The Hopf–Lax formula in Carnot groups: an approach with optimal control theory.

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Abstract

Let us consider the Heisenberg group $I\!H$, the simplest model of Carnot group: in this situation, the horizontal gradient of a function $v : I\!H(\sim I\!R^3) \to I\!R$ at the point $x = (x_1, x_2, x_3)$ is defined by

$$\mathbb{X}v(x_1, x_2, x_3) = \Big(v_{x_1}(x) - \frac{x_2}{2}v_{x_3}(x),\ v_{x_2}(x) + \frac{x_1}{2}v_{x_3}(x)\Big).$$

We are interested in the problem

$$\begin{cases} H\Big(\mathbb{X}u(x,t)\Big) + u_t(x,t) = 0, & \forall (x,t) \in I\!H \times (0,\infty) \\ u(x,0) = g(x), & \forall x \in I\!H \end{cases} \tag{1}$$

where the Hamiltonian $H : I\!R^2 \to I\!R$ is a convex, non negative and superlinear function and the initial date $g : I\!R^3 \to I\!R$ is Lipschitz.

In the particular case $H(\cdot) = \varphi(|\cdot|)$ where $\varphi : [0,\infty) \to I\!R$ is a convex, increasing, superlinear function, Manfredi and Stroffolini [3] proved that the unique viscosity solution for (1) is given by the Hopf–Lax formula

$$u(x,t) = \min_{y \in I\!H} \left\{ t\varphi^* \left(\frac{d_{cc}(x,y)}{t} \right) + g(y) \right\} \tag{2}$$

where φ^* is the Fenchel transform of the functoin φ and d_{CC} is the Carnot–Carathéodory distance in $I\!H$.

We prove a Hopf–Lax formula for the problem (1) that generalizes (2). Such result is obtained using an optimal control approach similar to the classical case (see for instance the book of Evan [2]). We are able to pass from the Heisenberg situation to a generic Carnot group and to generalize a recent result obtained by Dragoni [1].

These recent results are obtained with Z. Balogh and R. Pini.

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