A pointfree approach to topology over the minimalist foundation: whys and hows

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Abstract
I plan to survey some recent insights into the general reasons for developing pointfree topology over a minimalist foundation (and hence also for the enterprise of writing a book about it).

The aim of a new conception of mathematics is the search and management of information, rather than absolute truth. However, we do not limit ourselves to total information, as in some interpretation of constructivism, but look for preservation also of partial information, and even of our human intuition, in whatever way it can be expressed rigorously. This kind of knowledge is essential when instructing a computer to do mathematics for us.

It is a fact of life, apparently, that while points cannot be generated by effective rules fixed in advance, this is possible for opens, starting from a base indexed on an inductively generated set. This was fully clear to Martin-Löf in 1970, but can somehow be traced back to Dedekind in 1872. Our approach to constructive topology, called positive topology, develops this fact into a general and abstract treatment. So the notion of topological space is split into three distinct constructive notions:

1. concretes spaces $\text{CSpa}$, in which both and a base for topology are given effectively, and that impredicatively are the same as topological spaces $\text{Top}$,

2. positive topologies $\text{PTop}$, an abstract and rich formulation of the (point-free) structure of opens, which is totally effective,

3. ideal spaces $\text{ISpa}$, that is positive topologies equipped with an ideal, non-effective notion of point.

The notion of positive topology is obtained by abstracting the structure of opens and closed in a concrete space. Forgetting some properties is necessary, otherwise we get no new examples and are pushed back to concrete spaces. Following this method it becomes clear that the category $\text{CSpa}$ can be faithfully embedded in $\text{PTop}$, which is some sense fulfils Grothendieck’s dream of a generalization of topology.

Ideal spaces are obtained by dressing positive topologies with some redundant information, with the only purpose of pleasing our intuition of points (as it happens in real life with continuous vision). Doing this carefully, it becomes obvious that ideal spaces can be undressed at wish, and thus $\text{ISpa}$ and $\text{PTop}$ are isomorphic categories. This accomplishes a very general, although informal, mathematical form of Hilbert’s program: conservativity of ideal, pointwise aspects over real, pointfree ones.