Stability inequalities for Lebesgue constants via Markov-like inequalities

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Abstract

We prove that $L^\infty$-norming sets for finite-dimensional function spaces on compact sets admitting a Markov-like inequality, are stable under small perturbations. This implies stability of interpolation operator norms (Lebesgue constants), in spaces of algebraic and trigonometric polynomials.

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1 Introduction.

The purpose of this paper is to give a general setting in order to answer the following question: which is the response of Lebesgue constants (the projection operator norms) of interpolation to small perturbations of the sampling nodes?

The problem is of manifest practical interest, since in the applications not only the sampled values are affected by errors (and this essentially concerns stability of the operator via the Lebesgue constant), but also theoretically good sampling nodes are affected by nonnegligible measurement errors. Think, for instance, to sampling by GPS-like systems at prescribed locations on a region of the earth. In these cases, stability of the Lebesgue constant itself becomes relevant. For example, as it is well-known by the Runge phenomenon, point location is an essential feature with polynomials, in order to guarantee stability and convergence of the interpolation process.

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Embedding the problem in the general framework of norming set inequalities, we prove below that stability holds under small perturbations whenever the approximation domain admits a Markov-like inequality for the relevant function space. This allows to get a general stability result for Lebesgue constants of univariate as well as multivariate interpolation operators. We discuss examples concerning polynomial and trigonometric interpolation.

2 Small perturbations of $L^\infty$-norming sets

Below, we shall adopt the notation $\|f\|_D = \sup_{x \in D} |f(x)|$ for the uniform norm of bounded complex-valued functions defined on a compact (continuous or discrete) set $D \subset \mathbb{C}^d$.

**Proposition 1** (on the stability of norming sets)

Let $S \subset C^1(K; \mathbb{C})$, where $K \subset \mathbb{C}^d$ is a compact set.

Assume that

- there exist a compact subset $X \subset K$ and a constant $\lambda = \lambda(S, K, X) > 0$ such that
  \[ \|\phi\|_K \leq \lambda \|\phi\|_X , \forall \phi \in S , \]  
  i.e., $X$ is a $L^\infty$-norming set for $S$ on $K$;

- $K$ admits a Markov-like inequality for $S$, i.e., there exists a constant $\mu = \mu(S, K) > 0$ such that
  \[ |\nabla \phi(x)| \leq \mu \|\phi\|_K , \forall \phi \in S \forall x \in K . \]

Moreover, let $\tilde{X} \subset K$ be a perturbation of $X$ of size $\varepsilon > 0$, in the sense that

\[ \tilde{X} \subseteq X + B[0, \varepsilon] , \]

where $B[0, \varepsilon]$ denotes the closed ball (in the euclidean norm) centered at 0 with radius $\varepsilon$.

Then, for every $\alpha \in (0, 1)$ the following stability inequality holds

\[ \|\phi\|_K \leq \frac{\lambda}{1 - \alpha} \|\phi\|_{\tilde{X}} , \forall \phi \in S , \]

provided that

(i) $K$ is convex and $\varepsilon = \frac{\alpha}{K \mu}$;

or

(ii) $S$ is a subset of analytic functions closed under partial differentiation (i.e., $\partial_j \phi \in S, \forall \phi \in S, 1 \leq j \leq d$), and $\varepsilon = \frac{\log(1+\alpha/\lambda)}{\mu d}$.
Observe that under assumption (ii) the compact set $K$ can be nonconvex, or even disconnected.

**Proof.** Let us assume (i). Take any $\phi \in S$ and consider $\xi \in X$ such that $|\phi(\xi)| = \|\phi\|_X$. Due to (3) there exists $\tilde{\xi} \in \tilde{X}$ such that $|\xi - \tilde{\xi}| \leq \varepsilon$. Thus, denoting by $\langle \cdot, \cdot \rangle$ the euclidean scalar product in $\mathbb{C}^d$, assuming $\tilde{\xi} \neq \xi$ (otherwise (4) is obviously true) and setting $\tau = (\xi - \tilde{\xi})/(\xi - \tilde{\xi})$, we have

$$|\phi(\xi)| \leq |\phi(\tilde{\xi})| + |\phi(\xi) - \phi(\tilde{\xi})|$$

$$\leq |\phi(\tilde{\xi})| + \left| \int_0^{[\xi-\tilde{\xi}]} \left\langle \nabla \phi \left( \tilde{\xi} + t\tau \right) \right., \tau \left. \right\rangle \, dt \right|$$

$$\leq |\phi(\tilde{\xi})| + \int_0^{[\xi-\tilde{\xi}]} \left| \left\langle \nabla \phi \left( \tilde{\xi} + t\tau \right) \right. \left. \right\rangle \right| \, dt$$

$$\leq \|\phi\|_X + \|\nabla \phi\|_{\tilde{\xi},\xi} |\xi - \tilde{\xi}| \leq \|\phi\|_X + \mu \varepsilon \|\phi\|_K.$$  \hspace{1cm} (5)

Here we used the Markov inequality (2) and the convexity of $K$. Indeed, we have

$$\| \nabla \phi \|_{\tilde{\xi},\xi} \leq \| \nabla \phi \|_{\tilde{\xi},\xi} \leq \| \nabla \phi \|_K$$

since the line segment $[\tilde{\xi},\xi]$ lies in $K$.

Therefore we have

$$\|\phi\|_K \leq \lambda \|\phi\|_X \leq \lambda \|\phi\|_X + \mu \varepsilon \lambda \|\phi\|_K$$

and, since $\varepsilon = \frac{\alpha}{\lambda \mu}$ with $\alpha < 1$, (4) follows.

Now we assume (ii). Take $\phi, \xi, \tilde{\xi}$ as above. Since $\phi$ is analytic in a a neighbourhood of $K$ and both $\xi, \tilde{\xi}$ lie in $K$ we have

$$\phi(\xi) = \phi(\tilde{\xi}) + \sum_{\beta \in \mathbb{N}^d, |\beta| \geq 1} \frac{\partial^{\beta} \phi(\tilde{\xi})}{\beta!} (\xi - \tilde{\xi})^\beta.$$ 

Notice that, since $S$ is closed under partial differentiation, the Markov inequality (2) can be iterated to get

$$|\partial^{\beta} \phi(x)| \leq \mu^{[\beta]} \|\phi\|_K, \quad \forall \phi \in S, x \in K, \beta \in \mathbb{N}^d.$$ 

Therefore we can write

$$\|\phi\|_X \leq \|\phi\|_X + \sum_{|\beta| \geq 1} \frac{\mu^{[\beta]} \|\phi\|_K |\xi - \tilde{\xi}|^{[\beta]}}{\beta!} \leq \|\phi\|_X + \sum_{|\beta| \geq 1} \frac{(\mu \varepsilon)^{[\beta]} \|\phi\|_K}{\beta!}$$

$$\leq \|\phi\|_X + (\exp(\mu \varepsilon) - 1) \|\phi\|_K.$$ 

Here we used the fact that $\sum_{|\beta| \geq 0, \beta \in \mathbb{N}^d} \frac{z^{|\beta|}}{\beta!} = e^z$ for any $z \in \mathbb{C}$.

Finally, we have

$$\|\phi\|_K \leq \lambda \|\phi\|_X \leq \lambda \|\phi\|_X + (\exp(\mu \varepsilon) - 1) \lambda \|\phi\|_K,$$
and since \( \varepsilon = \frac{1}{\mu d} \log(1 + \alpha/\lambda) \) with \( \alpha < 1 \) we obtain
\[
\| \phi \|_K \leq \lambda \| \phi \|_\tilde{X} + \alpha \| \phi \|_K,
\]
from which equation (4) follows. □

Proposition 1, as it is stated, is a general matter of functional inequalities. On the other hand, we can immediately specialize it to the sensitivity study of interpolation operators in finite-dimensional spaces
\[
S = \text{span}\{g_1, \ldots, g_N\}, \quad N = \dim(S).
\] (6)

In fact, consider a finite set \( X = \{x_1, \ldots, x_N\} \) of unisolvent sampling nodes for interpolation in \( S \), a property equivalent to being an \( S \)-determining set (i.e., a function of \( S \) vanishing there vanishes everywhere on \( K \); this clearly implies that necessarily \( K \) itself is \( S \)-determining). Let
\[
L_X : (C(K), \| \cdot \|_K) \to (S, \| \cdot \|_K)
\]
the interpolation operator associated to \( X \),
\[
L_X f(x) = \sum_{i=1}^{N} f(x_i) \ell_{x_i}(x),
\] (7)
where the \( \ell_{x_i}(x) \) are the cardinal functions of the unisolvent interpolation set \( X \), defined by the generalized Vandermonde determinants
\[
VDM(x_1, \ldots, x_N) = \det[g_j(x_i)]_{1 \leq i,j \leq N}
\] (8)
as
\[
\ell_{x_i}(x) = \frac{VDM(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_N)}{VDM(x_1, \ldots, x_N)}, \quad 1 \leq j \leq N .
\] (9)

Now, consider the uniform norm of the interpolation operator
\[
\lambda = \| L_X \| = \max_{x \in K} \sum_{i=1}^{M} |\ell_{x_i}(x)|,
\] (10)
that by extension from polynomial interpolation we could term its “Lebesgue constant”.

As it is well-known, the Lebesgue constant plays a key role in the study of accuracy and stability of interpolation. Such a role can be summarized by the following estimate
\[
\| f - L_X \tilde{f} \|_K \leq (1 + \lambda) \text{dist}_K(f, S) + \lambda \| f - \tilde{f} \|_K,
\] (11)
where the first summand on the r.h.s. concerns accuracy (\( \text{dist}_K(f, S) = \inf_{\phi \in S} \| f - \phi \|_K \)) and the second one stability with respect to errors on
the sampled function. Thus the response of the Lebesgue constant itself to node perturbations, i.e., the stability analysis of the Lebesgue constant, is a relevant problem. Think for example to polynomial interpolation, where node location is an essential feature in view of the Runge phenomenon.

From Proposition 1 we get the following stability result

**Corollary 1** (on the stability of Lebesgue constants)

Let $S \subset C^1(K; \mathbb{C})$, where $K \subset \mathbb{C}^d$ is a ($S$-determining) compact set, and $X = \{x_1, \ldots, x_N\} \subset K$, $N = \dim(S)$, a finite $S$-determining (equivalently, unisolvent for interpolation) sampling set. Assume that (2) holds, and let $\tilde{X} = \{\tilde{x}_1, \ldots, \tilde{x}_N\} \subset K$ be a perturbation of $X$, such that (i) or (ii) holds with $\lambda = \|L_X\|$.

Then the set $\tilde{X}$ is unisolvent itself and, considering the interpolation operator $L_X$ in (7) and the “perturbed” operator $L_{\tilde{X}}$ defined on $\tilde{X}$

$$L_{\tilde{X}} f(x) = \sum_{i=1}^{N} f(\tilde{x}_i) \ell_{\tilde{x}_i}(x),$$

(12)

the following stability inequality for the Lebesgue constant holds

$$\frac{\|L_{\tilde{X}}\|}{\|L_X\|} \leq \frac{1}{1 - \alpha}.$$  

(13)

**Proof.** First observe that (11) holds with $\lambda = \|L_X\|$, since $\|L_X f\|_K \leq \lambda \|f\|_X$ for every $f \in C(K)$ and $L_X \phi = \phi$ for every $\phi \in S$ ($L_X$ being a projection). By Proposition 1 we get

$$\|\phi\|_K \leq \frac{\lambda}{1 - \alpha} \|\phi\|_{\tilde{X}}$$

for every $\phi \in S$, which shows that $\tilde{X}$ is $S$-determining and thus unisolvent for interpolation in $S$. Then, $L_{\tilde{X}}$ is well-defined and we can write the chain of inequalities

$$\|L_{\tilde{X}} f\|_K \leq \frac{\lambda}{1 - \alpha} \|L_{\tilde{X}} f\|_{\tilde{X}} = \frac{\lambda}{1 - \alpha} \|f\|_{\tilde{X}} \leq \frac{\lambda}{1 - \alpha} \|f\|_K,$$

i.e., (13) is verified. □

Observe that (13) holds true even by replacing $\|L_X\|$ with $\lambda \geq \|L_X\|$, which is the most common situation in applications, where Lebesgue constants are not exactly known but only (often roughly) estimated.

**Remark 1** We notice that a key feature of Proposition 1 and Corollary 1 is that the function space $S$ is independent of $X$ (the sampling set).
entails that the stability analysis naturally applies to spaces of algebraic or trigonometric polynomials, but cannot be used (at least in the present formulation) in $X$-dependent interpolation spaces, such as for example splines or radial basis functions. Concerning rational interpolation, a peculiar difficulty is given by the fact that Markov-like inequalities do not generally hold for rational functions, and are known only in special restricted instances, cf., e.g., [2].

**Remark 2** Proposition 1 and Corollary 1 can be extended to the case of $K$ compact and geodesically convex subset of a smooth $d$-dimensional submanifold $M \subset \mathbb{R}^m$, provided that a Markov-like tangential inequality for $S$ holds there, that is

$$|\partial_{\tau} \phi(x)| \leq \mu \|\phi\|_K, \quad \forall \phi \in S \quad \forall x \in K,$$

where $\tau$ is any unit vector in the tangent space at $x$. In this framework we have to replace the euclidean distance with the geodesic distance on $M$. The proof is similar to the first part of that of Proposition 1, where in (5) the integral can be made along a geodesic connecting $\xi$ and $\tilde{\xi}$, parametrized in the arclength (an intrinsically Lipschitz-continuous, and thus almost everywhere differentiable, parametrization).

### 2.1 Lipschitz-continuity of the Lebesgue constant

In this subsection we show that by Corollary 1 one obtains a Lipschitz continuity result for the Lebesgue constant, with respect to the Hausdorff distance of unisolvent interpolation sets. We recall that the Hausdorff distance of two $d$-dimensional (real or complex) compact sets is defined as

$$d_H(X,Y) = \inf\{\eta > 0 : Y \subseteq X + B[0,\eta], \ X \subseteq Y + B[0,\eta]\},$$

$B[0,\eta]$ denoting the closed euclidean ball centered at 0 with radius $\eta$.

Now, consider the Lebesgue constant $\lambda_X = \|L_X\|$ in (10) as a function of the unisolvent interpolation subset $X \subset K$. Note that such a function goes to infinity as $X$ approaches, in the Hausdorff distance, a subset where the Vandermonde determinant (8) vanishes (including collapse into a subset whose cardinality is less than $N$).

We can now state and prove the following

**Proposition 2** Let $S \subset C^1(K;\mathbb{C})$, where $K \subset \mathbb{C}^d$ is a compact set, and $\mathcal{U}_N(K)$ be the set of the $S$-unisolvent subsets of $K$ (endowed with the Hausdorff metric). Assume that $K$ admits a Markov-like inequality such as [2], and that

(i) $K$ is convex,
or

(ii) $S$ is a subset of analytic functions closed under partial differentiation.

Then, for any compact subset $E \subset U_N(K)$ and $X, Y \in E$, assuming (i) we have

$$|\lambda_X - \lambda_Y| \leq L_1 d_H(X, Y), \quad L_1 = 2\mu (\max_E \lambda)^2,$$

whereas assuming (ii)

$$|\lambda_X - \lambda_Y| \leq L_2 d_H(X, Y), \quad L_2 = 2\mu d \max_E \lambda (1 + \max_E \lambda).$$

Proof. Let us pick a compact subset $E \subset U_N(K)$, $\alpha \in (0, 1)$, and any $X, Y \in E$ such that $d_H(X, Y) \leq \varepsilon_0$, with

$$\varepsilon_0 = \frac{\alpha}{\mu m}$$

under assumption (i), or

$$\varepsilon_0 = \frac{\log (1 + \frac{\alpha}{m})}{\mu d}$$

under assumption (ii), where we set $m = \max_E \lambda$. Observe that such a maximum exists and is finite, since $\lambda$ is continuous in $E$, being the maximum in $x \in K$ of the Lebesgue function $F(x, X) = \sum_{i=1}^{N} |\ell_{x_i}(x)|$, which is continuous in $(x, X) \in K \times E$, and hence uniformly continuous $K \times E$ being compact.

Proceeding as in the proof of Proposition 1 with $\varepsilon = d_H(X, Y)$, we have that

$$\lambda_Y \leq \frac{1}{1 - \mu \lambda_X d_H(X, Y)} \lambda_X, \quad \lambda_X \leq \frac{1}{1 - \mu \lambda_Y d_H(X, Y)} \lambda_Y,$$

and thus

$$\lambda_Y - \lambda_X \leq \mu \lambda_X \lambda_Y d_H(X, Y), \quad \lambda_X - \lambda_Y \leq \mu \lambda_X \lambda_Y d_H(X, Y).$$

Hence we get

$$|\lambda_X - \lambda_Y| \leq \mu m^2 d_H(X, Y), \quad \forall X, Y \in E : d_H(X, Y) \leq \varepsilon_0.$$

(17)

Now, if on the contrary $d_H(X, Y) > \varepsilon_0$, we can write

$$\frac{|\lambda_X - \lambda_Y|}{d_H(X, Y)} < \frac{|\lambda_X - \lambda_Y|}{\varepsilon_0} \leq \frac{2m}{\varepsilon_0}.$$
Under assumption (i) we get
\[ |\lambda_X - \lambda_Y| \leq \frac{2m}{\alpha/(\mu m)} = \frac{2\mu m^2}{\alpha}, \]
whereas assuming (ii)
\[ |\lambda_X - \lambda_Y| \leq \frac{2\mu dm}{\log(1 + \frac{\alpha}{m})} < \frac{2\mu dm + \alpha}{\alpha}, \]
where we used the well-known inequality \( \log(1 + t) > t/(t + 1), \ t > 0. \) Then (15)-(16) follow by taking the limit as \( \alpha \to 1. \)

3 Applications

3.1 Polynomial interpolation

In the framework of total-degree polynomial approximation, \( S = \mathbb{P}^d_n, \) polynomial inequalities like (1) with \( X = X_n \) and \( \lambda = \lambda_n, \)
\[ \| p \|_K \leq \lambda_n \| p \|_{X_n}, \ \forall p \in \mathbb{P}^d_n, \]  
(18)
have begun to play a central role in the last years, in particular starting from a seminal paper by Calvi and Levenberg [13]. Indeed, when the cardinality of \( X_n \) increases algebraically (necessarily, \( \text{card}(X_n) \geq N = \dim(\mathbb{P}^d_n) \sim n^d/d! \)), and the quantities \( \lambda_n \) at most algebraically (even subexponentially when approximation of holomorphic functions is concerned), the norming sets \( X_n \) form a so-called “weakly admissible mesh” for polynomials on the compact set \( K. \) If \( \lambda_n \) can be taken constant in \( n, \) we speak of an “admissible mesh”, which is termed “optimal” when \( \text{card}(X_n) = \mathcal{O}(n^d). \)

Observe that unisolvent interpolation sets with slowly increasing Lebesgue constant are weakly admissible meshes, where \( \lambda_n \) is (an estimate of) the Lebesgue constant itself. We recall, among other properties, that weakly admissible meshes are preserved by affine transformations, and can be easily extended by finite union and product. Concerning theoretical and computational issues of (weakly) admissible polynomial meshes, we refer the reader, e.g., to [1, 5, 6, 8, 13, 16, 19, 20] and the references therein.

On the other hand, Markov polynomial inequalities, i.e. (2) with \( S = \mathbb{P}^d_n \) and \( \mu = \mu_n = Mn^r, \)
\[ |\nabla p(x)| \leq Mn^r \|p\|_K, \ \forall p \in \mathbb{P}^d_n \ \forall x \in K, \]  
(19)
play a key role in polynomial approximation theory, and are known to hold for several classes of compact sets \( K, \) typically with \( r = 2. \) For example, for convex compact sets with nonempty interior in \( \mathbb{R}^d \) the exponent is \( r = 2 \) and \( M \) can be taken as four times the reciprocal of the minimal distance.
between parallel supporting hyperplanes (or even two times such a reciprocal for centrally symmetric sets); cf. [30]. More generally, \( r = 2 \) for compact sets satisfying an interior cone condition, cf. e.g. [29 §3.3]. On the other hand, any nonpolar one-dimensional complex compact \( K \subset \mathbb{C} \) admits \( (19) \) with \( r = 2 \) and \( M = 2e/cap(K) \), where \( cap(K) \) is the capacity of \( K \), cf. [26] (but \( r = 1 \) in special instances, such as disks (circles) and ellipses). For a general view on multivariate polynomial inequalities we refer, e.g., to [24].

Stability of polynomial meshes and of Lebesgue constants of polynomial interpolation under small perturbations of the supporting discrete sets has been studied in [22, 23], by arguments similar to those in Proposition 1 and Corollary 1, so we do not go into details here. We only observe qualitatively that, by Corollary 1, Lebesgue constants of unisolvent interpolation sets are stable under node perturbations of size \( \varepsilon_n = \mathcal{O} \left( \frac{1}{n^2 \lambda_n} \right) \), whenever the compact set admits a Markov inequality \( (19) \).

For example, \( n + 1 \) equispaced nodes on the boundary of a complex disk \( D = \{ z \in \mathbb{C} \mid |z - c| \leq R \} \) have a \( \mathcal{O}(\log n) \) Lebesgue constant, and the classical Markov inequality \( \| p' \|_D \leq \frac{2}{\pi} \| p \|_D \) holds. In this case, we have stability under node perturbations of size \( \mathcal{O} \left( \frac{1}{n \log n} \right) \). On the other hand, on an interval \([a, b]\) we have the classical Markov inequality \( \| p' \|_{[a, b]} \leq \frac{2n^2}{b-a} \| p \|_{[a, b]} \), and the Lebesgue constant of the \( n + 1 \) Chebyshev points in \((a, b)\) is bounded by

\[
\lambda_{n, \text{cheb}} = 1 + \frac{2}{\pi} \log(n + 1),
\]

(cf., e.g., [12]). Then, stability holds under node perturbations of size \( \mathcal{O} \left( \frac{1}{n^2 \log n} \right) \), namely

\[
\varepsilon_n = \frac{\alpha(b - a)}{2n^2 \lambda_{n, \text{cheb}}}, \quad 0 < \alpha < 1.
\]

The latter property is naturally connected with the so-called “mock Chebyshev” approach to polynomial interpolation on the interval, i.e. the computational fact that the \( n + 1 \) points closest to Chebyshev nodes in a sufficiently dense uniform grid behave in interpolation processes like the exact Chebyshev points; cf. [10, 11, 15]. The density here is slightly higher than that adopted in [10, 11], which corresponds to a grid step of size \( \mathcal{O}(1/n^2) \) in order to mimic the density of Chebyshev points at the boundary, whereas the present choice provides a rigorous bound for stability of the Lebesgue constant, and thus of the interpolation process with “perturbed” Chebyshev nodes.

A bivariate example, concerning perturbation of the so-called Padua points [4] for total-degree polynomial interpolation on a square, was discussed in [23]. The perturbation size is \( \varepsilon_n = \mathcal{O} \left( \frac{1}{n^2 \log^2 n} \right) \) in this case, since \( (19) \) is satisfied with \( r = 2 \) and \( M = 2/L \) (with \( L \) length of the square side), and the Lebesgue constant of the Padua points has an optimal growth of
order $O(\log^2 n)$; cf. [4]. For the purpose of illustration, in Figure 1 we plot the Lebesgue constant $\lambda_n$ of the Padua points, and that of random perturbations of such points with radius $\varepsilon_n = \frac{\alpha}{\lambda_n}$, for some values of $\alpha$ at degrees $2, 3, \ldots, 20$. Such Lebesgue constants have been evaluated numerically on a suitable control mesh. Observe that for $\alpha = 0.5$ the Lebesgue constant is very close to the exact one, much closer than what is predicted by estimate (13), and even for $\alpha = 1, 2, 4$ (where (13) does not apply) its size increases less than 20% (except for $\alpha = 4$ with $n = 2$).

![Figure 1: Lebesgue constant of Padua points (solid line) compared to that of the perturbed points with $\alpha = 0.5$ (circles), $\alpha = 1$ (asterisks), $\alpha = 2$ (squares), $\alpha = 4$ (triangles).](image)

3.1.1 Interpolation on the 2-sphere

By Remark 2, the stability analysis of Lebesgue constants can be extended to polynomial interpolation on smooth $d$-dimensional submanifolds of $\mathbb{R}^n$. In this framework, $S = \mathbb{P}_m^n(K)$ (the subspace of $m$-variate polynomials restricted to $K$), with dimension $N \leq \dim(\mathbb{P}_m^n)$. In the case of compact algebraic varieties, like the 2-dimensional sphere and torus, a polynomial tangential Markov inequality holds with $\mu = Mn$, cf. (14), i.e. with exponent $r = 1$ (a property that indeed characterizes compact algebraic varieties, cf. [7]). For example, on a 2-sphere with radius $R$ we have $\mu = n/R$ and $N = (n + 1)^2$.

The problem of finding good points for polynomial interpolation on the 2-sphere has been studied in [31], producing numerically extensive tables of nodes (on the unit sphere) and associated quantities, such as Lebesgue
constants; cf. [27]. In particular, extremal-type nodes have been computed by numerical optimization, for example by maximizing the Vandermonde determinant (Fekete points) in the spherical harmonics basis, up to degree 165 (the Lebesgue constants are computed there up to degree 120).

From Remark 2 and the considerations above, we know that the geodesic perturbation radius of the interpolation nodes (cf. $\varepsilon$ in Proposition 1, (i) - (ii)) related to an (over)estimate of the perturbed Lebesgue constant by a factor $\rho = 1/(1 - \alpha)$, for a fixed $\alpha \in (0, 1)$, is here

$$\varepsilon = \frac{\alpha R}{n \lambda_n},$$

(22)
or conversely, for a fixed perturbation $\varepsilon$

$$\alpha = \alpha_n(\varepsilon) = \frac{\varepsilon n \lambda_n}{R}, \quad \rho = \rho_n(\varepsilon) = \frac{1}{1 - \alpha_n(\varepsilon)} = \frac{R}{R - \varepsilon n \lambda_n},$$

(23)
as long as $\alpha_n(\varepsilon) < 1$.

Let us consider, for instance, interpolation on the surface of the earth, approximated to a sphere, at the Fekete points tabulated in [27] (scaled by the earth radius), together with the corresponding numerically evaluated Lebesgue constants (whose growth rate, as observed in [27], turns out to be closer to a linear one in $n$ than to the theoretical quadratic overestimate $(n + 1)^2$ for Fekete points). Taking into account that the average positioning error by current GPS-like systems (cf. [21]) has a size of $\varepsilon \approx 5$ meters and that the earth radius is $R \approx 6371$ kilometers, the corresponding factors $\rho_n$ are displayed in Table 1, for a sequence of degrees. Observe that the size of the Lebesgue constant can be considered practically invariant for such positioning errors.

As a curiosity we consider also the factors obtainable by old-fashioned celestial navigation devices (sextant for the latitude and clock for the longitude, cf. [18]), whose average error is of 0.25 nautical miles (about 463 meters) in both the coordinates. Taking into account that at this scale the geodesic and euclidean distances practically coincide, we have $\varepsilon \approx 463\sqrt{2} \approx 655$ meters. The corresponding factors $\rho_n$ are displayed in Table 1. For $n \geq 82$ estimate (13) - (23) cannot be used, because $\alpha_n > 1$.

### 3.2 Subperiodic trigonometric interpolation

The problem of the stability of Lebesgue constants of trigonometric interpolation, and more generally of norming sets for trigonometric polynomials, with respect to perturbation of the nodal angles, does not seem to have been addressed in the literature. We show that the question can be posed in the more general setting of “subperiodic” trigonometric interpolation, i.e. interpolation by trigonometric polynomials on subintervals of the period.
Table 1: Polynomial interpolation at extremal points on the earth surface; $\rho_n$ is the ratio in (23) obtainable by GPS or by traditional Celestial Navigation devices.

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<tr>
<td>$\rho_n$ C.N.</td>
<td>1.1260</td>
<td>1.2518</td>
<td>1.5555</td>
<td>1.8086</td>
<td>3.3440</td>
<td>10.2741*</td>
<td>*</td>
<td>*</td>
</tr>
</tbody>
</table>

By no loss of generality we can consider the $(2n + 1)$-dimensional space of trigonometric polynomials of degree not exceeding $n$, restricted to the angular interval $[-\omega, \omega]$, $0 < \omega \leq \pi$, say

$$T_n([-\omega, \omega]) = \text{span}\{1, \cos(\theta), \sin(\theta), \ldots, \cos(n\theta), \sin(n\theta)\}, \ \theta \in [-\omega, \omega].$$

For an angular interval $[\eta_1, \eta_2]$, $\eta_2 - \eta_1 \leq 2\pi$, one simply takes the angular transformation

$$u = \left(\theta - \frac{\eta_2 + \eta_1}{2}\right) + \omega, \ \theta \in [-\omega, \omega], \ \omega = \frac{\eta_2 - \eta_1}{2}. \quad (25)$$

In the study of norming set inequalities and interpolation in $T_n([-\omega, \omega])$ a key role is played by the nonlinear transformation

$$\theta = \psi(x) = 2\arcsin(x \sin(\omega/2)), \ x \in [-1, 1], \quad (26)$$

with inverse $x = \psi^{-1}(\theta) = \sin(\theta/2)/\sin(\omega/2), \ \theta \in [-\omega, \omega]$.

In [28] the norming set inequality

$$\|t\|_{\omega, \omega} \leq \frac{\nu}{\nu - 1} \|t\|_{\psi(CL_{\lfloor\nu m\rfloor})}, \ \forall t \in T_n([-\omega, \omega]), \ \nu > 1, \quad (27)$$

has been proved by the classical Videnskii inequality (cf. [3, §5.1, E. 19]), where

$$CL_m = \left\{\cos\left(\frac{i\pi}{m}\right)\right\}, \ 0 \leq i \leq m \quad (28)$$

denotes the $m + 1$ Chebyshev-Lobatto points of degree $m$.

On the other hand, in [3, 14] interpolation at the nodal angles

$$\Theta_n(\omega) = \{\theta_i\} = \psi(C_{2n}),$$

$$C_{2n} = \{\xi_i\} = \left\{\cos\left(\frac{(2i + 1)\pi}{2(2n + 1)}\right)\right\}, \ 0 \leq i \leq 2n,$$

has been studied, where

$$C_m = \left\{\cos\left(\frac{(2i + 1)\pi}{2m + 2}\right)\right\}, \ 0 \leq i \leq m \quad (30)$$

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denotes the \( m + 1 \) Gauss-Chebyshev points, i.e. the zeros of \( T_{m+1}(x) \). Observe that \( \Theta_{n}(\pi) \) are \( 2n + 1 \) equally spaced angles in \([-\pi, \pi]\), whereas for \( \omega < \pi \) the nodal angles cluster at the endpoints.

In particular, denoting by \( \ell_{\xi_{i}}(x) = \frac{T_{2n+1}(x)}{T_{2n+1}'(\xi_{i})(x - \xi_{i})} \) the \( i \)-th Lagrange polynomial of the \( 2n + 1 \) Gauss-Chebyshev points, the trigonometric cardinal functions turn out to be

\[
\ell_{\theta_{n}+1}(\theta) = \frac{a_{i}(\theta)}{2} \left( 1 + \cos(\theta/2) \cos(\theta_{i}/2) \right), \quad b_{i}(\theta) = \frac{1}{2} \left( 1 - \frac{\cos(\theta/2)}{\cos(\theta_{i}/2)} \right) = 1 - a_{i}(\theta), \quad (32)
\]

cf. [8, Prop. 1]. Moreover, a nontrivial analysis shows that the Lebesgue constant can be bounded, uniformly in \( \omega \), by \( \lambda_{\text{cheb}}^{2n} \), where \( \lambda_{\text{cheb}}^{n} \) is defined in (20); cf. [14].

Concerning trigonometric Markov inequalities, it is known that

\[
\| t' \|_{[-\omega, \omega]} \leq A(\omega) n^{r(\omega)} \| t \|_{[-\omega, \omega]}, \quad \forall t \in T_{n}([-\omega, \omega]), \; n \geq 1 , \quad (33)
\]

where

\[
A(\omega) = \begin{cases} 
2 \cot(\omega/2) & n > \frac{1}{2} \sqrt{3 \tan^{2}(\omega/2) + 1} \\
1 + \frac{16\pi(\pi - \omega)}{\omega} & \text{otherwise}
\end{cases}
\]

and

\[
r(\pi) = 1 , \; r(\omega) = 2 , \; \omega < \pi ,
\]

cf. [3, §5.1.E. 14-19]. Notice that on the whole period \( (\omega = \pi) \), (33) is the classical inequality \( \| t' \|_{[-\pi, \pi]} \leq n \| t \|_{[-\pi, \pi]} \). This apparently surprising discontinuity in the Markov exponent \( r(\omega) \) comes from a deep result. Indeed, a trigonometric polynomial \( t \in T_{n}([-\omega, \omega]) \) can be identified with a bivariate algebraic polynomial of the same degree on an arc of the unit circle, and the trigonometric Markov inequality with a tangential Markov inequality for polynomials (cf. [14] with \( \mu = M n^{r} \)). By [7] a compact submanifold (possibly with boundary) of \( \mathbb{R}^{m} \) admits a tangential Markov inequality with exponent \( r = 1 \) if and only if it is algebraic. Note that, in particular, the unit circle has \( r = 1 \), being an algebraic curve, whereas any proper subarc has \( r = 2 \), cf. [7, Prop. 6.1].

In view of (33), by Proposition 1 we get that the norming set \( \psi(\mathcal{C}L_{[\nu \pi n]}) \) in (27) is stable under node perturbations not exceeding

\[
\varepsilon_{n} = \frac{\alpha \nu}{(\nu - 1) A(\omega)n^{r(\omega)}} , \quad 0 < \alpha < 1 .
\]
whose size is $O\left(\frac{1}{n^{\varepsilon n}}\right)$. On the other hand, by Corollary 1 the (estimate of the) Lebesgue constant of trigonometric interpolation is stable under node perturbations not exceeding

$$\varepsilon_n = \frac{\alpha}{A(\omega)n^{\varepsilon n}\lambda_{\text{cheb}}^{\max}} , \quad 0 < \alpha < 1 ,$$

whose size is $O\left(\frac{1}{n^{\varepsilon n} \log n}\right)$.

References


