On Compactness Estimates for Hamilton-Jacobi Equations

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Abstract. Consider a first-order Hamilton-Jacobi equation
\[ u_t(t, x) + H(\nabla u(t, x)) = 0, \quad x \in \mathbb{R}^N, \quad t > 0, \]  
with a strictly convex and coercive Hamiltonian \( H : \mathbb{R}^N \to \mathbb{R} \). For every \( \bar{u} \in W^{1,1}(\mathbb{R}^N, \mathbb{R}) \), let \( S_t \bar{u} = u(t, \cdot) \) denote the unique viscosity solution of (0.1) with initial data \( u(0, \cdot) = \bar{u} \). Having in mind the analysis recently developed for solutions to conservation laws [2-4], inspired by a question posed by Lax, we are interested in studying the compactifying effect of the operator \( S_t \) at any fixed time \( t > 0 \), w.r.t the \( W^{1,1} \)-topology. Namely, we wish to estimate the Kolmogorov \( \varepsilon \)-entropy in \( W^{1,1} \) of the image of bounded sets of initial data through the map \( S_t \). We recall that, given a metric space \((X, d)\), and a totally bounded subset \( K \) of \( X \), we let \( N_\varepsilon(K \mid X) \) denote the minimal number of sets in a cover of \( K \) by subsets of \( X \) having diameter \( \leq 2\varepsilon \), and define the Kolmogorov \( \varepsilon \)-entropy of \( K \) as \( H_\varepsilon(K \mid X) = \log_2 N_\varepsilon(K \mid X) \). Entropy numbers play a central role in various areas of information theory and statistics as well as of ergodic and learning theory. In the present setting, as suggested by Lax, this concept could provide a measure of the order of “resolution” of a numerical method for (0.1).

Our main result in [1] shows that, for every fixed \( L, M > 0 \), letting \( C_{[L,M]} \) denote the set of Lipschitz functions \( u : \mathbb{R}^N \to \mathbb{R} \) with Lipschitz constant \( L \) and with support contained in \([-M, M]^N\), there holds
\[ H_\varepsilon(S_T(C_{[L,M]}) \mid W^{1,1}(\mathbb{R}^N, \mathbb{R})) \approx (1/\varepsilon^N). \]  
Relying on fine properties of monotone operators we derive upper estimates on the \( \varepsilon \)-entropy of classes of semiconcave functions, which in turn yield upper estimates on \( H_\varepsilon(S_T(C_{[L,M]})) \). Instead, lower bounds on \( H_\varepsilon(S_T(C_{[L,M]}) \mid W^{1,1}(\mathbb{R}^N, \mathbb{R})) \) are established in two steps. We first introduce a class of semiconcave functions \( SF \) defined as combinations of suitable bump functions, and with a combinatorial argument we provide an optimal lower estimate on the \( \varepsilon \)-entropy of such a class. Next, we prove a controllability result showing that any element of \( SF \) can be obtained, at any given time \( T > 0 \), as the value \( u(t, \cdot) \) of a viscosity solution of (0.1), with initial data in \( C_{[L,M]} \).

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References