Global hypoellipticity for operators on compact manifolds and Diophantine phenomena

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We study the global hypoellipticity (GH) of a class of first order operators of type

\[ L = D_t + a(t)Q(x, D) + ib(t)P(x, D), \quad D_t = i^{-1} \partial_t, \quad t \in \mathbb{T}, \quad x \in M, \]

where \( \mathbb{T} = \mathbb{R}/(2\pi \mathbb{Z}) \) stands for the one-dimensional flat torus, \( M \) is a closed \( C^\infty \) manifold, \( a, b \) are real smooth functions on \( \mathbb{T} \), and \( P(x, D), Q(x, D) \) are self-adjoint first order pseudo-differential operators on \( M \). We recall that \( L \) is GH if \( Lu = f \in C^\infty (\mathbb{T} \times M), u \in D'(\mathbb{T} \times M) \) implies that \( u \in C^\infty (\mathbb{T} \times M) \).

We assume the existence of an elliptic normal differential operator \( E \) of order \( m > 0 \) such that

\[ [E, P(x, D)] = [E, Q(x, D)] = 0. \tag{1} \]

As particular cases we recapture constant vector fields on \( \mathbb{T} \times \mathbb{T}^n \) and vector fields on \( \mathbb{T} \times S^3 \).

We propose some new results on globally hypoelliptic operators under new Diophantine type conditions.

One of the main ingredients of our approach is the use of Fourier expansions in \( x \) defined by the basis of eigenfunctions associated to the elliptic operator \( E \) as outlined by R.T. Seeley in 1960’s.

The results are obtained in collaboration with and Fernando de Avila Silva (UFPR, Curitiba) and T. Gramchev (Univ. Cagliari)