Title

Some explicit solutions to a system of implicit partial differential equations

by Paolo Marcellini (University of Firenze)

Abstract

A rigid map $u: \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a Lipschitz-continuous map with the property that at every $x \in \Omega$ where $u$ is differentiable then its gradient $Du(x)$ is an orthogonal $m \times n$ matrix; i.e., $Du(x) \in O(n)$. If $\Omega$ is convex, then $u$ is globally a short map, in the sense that $|u(x) - u(y)| \leq |x - y|$ for every $x, y \in \Omega$; while locally, around any point of continuity of the gradient, $u$ is an isometry. Our motivation to introduce Lipschitz-continuous local isometric immersions (versus maps of class $C^1$) is based on the possibility of solving Dirichlet problems; i.e., we can impose boundary conditions. We also propose an approach to the analytical theory of origami, the ancient Japanese art of paper folding. An origami is a piecewise $C^1$ rigid map $u: \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ (plus a condition which exclude self intersections). If $u(\Omega) \subset \mathbb{R}^2$ we say that $u$ is a flat origami. In this case (and in general when $m = n$) we are able to describe the singular set $\Sigma_u$ of the gradient $Du$ of a piecewise $C^1$ rigid map: it turns out to be the boundary of the union of convex disjoint polyhedra, and some facet and edge conditions (angle condition) are satisfied. We show that these necessary conditions are also sufficient to recover a given singular set; i.e., that every polyhedral set $\Sigma$ which satisfies the angle condition is in fact the singular set $\Sigma_u$ of a map $u$, which is uniquely determined once we fix the value $u(x_0) \in \mathbb{R}^n$ and the gradient $Du(x_0) \in O(n)$ at a single point $x_0 \in \Omega \setminus \Sigma$. We use this characterization to explicitly solve a class of Dirichlet problems associated to some partial differential systems of implicit type.