$

satisfy

$$\underline{u}(x,t) > u_2(x,t) \;\Rightarrow\; \underline{v}(x,0,t) \geq v_1(x,0,t),$$
$$\underline{v}(x,0,t) > v_2(x,0,t) \;\Rightarrow\; \underline{u}(x,t) \geq u_1(x,t),$$

then, any supersolution $(\overline{u}, \overline{v})$ of (1.3) bounded from below and such that $\underline{u} \leq \overline{u}$ and $\underline{v} \leq \overline{v}$ at $t = 0$, satisfies $\underline{u} \leq \overline{u}$ and $\underline{v} \leq \overline{v}$ for all $t > 0$.

We now derive a result which gives a preliminary information about the long time behaviour of solutions.

Lemma 2.3. *Let (u, v) be the solution of (1.3) starting from a bounded initial datum $(u_0, v_0) \not\equiv (0, 0)$. Then, there exist two positive, bounded, x-independent, stationary solutions (U_1, V_1) and (U_2, V_2) of (1.3) such that*

$$U_1 \leq \liminf_{t \to +\infty} u(t,x) \leq \limsup_{t \to +\infty} u(t,x) \leq U_2,$$

$$V_1(y) \leq \liminf_{t \to +\infty} v(t,x,y) \leq \limsup_{t \to +\infty} v(t,x,y) \leq V_2(y),$$

locally uniformly in $(x, y) \in \overline{\Omega}$.

Proof. Let S be the constant in (1.5). The pair $(\overline{U}, \overline{V})$ defined by

$$(\overline{U}, \overline{V}) = \left[\max\left(\frac{\|u_0\|_\infty + S}{\nu}, \frac{\|v_0\|_\infty + 1}{\mu}\right)\right](\nu, \mu),$$

is a supersolution of (1.3) which is larger than (u, v) at $t = 0$. Let $(\overline{u}, \overline{v})$ be the solution of (1.3) with initial datum $(\overline{U}, \overline{V})$. Using the comparison principle given by Proposition 2.1, we see that $\overline{u}$ and $\overline{v}$ are nonincreasing in t. Thus, $(\overline{u}, \overline{v})$ converges as $t \to +\infty$ to a stationary solution (U_2, V_2) of (1.3) satisfying

$$\limsup_{t\to+\infty} u(t, x) \leq U_2(x), \qquad \limsup_{t\to+\infty} v(t, x, y) \leq V_2(x, y),$$

locally uniformly in $(x, y) \in \overline{\Omega}$. Furthermore, by translation invariance of the problem in the x-direction, we see that (U_2, V_2) does not depend on x.

We now construct the pair (U_1, V_1). Take $R > 0$ large enough in such a way that the principal eigenvalue of $-\Delta$ in $B_R \subset \mathbb{R}^2$ with Dirichlet boundary condition is less than $f'(0)/(2d)$. The associated principal eigenfunction φ_R satisfies

$$-d\Delta\varphi_R \leq \frac{1}{2} f'(0)\varphi_R \quad \text{in } B_R.$$

Hence, for $\varepsilon > 0$ small enough, the function $\varepsilon\varphi_R$ satisfies $-d\Delta(\varepsilon\varphi_R) \leq f(\varepsilon\varphi_R)$ in B_R. We extend φ_R to 0 outside B_R and we define $\underline{V}(x, y) := \varepsilon\varphi_R(x, y - R - 1)$. Thus, $(0, \underline{V})$ is a generalised subsolution of (1.3). The strong comparison principle given by 2.1 implies that u and v are positive for $t > 0$. Hence, up to decreasing ε if need be, we have that $(0, \underline{V})$ is below (u, v) at, say, $t = 1$. Let $(\underline{u}, \underline{v})$ be the solution of (1.3) starting from $(0, \underline{V})$ at $t = 1$. Using the comparison principle for generalised subsolutions - Proposition 2.2 - we see that $\underline{u}$ and $\underline{v}$ are nondecreasing in t. If one of them were not strictly increasing, the strong comparison principle of Proposition 2.1 would imply that $(\underline{u}, \underline{v})$ is constant in time, which is impossible because $(0, \underline{V})$ is not a solution of (1.3). Thus, as $t \to +\infty$, $(\underline{u}, \underline{v})$ converges to a stationary solution (U_1, V_1) of (1.3) satisfying

$$0 < U_1(x) \leq \liminf_{t\to+\infty} u(t, x), \qquad \underline{V}(x, y) < V_1(x, y) \leq \liminf_{t\to+\infty} v(t, x, y),$$

locally uniformly in $(x, y) \in \overline{\Omega}$. It remains to show that (U_1, V_1) does not depend on x. Since $\underline{V}$ is compactly supported, there exists $k > 0$ such that (U_1, V_1) is above the translated by any $h \in (-k, k)$ in the x-direction of $(0, \underline{V})$. By translation invariance of the problem, the solutions of (1.3) emerging from these initial data coincide with the translated by $h \in (-k, k)$ of $(\underline{u}, \underline{v})$. We then infer, by comparison, that (U_1, V_1) is above the translated by $h \in (-k, k)$ in the x-direction of itself. Namely, it does not depend on x. $\square$

3 Liouville-type result for 1-dimensional solutions

In this section, we derive a Liouville-type result for stationary solutions of (1.3) which do not depend on x. Namely, we will show that the problem

$$\begin{cases} U \equiv \text{constant}, \ V \equiv V(y) \\ -dV'' = f(V), & y > 0 \\ \nu V(0) = \mu U - g(U) \\ -dV'(0) = g(U). \end{cases} \tag{3.1}$$

admits a unique positive, bounded solution. This will imply the general Liouville-type result, Theorem 1.2 part (i), because, by Lemma 2.3, any positive, bounded, stationary solution of (1.3) lies between two positive, bounded solutions of (3.1).

We start with considering the pure mortality case $g(U) = -\rho U$. The proof is much simpler in this case. Problem (3.1) reduces to

$$\begin{cases} U \equiv \frac{\nu}{\mu+\rho} V(0) \\ -dV'' = f(V), & y > 0 \\ dV'(0) = \frac{\nu\rho}{\mu+\rho} V(0). \end{cases} \tag{3.2}$$

Proposition 3.1. *Under the assumption* (1.4), *problem* (3.2) *admits a unique positive, bounded solution.*

Proof. Let V be a positive, bounded solution of (3.2). It is straightforward to check that V necessarily satisfies $0 < V \leq 1$, $V(+\infty) = 1$ and $V'(+\infty) = 0$. Thus, multiplying the second equation of (3.2) by V' and integrating by parts between 0 and $+\infty$ we get

$$\int_{V(0)}^{1} f(s)ds = \frac{d}{2}(V'(0))^2 = \frac{\nu^2\rho^2}{2d(\mu+\rho)^2} V^2(0).$$

Examining the function θ defined by

$$\theta(\sigma) := \frac{\nu^2\rho^2}{2d(\mu+\rho)^2}\sigma^2 - \int_{\sigma}^{1} f(s)ds,$$

we see that

$$\theta(0) < 0, \qquad \theta(1) \geq 0, \qquad \theta' > 0 \text{ in } (0,1).$$

There exists then a unique value $\sigma_0 \in (0,1]$ such that $\theta(\sigma_0) = 0$. Hence, $V(0) = \sigma_0$. This proves the uniqueness of positive, bounded solutions of (3.2). It is also easy to verify that the solution V of (3.2) with initial datum $V(0) = \sigma_0$ is actually positive and bounded. Assume indeed by contradiction that this is not the case. One can then find $\xi \geq 0$ such that either $V(\xi) = 1$ and $V'(\xi) > 0$, or $V(\xi) < 1$ and $V'(\xi) = 0$. Owing to (1.4), both cases are ruled out by the following equality:

$$\int_{\sigma_0}^{V(\xi)} f(s)ds = \frac{d}{2}[(V'(0))^2 - (V'(\xi))^2] = \int_{\sigma_0}^{1} f(s)ds - \frac{d}{2}(V'(\xi))^2.$$

□

This concludes the proof of the Liouville-type result for stationary solutions of (1.1).

We now pass to the case of a general reaction term on the road.

Theorem 3.2. *Under the assumptions* (1.4)-(1.6), *problem* (3.1) *admits a unique positive, bounded solution.*

The existence result is contained in Lemma 2.3, that we proved using a sub and supersolution argument. Let us present a more explicit construction inspired by the shooting method.

Proof of the existence part of Theorem 3.2. For $U \in \mathbb{R}$, let V_U denote the associated solution of (3.1). Consider the following set:

$$\mathcal{U} := \{U > 0 \ : \ \forall y \geq 0, \ V_U(y) > 0\}.$$

This set is nonempty because, for $U \geq \max(\nu/\mu, S)$ (S being the constant in (1.5)) the function V_U satisfies $V_U(0) \geq 1$, $V_U'(0) \geq 0$. It follows that V_U is nondecreasing and then positive. We then define

$$U^* := \inf \mathcal{U}.$$

Suppose by contradiction that $U^* = 0$. We can then take $U \in \mathcal{U}$ close enough to 0 in such a way that $V_U(0) < 1$. Notice that $V_U'(0) > 0$, because otherwise V_U would not be positive. Call η the first point where V_U reaches the value 1, if it exists, else set $\eta := +\infty$. In the second case, the function V_U lies in $(0,1)$ and then, since $dV_U'' = -f(V_U) < 0$, it satisfies $V_U(+\infty) = 1$, $V_U'(+\infty) = 0$. Hence, in both cases, $V_U'(\eta) \geq 0$. We derive

$$V_U'(0) \geq \frac{1}{d}\int_0^\eta f(V_U(y))dy \geq \frac{1}{d}\int_0^\eta \frac{V_U'(y)}{V_U'(0)} f(V_U(y))dy = \frac{\int_{V_U(0)}^1 f(s)ds}{dV_U'(0)}.$$

Choosing $U \in \mathcal{U}$ small enough then leads to a contradiction, because $V_U(0), V_U'(0) \to 0$ as $U \to 0$. Therefore, $U^* > 0$.

We claim that V_{U^*} is positive and bounded. The continuity of solutions of the Cauchy problem with respect to the initial datum yields $V_{U^*} \geq 0$. Assume by way of contradiction that there exists $y_0 > 0$ such that $V_{U^*}(y_0) = 0$. We necessarily have that $V_{U^*}'(y_0) = 0$, because otherwise V_{U^*} would be negative somewhere. Hence, $V_{U^*} \equiv 0$ by the uniqueness of solutions of the Cauchy problem. But then $U^* = 0$, which is impossible. If $V_{U^*}(0) = 0$ then

$$V_{U^*}'(0) = -g(U^*)/d = -\mu U^*/d < 0,$$

which is again a contradiction. We have shown that $V_{U^*} > 0$ on $[0, +\infty)$.

Assume now by contradiction that V_{U^*} is not bounded from above. There exists then $y_0 \geq 0$ such that

$$V_{U^*}(y_0) > 1, \qquad V_{U^*}'(y_0) > 0.$$

Since $V_U \to V_{U^*}$ as $U \to U^*$ in $C^1_{loc}([0,+\infty))$, for U close enough to U^* we have that

$$\min_{[0,y_0]} V_U > \frac{1}{2}\min_{[0,y_0]} V_{U^*} > 0, \qquad V_U(y_0) > 1, \qquad V_U'(y_0) > 0.$$

It follows that V_U is increasing in $[y_0,+\infty)$ and then $\min V_U > 0$. This means that $U \in \mathcal{U}$, which contradicts the definition of U^*. Therefore, (U^*, V_{U^*}) is a positive, bounded solution of (3.1). $\square$

Proof of the uniqueness part of Theorem 3.2. Assume by way of contradiction that (3.1) admits two distinct positive, bounded solutions (U_1,V_1) and (U_2,V_2). The uniqueness of solutions of the Cauchy problem yields $U_1 \neq U_2$; say, $U_1 < U_2$. Since (1.6) implies that the function $G : \mathbb{R}_+ \to \mathbb{R}$ defined by $G(s) := \mu - g(s)/s$ is nondecreasing, we obtain

$$\nu V_1(0) = G(U_1)U_1 \leq G(U_2)U_1 \leq \nu V_2(0),$$

and equality holds if and only if $V_2(0) = 0$. But if $V_2(0) = 0$ then $-dV_2'(0) = \mu U_2 > 0$ and then V_2 would not be nonnegative. Thus, $V_1(0) < V_2(0)$. We argue differently depending on the fact that $V_1 = V_2$ somewhere or not.

Case 1) $V_1(y) = V_2(y)$ for some $y > 0$.
Let η be the smallest $y > 0$ such that $V_1(y) = V_2(y)$. Namely, $V_1 < V_2$ in $(0,\eta)$ and $V_1(\eta) = V_2(\eta)$. By the uniqueness of solutions of the Cauchy problem we infer that $V_1'(\eta) > V_2'(\eta)$. Multiplying by V_2 the equation satisfied by V_1'' and by V_1 the one satisfied by V_2'' and integrating over $(0,\eta)$, we get

$$\begin{aligned}\frac{1}{d}\int_0^\eta V_1V_2\left(\frac{f(V_1)}{V_1} - \frac{f(V_2)}{V_2}\right) &= \int_0^\eta (V_2''V_1 - V_1''V_2) = \left[V_2'V_1 - V_1'V_2\right]_0^\eta \\ &< V_1'(0)V_2(0) - V_2'(0)V_1(0).\end{aligned}$$

The first integral above is nonnegative by (1.6). Thus,

$$V_1'(0)V_2(0) > V_2'(0)V_1(0). \tag{3.3}$$

This inequality reads

$$g(U_1)(\mu U_2 - g(U_2)) < g(U_2)(\mu U_1 - g(U_1)),$$

that is,

$$g(U_1)U_2 < g(U_2)U_1.$$

This contradicts (1.6) because $U_1 < U_2$.

Case 2) $V_1(y) < V_2(y)$ for all $y > 0$.
Notice that V_1 and V_2 cannot attain the value 1 with a nonzero first derivative, because they would be either unbounded or negative somewhere. It follows from the uniqueness of solutions of the Cauchy problem that they are either identically equal to 1, or below 1, concave and increasing or above 1, and decreasing. In any case they satisfy

$$\forall i \in \{1,2\}, \quad \lim_{y\to+\infty} V_i(y) = 1, \quad \lim_{y\to+\infty} V_i'(y) = 0.$$

Thus, the same computation as in the case 1 yields

$$\frac{1}{d}\lim_{\eta\to+\infty}\int_0^\eta V_1V_2\left(\frac{f(V_1)}{V_1}-\frac{f(V_2)}{V_2}\right)=\lim_{\eta\to+\infty}\int_0^\eta (V_2''V_1-V_1''V_2)$$
$$=V_1'(0)V_2(0)-V_2'(0)V_1(0).$$

We know from (1.6) that the function $s\mapsto f(s)/s$ is nonincreasing. Moreover, since it is positive in $(0,1)$ and negative in $(1,+\infty)$, it cannot be constant in a left or in a right neighborhood of 1. It follows that $f(V_1)/V_1\geq f(V_2)/V_2$ on $[0,+\infty)$, with strict inequality somewhere. As a consequence, the first integral above is strictly positive. That is, (3.3) holds. We then get a contradiction by arguing as in the previous case. □

We now derive some properties of positive, bounded solutions of (3.1).

Proposition 3.3. *Assume that* (1.4), (1.5) *hold and that $s\mapsto g(s)/s$ is nonincreasing. Let $S_*:=\inf\{S>0\ :\ g(S)\leq 0\}$. Then, any positive, bounded solution (U,V) of* (3.1) *satisfies*

$$S_*\leq\frac{\nu}{\mu}\ \Rightarrow\ \begin{cases}S_*\leq U\leq\frac{\nu}{\mu}\\ V\leq 1,\end{cases}\qquad\qquad S_*\geq\frac{\nu}{\mu}\ \Rightarrow\ \begin{cases}\frac{\nu}{\mu}\leq U\leq S_*\\ V\geq 1.\end{cases}$$

Proof. Since $-dV''=f(V)$, V cannot attain the value 1 without being constant. The following implications are then easily obtained:

$$V\equiv 1\text{ on }[0,+\infty)\ \Leftrightarrow\ V(0)=1\ \Leftrightarrow\ V'(0)=0\ \Leftrightarrow\ g(U)=0\ \Rightarrow\ S_*\leq U=\frac{\nu}{\mu},$$
$$V<1\text{ on }[0,+\infty)\ \Leftrightarrow\ V(0)<1\ \Leftrightarrow\ V'(0)>0\ \Leftrightarrow\ g(U)<0\ \Rightarrow\ S_*<U<\frac{\nu}{\mu},$$
$$V>1\text{ on }[0,+\infty)\ \Leftrightarrow\ V(0)>1\ \Leftrightarrow\ V'(0)<0\ \Leftrightarrow\ g(U)>0\ \Rightarrow\ \frac{\nu}{\mu}<U<S_*.$$

The result follows. □

It is not hard to construct examples with $s\mapsto f(s)/s$ not nonincreasing for which the Liouville-type result fails. However, as shown by Proposition 3.1, this hypothesis can be dropped if g satisfies some suitable conditions. The weaker sufficient condition we are able to obtain is expressed in terms of the following quantity:

$$S_M:=\min\{s\geq 0\ :\ g'\leq 0\text{ in }[s,+\infty)\}.$$

Since $S_M=0$ when $g(s)=-\rho s$, the following result generalizes Proposition 3.1.

Proposition 3.4. *Assume that* (1.4), (1.5) *hold, that $s\mapsto g(s)/s$ is nonincreasing and that*

$$S_M\leq\frac{\nu+g(S_M)}{\mu}.$$

Then, problem (3.1) *admits a unique positive, bounded solution.*

Proof. Let (U,V) be a positive, bounded solution of (3.1). We know that $V(+\infty) = 1$ and $V'(+\infty) = 0$. The same integration by parts as in the proof of Proposition 3.1 yields $\theta(U) = 0$, with

$$\theta(\sigma) := g^2(\sigma) + 2d\int_1^{G(\sigma)} f(s)ds, \qquad G(\sigma) := (\mu\sigma - g(\sigma))/\nu.$$

Thus, U satisfies

$$\begin{cases} (G(U)-1)g(U) \geq 0 \\ \theta(U) = 0. \end{cases} \tag{3.4}$$

It is not hard to show that, conversely, if $U > 0$ satisfies (3.4) then the associated solution of (3.1) is positive and bounded. The function G satisfies $G(0) = 0$, $G(+\infty) = +\infty$ and it is strictly increasing in the set where it is positive. Let $S_1 > 0$ be such that $G(S_1) = 1$. Set $S_* := \inf\{S > 0 \ : \ g(S) \leq 0\}$ and then call $a := \min(S_1, S_*)$, $b := \max(S_1, S_*)$. For $U > b$, $G(U) > 1$ and $g(U) \leq 0$. Hence $U > b$ satisfies the first condition in (3.4) only if $g(U) = 0$, but then $\theta(U) < 0$ because $f < 0$ in $(1, +\infty)$. If $0 < U < a$ then $(G(U)-1)g(U) < 0$. This shows that (3.4) has no solution outside $[a,b]$. Since $\theta(S_*) \leq 0$ and $\theta(S_1) \geq 0$, the function θ vanishes somewhere on $[a,b]$. Moreover, $(G-1)g \geq 0$ there. Therefore, (3.4) admits solution on $[a,b]$. We conclude the proof by showing that θ is strictly monotone on $[a,b]$. If $S_* < S_1$ then g^2 is nondecreasing and G is strictly increasing and smaller than 1 on $[a,b]$, whence θ is strictly increasing. Consider now the case $S_* > S_1$. The function G is strictly increasing and larger than 1 on $[S_1, S_*]$. On the other hand, by hypothesis,

$$G(S_M) = \frac{\mu S_M - g(S_M)}{\nu} \leq 1.$$

Whence, $S_M \leq S_1$ and thus g^2 is decreasing on $[S_1, S_*]$. We eventually infer that θ is strictly decreasing on $[S_1, S_*]$. □

Remark 3.5. Under the assumptions of Proposition 3.3, if (3.1) admits multiple positive, bounded solutions, then there exists a minimal one $(\underline{U}, \underline{V})$ among them. Moreover, $(\underline{U}, \underline{V})$ satisfies $\underline{U} \geq \nu/\mu$, $\underline{V} \geq 1$ and it attracts solutions of (1.3) starting below $(\nu/\mu, 1)$. Indeed, if non-uniqueness occurs, then Proposition 3.4 yields

$$S_* \geq S_M > \frac{\nu + g(S_M)}{\mu} \geq \frac{\nu}{\mu}.$$

Whence any positive, bounded solution (U,V) of (3.1) satisfies $U \geq \nu/\mu$ and $V \geq 1$ due to Proposition 3.3. Therefore, by comparison, if (u,v) is a solution of (1.3) with an initial datum (u_0, v_0) below $(\nu/\mu, 1)$, then

$$\limsup_{t\to+\infty} u(t,x) \leq U, \quad \limsup_{t\to+\infty} v(t,x,y) \leq V(y),$$

locally uniformly in $(x,y) \in \overline{\Omega}$. On the other hand, if $(u_0, v_0) \not\equiv (0,0)$, Lemma 2.3 implies that equality holds in the above expressions for one of these solutions (U,V). This is the minimal solution.

4 Propagation

In this section we prove statements (ii) and (iii) of Theorem 1.2. We concentrate on propagation to the right, since propagation to the left is obtained by replacing q with $-q$. The general plan of the proof is that of [4]. First, we look for plane waves (or exponential solutions) for the linearised system around $(0,0)$. These have a critical velocity w_*. It then follows from the KPP assumption that solutions of the nonlinear system spread at most with velocity w_*. To prove that they spread at least with velocity w_*, we look at the problem in a wide strip $\mathbb{R}\times(0,L)$ and we construct compactly supported sub-solutions using again exponential solutions, but this time with complex exponents. Finally, we establish the condition under which the critical wave velocity is higher than the KPP speed. We do not repeat here all the arguments of [4], but we give details only of the points that are different.

4.1 Exponential solutions

Exponential solutions of the linearised system are looked under the form

$$(u(t,x),v(t,x,y)) = e^{-\alpha(x-ct)}(1,\gamma e^{-\beta y}), \tag{4.1}$$

with $c\geq 0$, $\gamma>0$ and $\alpha,\beta\in\mathbb{R}$ (not necessarily positive). Namely, we look for c, α, β and γ satisfying

$$\begin{cases} c\alpha - D\alpha^2 - q\alpha = \nu\gamma - \mu + g'(0) \\ c\alpha - d(\alpha^2+\beta^2) = f'(0) \\ d\gamma\beta = \mu - \nu\gamma. \end{cases}$$

The third equation yields $\gamma = \mu/(\nu+d\beta)$, which is positive iff $\beta > -\nu/d$. This will be assumed without further reference. Substituting we get

$$\begin{cases} -D\alpha^2 + (c-q)\alpha = -\dfrac{d\mu\beta}{\nu+d\beta} + g'(0) \\ c\alpha - d(\alpha^2+\beta^2) = f'(0). \end{cases} \tag{4.2}$$

The first equation of (4.2) in the unknown α has the roots

$$\alpha_D^{\pm}(c,\beta) = \frac{1}{2D}\left(c-q\pm\sqrt{(c-q)^2+4D(\chi(\beta)-g'(0))}\right).$$

where the function χ is defined on $(-\nu/d,+\infty)$ by

$$\chi(\beta) := \frac{d\mu\beta}{\nu+d\beta}.$$

Since χ is strictly increasing and tends to μ at $+\infty$, we see that $\alpha_D^{\pm}(c,\beta)$ are real iff

$$(c-q)^2 > 4D(g'(0)-\mu), \tag{4.3}$$

$$\beta \geq \underline{\beta}(c) := \chi^{-1}(g'(0)-(c-q)^2/4D).$$

Therefore, if c satisfies (4.3), the set of real solutions (β, α) of the first equation of (4.2) is given by $\Sigma(c) = \Sigma^-(c) \cup \Sigma^+(c)$, with

$$\Sigma^\pm(c) := \{(\beta, \alpha_D^\pm(c, \beta)) \ : \ \beta \geq \underline{\beta}(c)\}.$$

This is a smooth curve with leftmost point $(\underline{\beta}(c), (c-q)/2D)$. The second equation in (4.2) admits real solutions iff $c \geq c_K$, where $c_K := 2\sqrt{df'(0)}$ is the classical KPP critical speed for the second equation in (1.3). In the (β, α) plane, it represents the circle $\Gamma(c)$ of centre $(0, c/2d)$ and radius

$$r(c) = \frac{\sqrt{c^2 - c_K^2}}{2d}.$$

Let $\mathcal{S}(c)$ denote the closed set bounded from below by $\Sigma^-(c)$ and from above by $\Sigma^+(c)$ and let $\mathcal{G}(c)$ denote the closed disc with boundary $\Gamma(c)$. Exponential functions of the type (4.1) are supersolutions of the linearisation of (1.3) iff $(\beta, \alpha) \in \mathcal{S}(c) \cap \mathcal{G}(c)$. It is easy to check that, for $c_K \leq c < c'$, $\mathcal{G}(c) \subset \mathcal{G}(c')$. Moreover, as $c \to +\infty$, the radius $r(c)$ of the disc $\mathcal{G}(c)$ diverges and its bottom $(0, (c - \sqrt{c^2 - c_K^2})/2d)$ tends to $(0, 0)$. Consequently

$$\bigcup_{c \geq c_K} \mathcal{G}(c) = \mathbb{R} \times (0, +\infty).$$

On the other hand, the sets $\mathcal{S}(c)$ are not increasing with respect to c, but the sets $\mathcal{S}(c) \cap (\mathbb{R} \times \mathbb{R}_+)$ are. Indeed, under condition (4.3), $\alpha_D^+(c, \beta) \geq 0$ iff $c \geq q$ or $c \leq q$ and $\chi(\beta) \geq g'(0)$, and in both cases

$$2D\partial_c \alpha_D^+(c, \beta) = 1 + \frac{c - q}{\sqrt{(c-q)^2 + 4D(\chi(\beta) - g'(0))}} \geq 0,$$

whereas $\alpha_D^-(c, \beta) \geq 0$ iff $c \geq q$ and $\chi(\beta) \leq g'(0)$, which yields

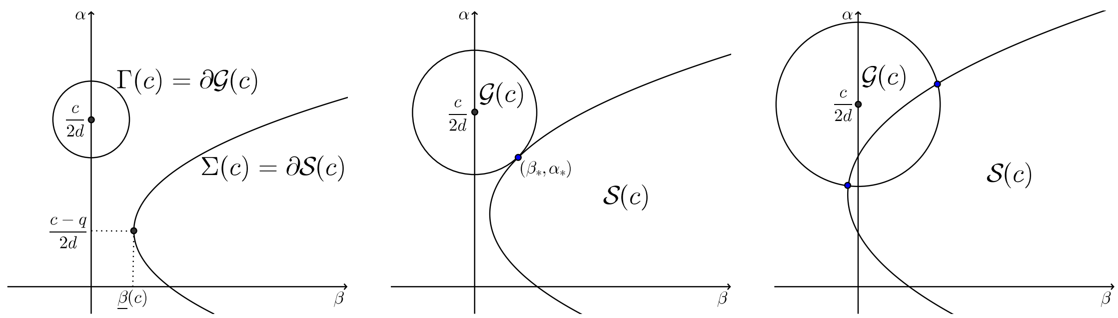
$$2D\partial_c \alpha_D^-(c, \beta) = 1 - \frac{c - q}{\sqrt{(c-q)^2 + 4D(\chi(\beta) - g'(0))}} \leq 0.$$

It follows that there exists $w_* \geq c_K$ such that $\mathcal{S}(c) \cap \mathcal{G}(c) \neq \emptyset$ iff $c \geq w_*$. Moreover, $w_* = c_K$ iff $(0, c_K/2d) \in \mathcal{S}(c_K)$. Otherwise $\mathcal{S}(w_*) \cap \mathcal{G}(w_*)$ reduces to a point in $(0, +\infty)^2$, denoted by (β_*, α_*). These notations will be kept in the following sections.

4.2 Spreading

Proof of Theorem 1.2 part (ii). Let (u, v) be the solution of (1.3) starting from a nonnegative, compactly supported initial datum $(u_0, v_0) \not\equiv (0, 0)$. We prove separately that (u, v) spreads (towards right) at most and then at least at the critical velocity of plane waves w_*.

Step 1. (u, v) *spreads at most with velocity* w_*.
The definition of w_* implies the existence of an exponential supersolution of the linearization of (1.3) (thus a supersolution of (1.3) by (1.6)) of the type (4.1), with



(a) $c < w_*$ (b) $c = w_*$ (c) $c > w_*$

Figure 1: Case $\frac{c_K}{2d} \notin \mathcal{S}(c_K)$.

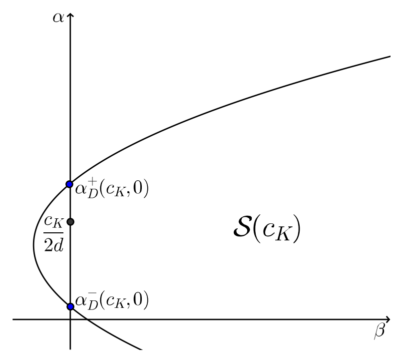


Figure 2: Case $\frac{c_K}{2d} \in \mathcal{S}(c_K)$.

$c = w_*$, $\gamma, \alpha > 0$ and $\beta \geq 0$. Since, up to translation in the direction e_1, this supersolution is above (u, v) at time $t = 0$, the comparison principle yields

$$\lim_{t \to +\infty} \sup_{\substack{x > (w_* + \varepsilon)t \\ y \geq 0}} |(u(x,t), v(x,y,t))| = 0.$$

Step 2. (u, v) *spreads at least with velocity* w_*.
We need to show that

$$\forall \varepsilon, a > 0, \qquad \lim_{t \to +\infty} \sup_{\substack{0 \leq x \leq (w_* - \varepsilon)t \\ 0 \leq y < a}} |(u(x,t), v(x,y,t) - (U, V(y))| = 0,$$

where (U, V) is the unique positive, bounded, stationary solution of (1.3). We first show that

$$\forall \varepsilon > 0, \ \exists c \in (w_* - \varepsilon, w_*), \quad \lim_{t \to +\infty} (u(x + ct, t), v(x + ct, y, t)) = (U, V(y)), \quad (4.4)$$

locally uniformly in $(x, y) \in \overline{\Omega}$. Then we will conclude applying the following lemma with $c^- = 0$, $c^+ = c$.

Lemma 4.1. *Let* $c^- < c^+$ *be such that any nonnegative, bounded, solution* $(u, v) \not\equiv (0, 0)$ *of* (1.3) *satisfies*

$$\lim_{t \to +\infty} (u(x + c^\pm t, t), v(x + c^\pm t, y, t)) = (U, V(y)),$$

locally uniformly in $(x,y) \in \overline{\Omega}$, *with* $U, V > 0$. *Then, such solutions satisfy*

$$\forall a > 0, \qquad \lim_{t\to+\infty} \sup_{\substack{c^- t \le x \le c^+ t \\ 0 \le y < a}} |(u(x,t), v(x,y,t) - (U, V(y))| = 0.$$

Let us postpone the proof of Lemma 4.1. In order to prove (4.4) it is sufficient to derive the analogue of Lemma 2.3 for the problem in the frame moving with speed c in the direction e_1, for c arbitrarily close to w_*. Then one concludes using Theorem 3.2 (or Proposition 3.1 in the case of (1.1) under the only assumption (1.2)). The only argument in the proof of Lemma 2.3 which is modified by the change of frame is the existence of a compactly supported, stationary, strict subsolution $(\underline{U}, \underline{V})$ for the linearised problem. If $w_* = c_K$ then the construction of a function $\underline{V}$ such that $(0, \underline{V})$ satisfies the desired properties is standard (see Lemma 6.2 of [4]). In the case $w_* > c_K$, the bifurcation analysis of Lemma 6.1 of [4] applies and provides the subsolution $(\underline{U}, \underline{V})$ for c close enough to w_*. Since this is the core of the argument, for the sake of completeness we sketch below the proof of that lemma.

One starts with penalizing $f'(0)$ and $g'(0)$ and considering the problem in the strip $\mathbb{R} \times (0, L)$, with a Dirichlet condition on $\mathbb{R} \times \{L\}$:

$$\begin{cases} -D\partial_{xx}u + (q-c)\partial_x u = \nu v(x,0,t) + (g'(0) - \delta + \mu)u & x \in \mathbb{R} \\ -d\Delta v - c\partial_x v = (f'(0) - \delta)v & (x,y) \in \mathbb{R} \times (0,L) \\ v(x,L) = 0 & x \in \mathbb{R} \\ -d\partial_y v(x,0,t) = \mu u(x) - \nu v(x,0,t) & x \in \mathbb{R}. \end{cases} \tag{4.5}$$

The presence of $\delta << 1$ can be viewed as a perturbation of the terms $f'(0)$, $g'(0)$; thus we can continue the discussion with $\delta = 0$. Exponential solutions are sought for in the form

$$e^{-\alpha x}(1, \gamma e^{-\beta y} + \tilde{\gamma} e^{\beta y}).$$

They exist for $(\beta, \alpha) \in \Gamma(c) \cap \Sigma_L(c)$, where $\Gamma(c)$ is the same as in the previous section and $\Sigma_L(c)$ is the union of two curves $\Sigma_L^{\pm}(c)$ parametrized by $\alpha = \alpha_L^{\pm}(c, \beta)$. The functions $\alpha_L^{\pm}$ have the same monotonicity in c as the functions $\alpha^{\pm}$ defining $\Sigma(c)$; moreover, as $L \to \infty$, $\alpha_L^{\pm} \to \alpha^{\pm}$ locally uniformly in $\beta > 0$, together with their derivatives. Calling $\mathcal{S}_L(c)$ the set between $\Sigma_L^-(c)$ and $\Sigma_L^+(c)$, it follows that $\mathcal{S}^L(c) \cap \mathcal{G}(c) \neq \emptyset$ iff $c \ge w_*^L$, with $w_*^L \to w_*$ as $L \to \infty$. In particular, $w_*^L > c_K$ for L large enough (and δ small enough). By using Rouché's theorem one eventually finds, for $c < w_*^L$ close enough to w_*^L, two values $\alpha, \beta \in \mathbb{C}\backslash\mathbb{R}$ such that the associated exponential function satisfies (4.5). Using its real part, one eventually obtain the compactly supported subsolution $(\underline{U}, \underline{V})$ for c arbitrarily close to w_*. $\square$

Proof of Lemma 4.1. Let (u, v) be as in the statement of the lemma and fix $a > 0$, $0 < \varepsilon < U$. Consider the solution (u^1, v^1) emerging from the initial datum $(\sup u, \sup v)$, and a solution $(u^2, v^2) \not\equiv (0,0)$ with an initial datum (u_0^2, v_0^2) satisfying

$$0 \le u_0^2 \le U - \varepsilon, \qquad \operatorname{supp} u_0^2 \subset [-1, 1], \qquad v_0^2 \equiv 0.$$

By hypothesis, for $a > 0$, there exists $T > 0$ such that

$$\forall t \geq T,\ 0 \leq y \leq a, \quad |(u^i(c^- t, t), v^i(c^- t, y, t)) - (U, V(y))| < \varepsilon. \tag{4.6}$$

Set $k := \max(1, |c^+ - c^-|T)$ and let $T' > 0$ be such that

$$\forall t \geq T',\ |x| \leq k,\ 0 \leq y \leq a, \quad |(u(x + c^+ t, t), v(x + c^+ t, y, t)) - (U, V(y))| < \varepsilon.$$

For $1/2 \leq \lambda \leq 1$, call $c := (1 - \lambda)c^- + \lambda c^+$. Let $t \geq 2T'$. If $(1 - \lambda)t \leq T$, then applying the previous estimate with $x = (c - c^+)t \in [-k, k]$, we derive

$$\forall y \in [0, a], \quad |(u(ct, t), v(ct, y, t)) - (U, V(y))| < \varepsilon.$$

Consider the case $(1 - \lambda)t > T$. For $(x, y) \in \overline{\Omega}$, we see that

$$u_0^2(x) \leq (U - \varepsilon)\mathbb{1}_{[-1,1]}(x) \leq u(x + c^+\lambda t, \lambda t) \leq \sup u, \quad v_0^2 \leq v(x + c^+\lambda t, y, \lambda t) \leq \sup v,$$

and then the comparison principle yields

$$u^2(x + (1 - \lambda)c^- t, (1 - \lambda)t) \leq u(x + ct, t) \leq u^1(x + (1 - \lambda)c^- t, (1 - \lambda)t)$$

$$v^2(x + (1 - \lambda)c^- t, y, (1 - \lambda)t) \leq v(x + ct, y, t) \leq v^1(x + (1 - \lambda)c^- t, y, (1 - \lambda)t).$$

Whence, by (4.6), we get again

$$\forall y \in [0, a], \quad |(u(ct, t), v(ct, y, t)) - (U, V(y))| < \varepsilon.$$

We have therefore shown that (u, v) converges to (U, V) as $t \to +\infty$ uniformly in the set $(c^- + c^+)t/2 \leq x \leq c^+ t$, $0 \leq y \leq a$. The proof is completed by exchanging the roles of c^- and c^+. □

4.3 Spreading velocity

Proof of Theorem 1.2 part (iii). We have shown in the previous section that the asymptotic speed of spreading coincides with the critical velocity of plane waves w_*. With the same notation as in Section 4.1, we know that $w_* = c_K$ iff $(0, c_K/2d) \in \mathcal{S}(c_K)$, that is, iff $\alpha_D^\pm(c_K, 0)$ are real and satisfy

$$\alpha_D^-(c_K, 0) \leq \frac{c_K}{2d} \leq \alpha_D^+(c_K, 0). \tag{4.7}$$

Requiring (4.7) is equivalent to

$$\left| \frac{c_K}{2d} - \frac{c_K - q}{2D} \right| \leq \frac{1}{2D}\sqrt{(c_K - q)^2 - 4Dg'(0)},$$

which, in turn, rewrites

$$\left(c_K \left(\frac{D}{d} - 1 \right) + q \right)^2 \leq (c_K - q)^2 - 4Dg'(0).$$

Notice that this condition automatically implies that $\alpha_D^\pm(c_K, 0)$ are real. Whence, recalling that $c_K = 2\sqrt{df'(0)}$, we find that (4.7) holds iff

$$4\frac{D}{d}f'(0) + 4\sqrt{\frac{f'(0)}{d}}\,q - 8f'(0) \leq -4g'(0).$$

This concludes the proof. □

5 The large diffusion and transport limits of $w_*^\pm$

We now prove Theorem 1.3. As before, we focus on the speed towards right w_*^+, recalling that it coincides with the critical wave speed w_* defined in Section 4.1.

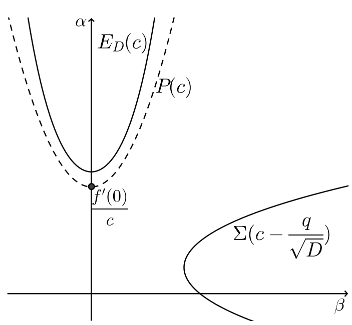
5.1 Large diffusion

Replacing c with $c = \sqrt{D}c$ and α with $\alpha/\sqrt{D}$ reduces (4.2) to the following system:

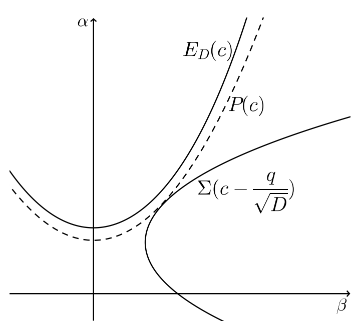
$$\begin{cases} -\alpha^2 + \left(c - \dfrac{q}{\sqrt{D}}\right)\alpha = -\dfrac{d\beta\mu}{\nu + d\beta} + g'(0) \\ \alpha = \dfrac{1}{c}(f'(0) + d\beta^2) + \dfrac{d}{Dc}\alpha^2. \end{cases} \tag{5.1}$$

The first equation is satisfied for $(\beta, \alpha) \in \Sigma(c - q/\sqrt{D})$, where $\Sigma = \Sigma^- \cup \Sigma^+$ is the curve defined in Section 4.1 with $D = 1$ and $q = 0$. The second equation is that of an ellipse $E_D(c)$ in the (β, α) plane, which is above the parabola $P(c)$ of equation $\alpha = (f'(0) + d\beta^2)/c$. For c close to 0, $\Sigma(c)$ is below $P(c)$. Then, increasing c, $\Sigma^+(c)$ moves upward while $P(c)$ moves downward and tends to the β axis as $c \to +\infty$. There exists then a positive value h such that $\Sigma(c) \cap P(c) \neq \emptyset$ if and only if $c \geq h$. Since $\Sigma(c - q/\sqrt{D})$ and $E_D(c)$ converge locally uniformly to $\Sigma(c)$ and $P(c)$ respectively as $D \to \infty$, and Σ is bounded in the α direction, it follows that if $c < h$ then (5.1) has no solution for D large enough, whereas if $c > h$ then (5.1) has solution for D large enough. Reverting to the original variable, we eventually infer that

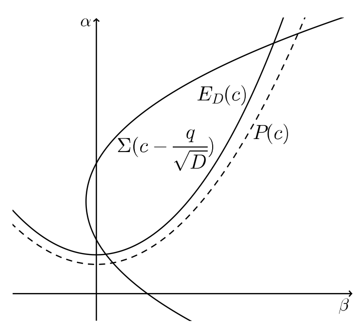
$$\lim_{D\to\infty} \frac{w_*}{\sqrt{D}} = h.$$



(a) $c < h$



(b) $c = h$



(c) $c > h$

Figure 3: The asymptotics of w_* as $D \to \infty$.

5.2 Large transport

We consider now w_* as a function of q. We know that $w_* = c_K$ for $-q$ large enough. Let us investigate the behaviour as $q \to +\infty$. Notice first that, as $q \to +\infty$, $\alpha_D^+(c, \beta) \to 0$ locally uniformly in c and uniformly in $\beta \geq 0$, from which we deduce that $w_* \to +\infty$. Let us take $c = \kappa q$ with $\kappa > 0$ and $q \to +\infty$. The set $\Gamma(\kappa q)$ intersect $\Sigma(\kappa q)$ if and only if there exists $\beta \geq 0$ such that

$$\frac{1}{2D}\left(q(\kappa-1)+\sqrt{q^2(\kappa-1)^2+4D(\chi(\beta)-g'(0))}\right) \geq \frac{\kappa q-\sqrt{\kappa^2q^2-c_K^2-4d^2\beta^2}}{2d}.$$

If $\kappa > 1$, this inequality holds for any given $\beta \geq 0$, provided q is large enough. This shows that $\limsup_{q\to+\infty} w_*/q \leq 1$. If $\kappa < 1$, the inequality implies that, as $q \to +\infty$,

$$\frac{\chi(\beta)-g'(0)}{1-\kappa}+o(1) \geq \frac{2f'(0)+2d\beta^2}{\kappa+\sqrt{\kappa^2-(c_K^2+4d^2\beta^2)q^{-2}}}.$$

In particular, β cannot diverge as $q \to +\infty$, whence

$$\frac{\chi(\beta)-g'(0)}{1-\kappa}+o(1) \geq \frac{f'(0)+d\beta^2}{\kappa}.$$

It follows that, if $g'(0) \geq \mu$ and $\kappa < 1$, $\Gamma(\kappa q)\cap\Sigma(\kappa q) = \emptyset$ for q large enough. Instead, if $g'(0) < \mu$, there is a threshold value

$$k := \left(1+\max_{\beta\geq 0}\left(\frac{\chi(\beta)-g'(0)}{f'(0)+d\beta^2}\right)\right)^{-1} < 1$$

such that $\Gamma(\kappa q)\cap\Sigma(\kappa q)$ is empty if $\kappa < k$ and q is large enough and is nonempty if $\kappa > k$ and q is large enough. Theorem 1.3 follows.

Remark 5.1. The computations in both the large diffusion and transport cases show that the scale at which the limit of w_* as $D, q \to +\infty$ is affected by both terms is $D \sim q^2$. This was of course expected by dimensional considerations.

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