Stability inequalities for Lebesgue constants via Markov-like inequalities *

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February 11, 2018

Abstract

We prove that L^∞-norming sets for finite-dimensional multivariate function spaces on compact sets are stable under small perturbations. This implies stability of interpolation operator norms (Lebesgue constants), in spaces of algebraic and trigonometric polynomials.

2010 AMS subject classification: 41A10, 41A63, 42A15, 65D05.

Keywords: Norming sets, Markov-like inequalities, stability inequalities, polynomial interpolation, (subperiodic) trigonometric interpolation, Lebesgue constant.

1 Introduction.

The purpose of this paper is to give a general setting in order to answer the following question: which is the response of Lebesgue constants (the projection operator norms) of interpolation to small perturbations of the sampling nodes?

The problem is of manifest practical interest, since in the applications not only the sampled values are affected by errors (and this essentially concerns stability of the operator via the Lebesgue constant), but also theoretically good sampling nodes are affected by nonnegligible measurement errors. For example, as it is well-known by the Runge phenomenon, point location is an essential feature with polynomials, in order to guarantee stability and convergence of the interpolation process.

*Work partially supported by the Horizon 2020 ERA-PLANET European project "GEOEssential", by the DOR funds and the biennial project BIRD163015 of the University of Padova, and by the GNCS-INdAM. This research has been accomplished within the RITA "Research ITalian network on Approximation".

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Embedding the problem in the general framework of norming set inequalities for finite-dimensional smooth function spaces, we prove below that stability holds under small perturbations, where the perturbation size depends on the norm of the gradient operator. This allows to get a general stability result for Lebesgue constants of univariate as well as multivariate interpolation operators. We discuss examples concerning polynomial and trigonometric interpolation, where Markov-like inequalities play a key role.

2 Small perturbations of L^∞-norming sets

Below, we shall adopt the notation $\|f\|_D = \sup_{x\in D} |f(x)|$ for the uniform norm of bounded complex-valued functions defined on a compact (continuous or discrete) set $D \subset \mathbb{C}^d$. Moreover, the notion of convexity in $\mathbb{C}^d$ is the one inherited from $\mathbb{R}^{2d}$.

Proposition 1 (on the stability of norming sets)
Let S be a finite-dimensional subspace of $C^1(\Omega;\mathbb{C})$, with Ω open subset of $\mathbb{C}^d$, and K a compact subset of Ω. Assume that there exist a compact subset $X \subset K$ and a constant $\lambda = \lambda(S, K, X) > 0$ such that

$$\|\phi\|_K \le \lambda\, \|\phi\|_X \ , \ \ \forall \phi \in S \ , \tag{1}$$

i.e., X is an L^∞-norming set for S on K. Moreover, let μ be a constant such that

$$\|\nabla\| = \sup_{S\ni\phi\neq 0} \frac{\|\,|\nabla\phi|\,\|_K}{\|\phi\|_K} \le \mu \tag{2}$$

(observe that $\nabla|_S$ is a linear operator between finite-dimensional spaces and hence is bounded), and let $\widetilde{X} \subset K$ be a perturbation of X of size $\varepsilon > 0$, in the sense that

$$\widetilde{X} \subseteq X + B[0,\varepsilon] \ , \tag{3}$$

where $B[0,\varepsilon]$ denotes the closed ball (in the euclidean norm) centered at 0 with radius ε.

Then, for every $\alpha \in (0,1)$ the following stability inequality holds

$$\|\phi\|_K \le \frac{\lambda}{1-\alpha}\, \|\phi\|_{\widetilde{X}} \ , \ \ \forall \phi \in S \ , \tag{4}$$

provided that

(i) K is convex and $\varepsilon = \frac{\alpha}{\lambda\,\mu}$;

or

(ii) S is a subspace of analytic functions on Ω, closed under partial differentiation (i.e., $\partial_j\phi \in S$, $\forall \phi \in S$, $1 \le j \le d$), and

$$\varepsilon < \min\left\{\frac{\log(1+\alpha/\lambda)}{\mu\, d}, dist(K,\partial\Omega)\right\} \ .$$

Observe that under assumption (ii) the compact set K can be nonconvex, or even totally disconnected.

Proof. Let us assume (i). Take any $\phi \in S$ and consider $\xi \in X$ such that $|\phi(\xi)| = \|\phi\|_X$. Due to (3) there exists $\widetilde{\xi} \in \widetilde{X}$ such that $|\xi - \widetilde{\xi}| \leq \varepsilon$. Thus, denoting by $\langle \cdot, \cdot \rangle$ the euclidean scalar product in $\mathbb{C}^d$, assuming $\widetilde{\xi} \neq \xi$ (otherwise (4) is obviously true) and setting $\tau = (\xi - \widetilde{\xi})/|\xi - \widetilde{\xi}|$, we have

$$|\phi(\xi)| \leq |\phi(\widetilde{\xi})| + |\phi(\xi) - \phi(\widetilde{\xi})|$$

$$= |\phi(\widetilde{\xi})| + \left| \int_0^{|\xi - \widetilde{\xi}|} \partial_\tau \phi(\widetilde{\xi} + t\tau)\, dt \right| \leq |\phi(\widetilde{\xi})| + \int_0^{|\xi - \widetilde{\xi}|} |\partial_\tau \phi(\widetilde{\xi} + t\tau)|\, dt$$

$$= |\phi(\widetilde{\xi})| + \int_0^{|\xi - \widetilde{\xi}|} \left| \langle \nabla \phi(\widetilde{\xi} + t\tau), \tau \rangle \right| \, dt$$

$$\leq \|\phi\|_{\widetilde{X}} + \| \, |\nabla \phi| \, \|_{[\widetilde{\xi}, \xi]} \, |\xi - \widetilde{\xi}| \leq \|\phi\|_{\widetilde{X}} + \mu\varepsilon \|\phi\|_K \, . \tag{5}$$

Here we used the bound (2) and the convexity of K. Indeed, we have $\| \, |\nabla \phi| \, \|_{[\widetilde{\xi}, \xi]} \leq \| \, |\nabla \phi| \, \|_K$ since the line segment $[\widetilde{\xi}, \xi]$ lies in K.

Therefore we have

$$\|\phi\|_K \leq \lambda \|\phi\|_X \leq \lambda \|\phi\|_{\widetilde{X}} + \mu\varepsilon\lambda \|\phi\|_K$$

and, since $\varepsilon = \frac{\alpha}{\lambda\mu}$ with $\alpha < 1$, (4) follows.

Now we assume (ii). Take $\phi \in S$, $\xi \in X$. By (3) there exists $\widetilde{\xi} \in \widetilde{X}$ such that $\|\xi - \widetilde{\xi}\|_\infty \leq |\xi - \widetilde{\xi}| \leq \varepsilon$. Since ϕ is analytic in Ω and the polydisc centered at $\widetilde{\xi}$ with radius ε lies in Ω, we have

$$\phi(\xi) = \phi(\widetilde{\xi}) + \sum_{\beta \in \mathbb{N}^d, |\beta| \geq 1} \frac{\partial^\beta \phi(\widetilde{\xi})}{\beta!} (\xi - \widetilde{\xi})^\beta \, .$$

Notice that, S being closed under partial differentiation, the inequality (2) can be iterated to get

$$|\partial^\beta \phi(x)| \leq \mu^{|\beta|} \|\phi\|_K \, , \ \ \forall \phi \in S, x \in K, \beta \in \mathbb{N}^d \, .$$

Therefore we can write

$$\|\phi\|_X \leq \|\phi\|_{\widetilde{X}} + \sum_{|\beta| \geq 1} \frac{\mu^{|\beta|} \|\phi\|_K}{\beta!} |\xi - \widetilde{\xi}|^{|\beta|} \leq \|\phi\|_{\widetilde{X}} + \sum_{|\beta| \geq 1} \frac{(\mu\varepsilon)^{|\beta|}}{\beta!} \|\phi\|_K$$

$$\leq \|\phi\|_{\widetilde{X}} + (\exp(\mu d \varepsilon) - 1) \|\phi\|_K \, .$$

Here we used the fact that $\sum_{|\beta| \geq 0, \beta \in \mathbb{N}^d} \frac{z^{|\beta|}}{\beta!} = e^{dz}$ for any $z \in \mathbb{C}$.

Finally, we have

$$\|\phi\|_K \le \lambda\|\phi\|_X \le \lambda\|\phi\|_{\widetilde{X}} + (\exp(\mu d\varepsilon) - 1)\lambda\|\phi\|_K \ ,$$

and since $\varepsilon < \frac{1}{\mu d}\log(1 + \alpha/\lambda)$ with $\alpha < 1$ we obtain

$$\|\phi\|_K < \lambda\|\phi\|_{\widetilde{X}} + \alpha\|\phi\|_K \ ,$$

from which equation (4) follows. $\square$

Proposition 1, as it is stated, is a general matter of functional inequalities. On the other hand, we can immediately specialize it to the sensitivity study of interpolation operators in finite-dimensional spaces. Let be

$$S = \mathrm{span}\{g_1, \ldots, g_N\} \ , \quad N = \dim(S) \ , \tag{6}$$

and let K be S-determining, i.e., a function of S vanishing there vanishes everywhere in Ω, or equivalently $\dim(S|_K) = \dim(S)$. Consider a finite set $X = \{x_1, \ldots, x_N\} \subset K$ of unisolvent sampling nodes for interpolation in S, a property equivalent to being an S-determining set. Let

$$\mathcal{L}_X : (C(K), \|\cdot\|_K) \to (S, \|\cdot\|_K)$$

be the interpolation operator associated to X,

$$\mathcal{L}_X f(x) = \sum_{i=1}^{N} f(x_i)\ell_{x_i}(x) \ , \tag{7}$$

where the ℓ_i are the cardinal functions of the unisolvent interpolation set X, defined by the generalized Vandermonde determinants

$$VDM(x_1, \ldots, x_N) = \det[g_j(x_i)]_{1 \le i,j \le N}$$

as

$$\ell_{x_i}(x) = \frac{VDM(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_N)}{VDM(x_1, \ldots, x_N)} \ , \quad 1 \le i \le N \ .$$

Now, consider the uniform norm of the interpolation operator

$$\lambda = \|\mathcal{L}_X\| = \max_{x \in K} \sum_{i=1}^{N} |\ell_{x_i}(x)| \ , \tag{8}$$

that, by extension from polynomial interpolation, we could term its “Lebesgue constant”.

As it is well-known, the Lebesgue constant plays a key role in the study of accuracy and stability of interpolation. Such a role can be summarized by the following estimate

$$\|f - \mathcal{L}_X \widetilde{f}\|_K \le (1 + \lambda)\,\mathrm{dist}_K(f, S) + \lambda\,\|f - \widetilde{f}\|_K \ , \tag{9}$$

where the first summand on the r.h.s. concerns accuracy ($\mathrm{dist}_K(f,S) = \inf_{\phi\in S} \|f-\phi\|_K$) and the second one stability with respect to errors on the sampled function. Thus the response of the Lebesgue constant itself to node perturbations, i.e. the stability analysis of the Lebesgue constant, is a relevant problem.

From Proposition 1 we get the following stability result

Corollary 1 (on the stability of Lebesgue constants)

Let S and $K \subset \Omega$ be as in Proposition 1, $N = dim(S) < \infty$, and let K be S-determining. Moreover, let $X = \{x_1,\dots,x_N\} \subset K$ be a finite S-determining (equivalently, unisolvent for interpolation) sampling set. Assume that $\widetilde{X} = \{\widetilde{x}_1,\dots,\widetilde{x}_N\} \subset K$ be a perturbation of X as in Proposition 1, such that (i) or (ii) holds with $\lambda = \|\mathcal{L}_X\|$.

Then the set $\widetilde{X}$ is unisolvent itself and, considering the interpolation operator $\mathcal{L}_X$ in (7) and the "perturbed" operator $\mathcal{L}_{\widetilde{X}}$ defined on $\widetilde{X}$

$$\mathcal{L}_{\widetilde{X}} f(x) = \sum_{i=1}^{N} f(\widetilde{x}_i)\ell_{\widetilde{x}_i}(x) \,, \tag{10}$$

the following stability inequality for the Lebesgue constant holds

$$\frac{\|\mathcal{L}_{\widetilde{X}}\|}{\|\mathcal{L}_X\|} \leq \frac{1}{1-\alpha} \,. \tag{11}$$

Proof. First observe that (1) holds with $\lambda = \|\mathcal{L}_X\|$, since $\|\mathcal{L}_X f\|_K \leq \lambda \|f\|_X$ for every $f \in C(K)$ and $\mathcal{L}_X \phi = \phi$ for every $\phi \in S$ ($\mathcal{L}_X$ being a projection). By Proposition 1 we get

$$\|\phi\|_K \leq \frac{\lambda}{1-\alpha}\,\|\phi\|_{\widetilde{X}}$$

for every $\phi \in S$, which shows that $\widetilde{X}$ is S-determining and thus unisolvent for interpolation in S. Then, $\mathcal{L}_{\widetilde{X}}$ is well-defined and we can write the chain of inequalities

$$\|\mathcal{L}_{\widetilde{X}} f\|_K \leq \frac{\lambda}{1-\alpha}\,\|\mathcal{L}_{\widetilde{X}} f\|_{\widetilde{X}} = \frac{\lambda}{1-\alpha}\,\|f\|_{\widetilde{X}} \leq \frac{\lambda}{1-\alpha}\,\|f\|_K \,,$$

i.e., (11) is verified. $\square$

Observe that (11) holds true even by replacing $\|\mathcal{L}_X\|$ with $\lambda \geq \|\mathcal{L}_X\|$, which is the most common situation in applications, where Lebesgue constants are not exactly known but only (often roughly) estimated.

Remark 1 We notice that a key feature of Proposition 1 and Corollary 1 is that the function space S is independent of X (the sampling set). This entails that the stability analysis naturally applies to spaces of algebraic or trigonometric polynomials, but cannot be used (at least in the present formulation) in X-dependent interpolation spaces, such as for example splines or radial basis functions.

2.1 Lipschitz-continuity of the Lebesgue constant

In this subsection we show that by Corollary 1 one obtains a Lipschitz continuity result for the Lebesgue constant, with respect to the Hausdorff distance of unisolvent interpolation sets. We recall that the Hausdorff distance of two d-dimensional (real or complex) compact sets is defined as

$$d_H(X,Y) = \inf\{\eta > 0 : Y \subseteq X + B[0,\eta],\ X \subseteq Y + B[0,\eta]\} ,$$

$B[0,\eta]$ denoting the closed euclidean ball centered at 0 with radius η.

Now, consider the Lebesgue constant $\lambda_X = \|\mathcal{L}_X\|$ in (8) as a function of the unisolvent interpolation subset $X \subset K$. Note that such a function goes to infinity as X approaches, in the Hausdorff distance, a subset where the generalized Vandermonde determinant vanishes (including collapse into a subset whose cardinality is less than N).

We can now state and prove the following

Proposition 2 *Let S and $K \subset \Omega$ be as in Proposition 1, $N = dim(S) < \infty$, and let $\mathcal{U}_N(K)$ be the set of the S-unisolvent interpolation subsets of K (endowed with the Hausdorff metric). Assume that*

(i) K is convex,

or

(ii) S is a subset of analytic functions on Ω, closed under partial differentiation.

Then, for any compact subset $E \subset \mathcal{U}_N(K)$ and $X, Y \in E$, assuming (i) we have

$$|\lambda_X - \lambda_Y| \leq L_1\, d_H(X,Y) ,\ \ L_1 = 2\mu\, (\max_E \lambda)^2 , \tag{12}$$

whereas assuming (ii)

$$|\lambda_X - \lambda_Y| \leq L_2\, d_H(X,Y) ,$$

$$L_2 = 2(\max_E \lambda)\ \max\left\{\mu d\,(1 + \max_E \lambda),\ \frac{1}{dist(K, \partial\Omega)}\right\} . \tag{13}$$

Proof. Let us pick a compact subset $E \subset \mathcal{U}_N(K)$, $\alpha \in (0,1)$, and any $X, Y \in E$ such that $d_H(X,Y) < \varepsilon_0$, with

$$\varepsilon_0 = \frac{\alpha}{\mu m}$$

under assumption (i), or

$$\varepsilon_0 = \min\left\{ \frac{\log\left(1+\frac{\alpha}{m}\right)}{\mu d}\,,\ \mathrm{dist}(K,\partial\Omega) \right\}$$

under assumption (ii), where we set $m = \max_E \lambda$. Observe that such a maximum exists and is finite, since λ is continuous in E, being the maximum in $x \in K$ of the Lebesgue function $F(x,X) = \sum_{i=1}^N |\ell_{x_i}(x)|$, which is continuous in $K \times E$, and hence uniformly continuous $K \times E$ being compact.

Proceeding as in the proof of Corollary 1 with $\varepsilon = d_H(X,Y)$, we have that

$$\lambda_Y \le \frac{1}{1-\mu\lambda_X d_H(X,Y)}\,\lambda_X\ ,\ \ \lambda_X \le \frac{1}{1-\mu\lambda_Y d_H(X,Y)}\,\lambda_Y\ ,$$

and thus

$$\lambda_Y - \lambda_X \le \mu\lambda_X\lambda_Y d_H(X,Y)\ ,\ \ \lambda_X - \lambda_Y \le \mu\lambda_X\lambda_Y d_H(X,Y)\ .$$

Hence we get

$$|\lambda_X - \lambda_Y| \le \mu m^2 d_H(X,Y),\ \forall X, Y \in E\ : d_H(X,Y) < \varepsilon_0\ . \tag{14}$$

Now, if on the contrary $d_H(X,Y) \ge \varepsilon_0$, we can write

$$\frac{|\lambda_X - \lambda_Y|}{d_H(X,Y)} \le \frac{|\lambda_X - \lambda_Y|}{\varepsilon_0} \le \frac{2m}{\varepsilon_0}\ .$$

Under assumption (i) we get

$$\frac{|\lambda_X - \lambda_Y|}{d_H(X,Y)} \le \frac{2m}{\alpha/(\mu m)} = \frac{2\mu m^2}{\alpha}\ ,$$

whereas assuming (ii)

$$\frac{|\lambda_X - \lambda_Y|}{d_H(X,Y)} \le \frac{2m}{\min\left\{\frac{\alpha}{\mu d(m+\alpha)}\,,\ \mathrm{dist}(K,\partial\Omega)\right\}}$$

$$= 2m\ \max\left\{\mu d\,\frac{m+\alpha}{\alpha}\,,\ \frac{1}{\mathrm{dist}(K,\partial\Omega)}\right\}\ ,$$

where we used the well-known inequality $\log(1+t) > t/(t+1)$, $t > 0$. Then (12)-(13) follow by taking the limit as $\alpha \to 1$. $\square$

3 Applications

3.1 Polynomial interpolation

In the framework of total-degree polynomial approximation, $S = \mathbb{P}_n^d$ (the subspace of d-variate polynomials with degree not exceeding n), polynomial inequalities like (1) with $X = X_n$ and $\lambda = \lambda_n$,

$$\|p\|_K \leq \lambda_n \|p\|_{X_n} \ , \ \ \forall p \in \mathbb{P}_n^d \ , \tag{15}$$

have been playing a central role in the last years, in particular starting from a seminal paper by Calvi and Levenberg [10]. Indeed, when the cardinality of X_n increases algebraically (necessarily, $\mathrm{card}(X_n) \geq N = \dim(\mathbb{P}_n^d) \sim n^d/d!$), and the quantities λ_n at most algebraically (even subexponentially when approximation of holomorphic functions is concerned), the norming sets X_n form a so-called "weakly admissible polynomial mesh" on the compact set K. If λ_n can be taken constant in n, we speak of an "admissible polynomial mesh", which is termed "optimal" when $\mathrm{card}(X_n) = \mathcal{O}(n^d)$. Among their applications, we recall that polynomial meshes are nearly-optimal for discrete Least Squares, can be considered as nice discrete models of Bernstein-Markov measures, and can be used also in the framework of polynomial optimization; cf., e.g., [2, 5, 10, 20, 23].

Observe that unisolvent interpolation sets with slowly increasing Lebesgue constant are weakly admissible polynomial meshes, where λ_n is (an estimate of) the Lebesgue constant itself. We recall, among other properties, that polynomial meshes are preserved by affine transformations, and can be easily extended by finite union and product. Concerning other theoretical and computational issues of polynomial meshes, we refer the reader, e.g., to [5, 10, 13, 16, 17, 19, 22, 25] and the references therein.

On the other hand, Markov polynomial inequalities, i.e. (2) with $S = \mathbb{P}_n^d$ and $\mu = \mu_n = Mn^r$,

$$|\nabla p(x)| \leq Mn^r \|p\|_K \ , \ \ \forall p \in \mathbb{P}_n^d \ \ \forall x \in K \ , \tag{16}$$

play a key role in polynomial approximation theory, and are known to hold for several classes of compact sets K, typically with $r = 2$. For example, for convex bodies in $\mathbb{R}^d$ the exponent is $r = 2$ and M can be taken as four times the reciprocal of the minimal distance between parallel supporting hyperplanes (or even two times such a reciprocal for centrally symmetric sets); cf. [29]. More generally, $r = 2$ for compact sets satisfying an interior cone condition. On the other hand, any nonpolar one-dimensional complex connected compact $K \subset \mathbb{C}$ admits (16) with $r = 2$ and $M = 2e/cap(K)$, where $cap(K)$ is the capacity of K, cf. [26] (but $r = 1$ in special instances, such as disks (circles) and ellipses). For a general view on multivariate polynomial inequalities we refer, e.g., to [24].

Stability of polynomial meshes and of Lebesgue contants of polynomial interpolation under small perturbations of the supporting discrete sets has been studied in [21], by arguments similar to those in Proposition 1 and Corollary 1, so we do not go into details here. We only observe qualitatively that, by Corollary 1, Lebesgue constants of unisolvent interpolation sets are stable under node perturbations of size $\varepsilon_n = \mathcal{O}\left(\frac{1}{n^r \lambda_n}\right)$, whenever the compact set admits a Markov inequality (16).

For example, $n+1$ equispaced nodes on the boundary of a complex disk $D = \{z \in \mathbb{C} : |z - c| \leq R\}$ have a $\mathcal{O}(\log n)$ Lebesgue constant, and the classical Markov inequality $\|p'\|_D \leq \frac{n}{R} \|p\|_D$ holds. In this case, we have stability under node perturbations of size $\mathcal{O}\left(\frac{1}{n \log n}\right)$. On the other hand, on an interval $[a, b]$ we have the classical Markov inequality $\|p'\|_{[a,b]} \leq \frac{2n^2}{b-a} \|p\|_{[a,b]}$, and the Lebesgue constant of the $n+1$ Chebyshev points in (a, b) is bounded by

$$\lambda_n^{\mathrm{cheb}} = 1 + \frac{2}{\pi} \log(n+1) \;, \tag{17}$$

cf., e.g., [9]. Then, stability holds under node perturbations of size $\mathcal{O}\left(\frac{1}{n^2 \log n}\right)$, namely

$$\varepsilon_n = \frac{\alpha(b-a)}{2n^2 \lambda_n^{\mathrm{cheb}}} \;, \quad 0 < \alpha < 1 \;. \tag{18}$$

The latter property is naturally connected with the so called "mock Chebyshev" approach to polynomial interpolation on the interval, i.e. the computational fact that the $n+1$ points closest to Chebyshev nodes in a sufficiently dense uniform grid behave in interpolation processes like the exact Chebyshev points; cf. [8, 12]. The density here is slightly higher than that adopted in [8], which corresponds to a grid step of size $\mathcal{O}(1/n^2)$ in order to mimic the density of Chebyshev points at the boundary, whereas the present choice provides a rigorous bound for stability of the Lebesgue constant, and thus of the interpolation process with "perturbed" Chebyshev nodes.

A bivariate example, concerning perturbation of the so-called Padua points [4] for total-degree polynomial interpolation on a square, was discussed in [21]. The perturbation size is $\varepsilon_n = \mathcal{O}\left(\frac{1}{n^2 \log^2 n}\right)$ in this case, since (16) is satisfied with $r = 2$ and $M = 2/L$ (with L length of the square side), and the Lebesgue constant of the Padua points has an optimal growth of order $\mathcal{O}(\log^2 n)$; cf. [4]. For the purpose of illustration, in figure 1 we plot the Lebesgue constant λ_n of the Padua points, and that of random perturbations of such points with radius $\varepsilon_n = \frac{\alpha}{n^2 \lambda_n}$, for some values of α at degrees $2, 3, \ldots, 20$. Such Lebesgue constants have been evaluated numerically on a suitable control mesh. Observe that for $\alpha = 0.5$ the Lebesgue constant is very close to the exact one, much closer than what is predicted by estimate

(11), and even for $\alpha = 1, 2, 4$ (where (11) does not apply) its size increases less than 20% (except for $\alpha = 4$ with $n = 2$).

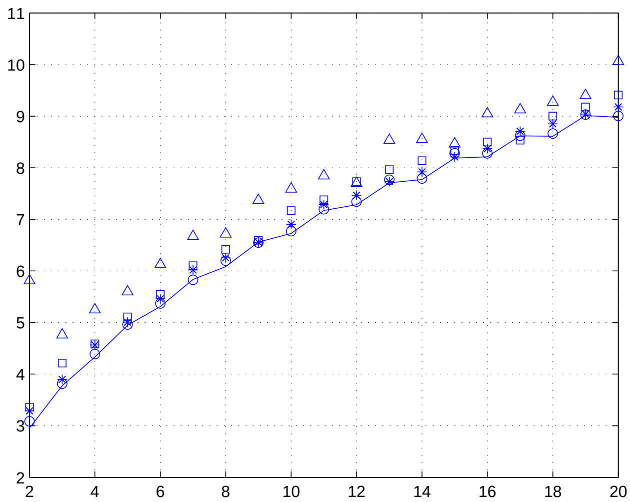

Figure 1: Lebesgue constant of Padua points (solid line) compared to that of the perturbed points with $\alpha = 0.5$ (circles), $\alpha = 1$ (asterisks), $\alpha = 2$ (squares), $\alpha = 4$ (triangles).

3.1.1 Interpolation on the 2-sphere

First, we observe that Proposition 1, Corollary 1 and Proposition 2 can be extended to the case of K compact and *geodesically convex* subset of a smooth d-dimensional submanifold $\mathcal{M} \subset \mathbb{R}^m$. We recall that geodesically convex means that any two points of K can be connected by a geodesic of $\mathcal{M}$ which lies entirely on K. Observe that a tangential inequality for S holds there, that is

$$|\partial_\tau \phi(x)| \le \mu \, \|\phi\|_K \; , \;\; \forall \phi \in S \;\; \forall x \in K \; , \tag{19}$$

where τ is any unit vector in the tangent space at x. Indeed, $\|\partial_\tau\| = \sup_{\phi \neq 0} \|\partial_\tau \phi\|_K / \|\phi\|_K$ is bounded independently of τ, by choosing a finite cover of K by local charts and considering the maximum of the norms of the partial derivatives in these local coordinates, that are bounded linear operators since S is finite-dimensional.

In this framework we have to replace the euclidean distance with the geodesic distance on $\mathcal{M}$. For example, in (5) the integral can be made along a geodesic connecting ξ and $\widetilde{\xi}$, parametrized in the arclength (an intrinsically Lipschitz-continuous, and thus almost everywhere differentiable, parametrization).

Now, let $S = \mathbb{P}_n^m(\mathcal{M})$ be the restriction to $\mathcal{M}$ of m-variate polynomials with degree not exceeding n, where $N = \dim(\mathbb{P}_n^m(\mathcal{M})) \leq \dim(\mathbb{P}_n^m)$. In the case of compact algebraic varieties, like the 2-dimensional sphere and torus, a polynomial tangential Markov inequality with exponent $r = 1$ holds, namely $\mu = Mn$ in (19. For example, on a 2-sphere with radius R we have $\mu = n/R$ and $N = (n+1)^2$.

The problem of finding good points for polynomial interpolation on the 2-sphere has been studied in [30], producing numerically extensive tables of nodes (on the unit sphere) and associated quantities, such as Lebesgue constants; cf. [27]. In particular, extremal-type nodes have been computed by numerical optimization, for example by maximizing the Vandermonde determinant (Fekete points) in the spherical harmonics basis, up to degree 165 (the Lebesgue constants are computed there up to degree 120).

From the considerations above, we know that the geodesic perturbation radius of the interpolation nodes (cf. ε in Proposition 1, *(i)* -(ii)) related to an (over)estimate of the perturbed Lebesgue constant by a factor $\rho = 1/(1-\alpha)$, for a fixed $\alpha \in (0,1)$, is here

$$\varepsilon = \frac{\alpha R}{n\lambda_n} \ , \tag{20}$$

or conversely, for a fixed perturbation ε

$$\alpha = \alpha_n(\varepsilon) = \frac{\varepsilon\, n\lambda_n}{R} \ , \quad \rho = \rho_n(\varepsilon) = \frac{1}{1-\alpha_n(\varepsilon)} = \frac{R}{R - \varepsilon\, n\lambda_n} \ , \tag{21}$$

as long as $\alpha_n(\varepsilon) < 1$.

Let us consider, for instance, interpolation on the surface of the earth, approximated to a sphere, at the Fekete points tabulated in [27] (scaled by the earth radius), together with the corresponding numerically evaluated Lebesgue constants (whose growth rate, as observed in [27], turns out to be closer to a linear one in n than to the theoretical quadratic overestimate $(n+1)^2$ for Fekete points). Taking into account that the average positioning error by current GPS-like systems (cf. [18]) has a size of $\varepsilon \approx 5$ meters and that the earth radius is $R \approx 6371$ kilometers, the corresponding factors ρ_n are displayed in Table 1, for a sequence of degrees. Observe that the size of the Lebesgue constant can be considered practically invariant for such positiong errors.

As a curiosity we consider also the factors obtainable by old-fashioned celestial navigation devices (sextant for the latitude and clock for the longitude, cf. [15]), whose average error is of 0.25 nautical miles (about 463 meters) in both the coordinates. Taking into account that at this scale the geodesic and euclidean distances practically coincide, we have $\varepsilon \approx 463\sqrt{2} \approx 655$ meters. The corresponding factors ρ_n are displayed in Table 1. For $n \geq 82$ estimate (11)-(21) cannot be used, because $\alpha_n > 1$.

Table 1: Polynomial interpolation at extremal points on the earth surface; ρ_n is the ratio in (21) obtainable by GPS or by traditional Celestial Navigation devices.

deg. n	30	40	50	60	70	80	100	120
pts.card.	961	1681	2601	3721	5041	6561	10201	14641
Leb.c. λ_n	36.28	48.91	69.47	72.48	97.40	109.75	160.69	220.45
ρ_n GPS	1.0009	1.0015	1.0027	1.0034	1.0054	1.0069	1.0128	1.0212
ρ_n C.N.	1.1260	1.2518	1.5555	1.8086	3.3440	10.2741	$*$	$*$

3.2 Subperiodic trigonometric interpolation

The problem of the stability of Lebesgue constants of trigonometric interpolation seems to have been considered in the literature only very recently [1]. We show here that the question can be posed in the more general setting of "subperiodic" trigonometric interpolation, i.e. interpolation by trigonometric polynomials on subintervals of the period, obtaining results that are in some sense complementary with respect to those in [1].

By no loss of generality we can consider the $(2n+1)$-dimensional space of trigonometric polynomials of degree not exceeding n, restricted to the angular interval $[-\omega,\omega]$, $0<\omega\leq\pi$, say

$$\mathbb{T}_n([-\omega,\omega]) = \text{span}\{1,\cos(\theta),\sin(\theta),\dots,\cos(n\theta),\sin(n\theta)\}\ ,\ \ \theta\in[-\omega,\omega]\ .$$

For an angular interval $[\eta_1,\eta_2]$, $\eta_2-\eta_1\leq 2\pi$, one simply takes the angular transformation

$$u=\left(\theta-\frac{\eta_2+\eta_1}{2}\right)+\omega\ ,\ \ \theta\in[-\omega,\omega]\ ,\ \omega=\frac{\eta_2-\eta_1}{2}\ .$$

In the study of norming set inequalities and interpolation in $\mathbb{T}_n([-\omega,\omega])$ a key role is played by the nonlinear transformation

$$\theta=\psi(x)=2\arcsin(x\,\sin(\omega/2))\ ,\ \ x\in[-1,1]\ , \tag{22}$$

with inverse $x=\psi^{-1}(\theta)=\sin(\theta/2)/\sin(\omega/2)$, $\theta\in[-\omega,\omega]$.

In [28] the norming set inequality

$$\|t\|_{[\omega,\omega]}\leq\frac{\nu}{\nu-1}\,\|t\|_{\psi(CL_{\lceil\nu\pi n\rceil})}\ ,\ \ \forall t\in\mathbb{T}_n([-\omega,\omega])\ ,\ \ \nu>1\ , \tag{23}$$

has been proved by the classical Videnskii inequality (cf. [3, §5.1, E. 19]), where

$$CL_m=\left\{\cos\left(\frac{i\pi}{m}\right)\right\}\ ,\ \ 0\leq i\leq m$$

denotes the $m+1$ Chebyshev-Lobatto points of degree m.

On the other hand, in [7, 11] interpolation at the nodal angles

$$\Theta_n(\omega) = \{\theta_i\} = \psi(C_{2n}) \;, \tag{24}$$

$$C_{2n} = \{\xi_i\} = \left\{\cos\left(\frac{(2i+1)\pi}{2(2n+1)}\right)\right\} \;,\;\; 0 \le i \le 2n \;,$$

has been studied, where

$$C_m = \left\{\cos\left(\frac{(2i+1)\pi}{2m+2}\right)\right\} \;,\;\; 0 \le i \le m$$

denotes the $m+1$ Gauss-Chebyshev points, i.e. the zeros of $T_{m+1}(x)$. Observe that $\Theta_n(\pi)$ are $2n+1$ equally spaced angles in $[-\pi,\pi]$, whereas for $\omega < \pi$ the nodal angles cluster at the endpoints.

In particular, denoting by $\ell_{\xi_i}(x) = T_{2n+1}(x)/(T'_{2n+1}(\xi_i)(x-\xi_i))$ the i-th Lagrange polynomial of the $2n+1$ Gauss-Chebyshev points, the trigonometric cardinal functions turn out to be

$$\ell_{\theta_i}(\theta) = a_i(\theta)\ell_{\xi_i}(\psi^{-1}(\theta)) + b_i(\theta)\ell_{\xi_{2n+2-i}}(\psi^{-1}(\theta)) \;,\;\; i \ne n+1 \;,$$

and $\ell_{\theta_{n+1}}(\theta) = \ell_{\xi_{n+1}}(\psi^{-1}(\theta))$, where

$$a_i(\theta) = \frac{1}{2}\left(1 + \frac{\cos(\theta/2)}{\cos(\theta_i/2)}\right) \;,\;\; b_i(\theta) = \frac{1}{2}\left(1 - \frac{\cos(\theta/2)}{\cos(\theta_i/2)}\right) = 1 - a_i(\theta) \;,$$

cf. [7, Prop. 1]. Moreover, a nontrivial analysis shows that the Lebesgue constant can be bounded, uniformly in ω, by $\lambda_{2n}^{\mathrm{cheb}}$, where $\lambda_n^{\mathrm{cheb}}$ is defined in (17); cf. [11].

Concerning trigonometric Markov inequalities, it is known that

$$\|t'\|_{[-\omega,\omega]} \le A(\omega) n^{r(\omega)} \|t\|_{[-\omega,\omega]} \;,\;\; \forall t \in \mathbb{T}_n([-\omega,\omega]) \;,\;\; n \ge 1 \;, \tag{25}$$

where

$$A(\omega) = \begin{cases} 2\cot(\omega/2) & n > \frac{1}{2}\sqrt{3\tan^2(\omega/2)+1} \\ 1 + \frac{16\pi(\pi-\omega)}{\omega} & \text{otherwise} \end{cases}$$

and

$$r(\pi) = 1 \;,\;\; r(\omega) = 2 \;,\; \omega < \pi \;,$$

cf. [3, §5.1,E. 14-19]. Notice that on the whole period ($\omega = \pi$), (25) is the classical inequality $\|t'\|_{[-\pi,\pi]} \le n\,\|t\|_{[-\pi,\pi]}$. This apparently surprising discontinuity in the Markov exponent $r(\omega)$ comes from a deep result. Indeed, a trigonometric polynomial $t \in \mathbb{T}_n([-\omega,\omega])$ can be identified with a bivariate algebraic polynomial of the same degree on an arc of the unit circle, and the trigonometric Markov inequality with a tangential Markov inequality for polynomials (cf. (19) with $\mu = Mn^r$). By [6] a compact submanifold of $\mathbb{R}^m$

admits a tangential Markov inequality with exponent $r = 1$ if and only if it is algebraic. Note that, in particular, the unit circle has $r = 1$, being an algebraic curve, whereas any proper subarc has $r = 2$, cf. [6, Prop. 6.1].

In view of (25), by Proposition 1 we get that the norming set $\psi(CL_{\lceil \nu\pi n\rceil})$ in (23) is stable under node perturbations not exceeding

$$\varepsilon_n = \frac{\alpha\nu}{(\nu-1)A(\omega)n^{r(\omega)}}\ ,\ \ 0<\alpha<1\ .$$

whose size is $\mathcal{O}\left(\frac{1}{n^{r(\omega)}}\right)$. On the other hand, by Corollary 1 the (estimate of the) Lebesgue constant of trigonometric interpolation is stable under node perturbations not exceeding

$$\varepsilon_n = \frac{\alpha}{A(\omega)n^{r(\omega)}\lambda_{2n}^{\rm cheb}}\ ,\ \ 0<\alpha<1\ , \tag{26}$$

whose size is $\mathcal{O}\left(\frac{1}{n^{r(\omega)}\log n}\right)$.

In the periodic case, $\omega = \pi$, It is worth comparing such a result with the recent study of trigonometric interpolation in perturbed points, developed in [1]. By some deep technical results it is there essentially shown that perturbing $N = 2n+1$ equally spaced nodes in $[-\pi, \pi)$ by arbitrary amounts not exceeding $2\pi\beta/N \sim \beta\pi/n$, $0 < \beta < 1/2$, the resulting Lebesgue constant of trigonometric interpolation is bounded by $C\,(N^{4\beta}-1)/(\beta(1-2\beta))$, where C is a universal constant (for $\beta \to 0$ a logarithmic bound is recovered); cf. [1, Thm. 2]. Moreover, it is conjectured on the base of numerical results that the sharp exponent be 2β instead of 4β, and that $C \approx 0.8$.

Our analysis is in some sense complementary to that quoted above, concerning "small perturbations". Indeed, by (26) we have that the Lebesgue constant is still logarithmic and bounded by $\lambda_{2n}^{\rm cheb}/(1-\alpha)$, $0 \le \alpha < 1$, for perturbations of $2n+1$ equally spaced nodes by arbitrary amounts not exceeding $\alpha/(n\lambda_{2n}^{\rm cheb}) \sim \alpha\pi/(2n\log(n))$, cf. (17).

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