

# Interest Rate Modeling

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*Main reference :*

Tomas Björk, *Arbitrage Theory in Continuous Time*,  
Oxford University Press, 2nd Ed. 2004.  
(Chapters 20-25).

*Additional reference :*

D. Brigo, and F. Mercurio, *Interest Rate Models - Theory  
and Practice*, Springer Verlag, Berlin, Heidelberg, 2nd Ed.  
2006.

## Basic concepts

- $T$ – bonds as main actors : *a contract guaranteeing a unit amount at a given maturity  $T > 0$*
- $p(t, T)$  : price, at  $t \leq T$ , of a  $T$ –bond

### Assumptions :

i)  $\exists T$ – bonds for every  $T > 0$ ;

ii)  $p(t, t) = p(T, T) = 1$ ;

iii)  $\forall t$ ,  $p(t, T)$  is differentiable in  $T$ .



- $L(t; T, S)$  : *simple forward rate for  $[T, S]$  contracted at  $t$  (LIBOR rate)*

$$\rightarrow 1 + (S - T) L(t; T, S) = \frac{p(t, T)}{p(t, S)}$$

$$\Rightarrow L(t; T, S) = \frac{p(t, T) - p(t, S)}{(S - T) p(t, S)} = \frac{1}{S - T} \left\{ \frac{p(t, T)}{p(t, S)} - 1 \right\}$$

- $f(t, T)$  : *instantaneous forward rate with maturity  $T$  contracted at  $t$*  (rate, contracted in  $t$  for an instantaneous investment at  $T$ )

$$\begin{aligned} \rightarrow f(t, T) &= \lim_{\Delta \downarrow 0} L(t; T, T + \Delta) \\ &= \lim_{\Delta \downarrow 0} \frac{1}{\Delta} \frac{p(t, T) - p(t, T + \Delta)}{p(t, T + \Delta)} = -\frac{\partial}{\partial T} \log p(t, T) \end{aligned}$$

$$\rightarrow p(t, T) = \exp \left[ - \int_t^T f(t, u) du \right]$$

- $r_t := f(t, t)$  : *instantaneous short (spot) rate at  $t$*

- *Money market account*

(An investment that continuously matures and is reinvested at the current short rate  $r_t$ ).

$$B_{t+\Delta} = B_t(1 + \Delta r_t)$$

$$\downarrow \Delta \downarrow 0$$

$$dB_t = B_t r_t dt$$

$$\downarrow$$

$$B_T = B_t \exp \left[ \int_t^T r_s ds \right]$$

- *Absence of arbitrage implies that*

$$\frac{p(t, T)}{B_t} = E^Q \left\{ \frac{1}{B_T} \mid \mathcal{F}_t \right\}$$

$$\rightarrow p(t, T) = E^Q \left\{ e^{-\int_t^T r_s ds} \mid \mathcal{F}_t \right\}$$

→ If  $r_t$  is deterministic, then

$$p(t, T) = e^{-\int_t^T r_s ds} = \frac{B_t}{B_T}$$

→ Notice that

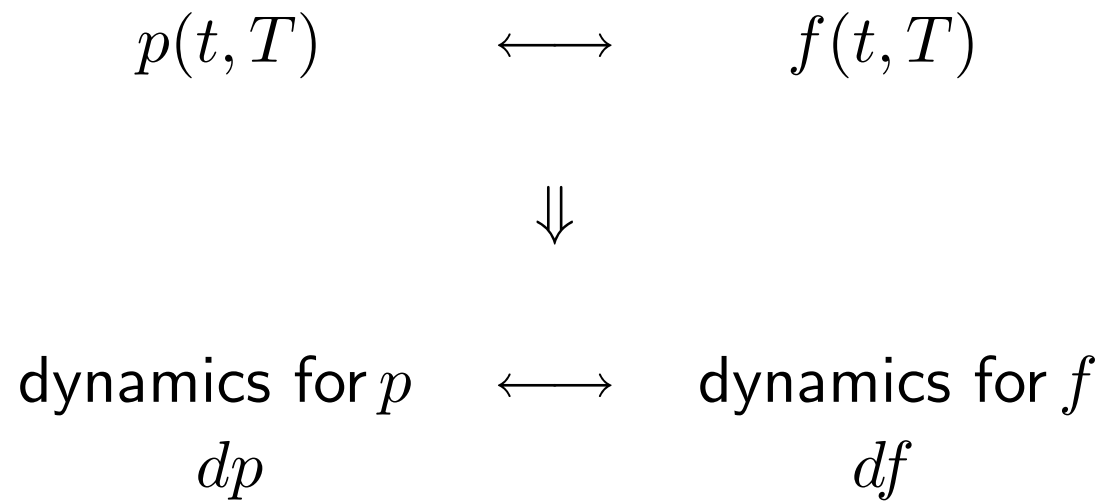
$$D(t, T) := \exp \left[ - \int_t^T r_s ds \right]$$

is the value in  $t$  of a unit amount in  $T$  (*from an amount  $D(t, T)$  in  $t$  one arrives at 1 in  $T$  with a continuously compounded investment at the current short rate  $r_t$ ) and can thus be interpreted as *(stochastic) “discount factor”**

→ *On the other hand,  $p(t, T)$  is the value in  $t$  of a contract that concerns a unitary amount in  $T$ .*

→ From the above one has that, if  $r_t$  is deterministic, then  $D(t, T) = p(t, T)$ .

- *One now has the following scheme*



*Exemplifying the implication  $df \rightarrow dp$  let*

$$df(t, T) = \alpha(t, T) + \sigma(t, T)dw_t$$

Putting

$$Y(t, T) := - \int_t^T f(t, s)ds$$

one has

$$p(t, T) = \exp\{Y(t, T)\}$$

with

$$dY(t, T) = f(t, t)dt - \int_t^T df(t, s)ds$$

Now we have

$$\begin{aligned} - \int_t^T df(t, s) ds &= - \int_t^T \alpha(t, s) dt ds - \int_t^T \sigma(t, s) dw_t ds \\ &= - \left( \int_t^T \alpha(t, s) ds \right) dt - \left( \int_t^T \sigma(t, s) ds \right) dw_t \\ &:= A(t, T) dt + S(t, T) dw_t \end{aligned}$$

$$\rightarrow dY(t, T) = r_t dt + A(t, T)dt + S(t, T)dw_t$$

and, finally,

$$\begin{aligned} dp(t, T) &= \\ &= p(t, T) \left[ r_t + A(t, T) + \frac{1}{2}S^2(t, T) \right] dt + S(t, T)dw_t \end{aligned}$$

## Short rate models

$r_t = f(t, t)$  a scalar process modeled as a Markov diffusion

$$dr_t = \mu(t, r_t)dt + \sigma(t, r_t)dw_t$$

$$r_t \xrightarrow{?} \begin{cases} p(t, T) \\ f(t, T) \end{cases}$$

↑

↑

finite dim. “factor”

$\infty$  – dim.

- Postulate

$$p(t, T) = F^T(t, r_t)$$

*i.e. the bond is seen as a “derivative” with underlying  $r_t$ .*

- How to determine  $F^T(t, r_t)$  ?

→ Use *Principle of absence of arbitrage (AOA)*

→ Since the “underlying”  $r_t$  has no market, pricing is not relative to  $r_t$  but relative to other bonds.

*Idea* : Take two bonds

$$p(t, T) = F^T(t, r_t) \quad ; \quad p(t, S) = F^S(t, r_t)$$

and with them form a self financing portfolio that is locally riskless, i.e. such that

$$dV_t = V_t k_t dt$$

→ *Since we have already the locally riskless asset*

$$dB_t = B_t r_t dt$$

*absence of arbitrage implies  $k_t = r_t$ .*

- How to *implement this idea* ?
- By Ito's formula

$$dF^T(t, r_t) = F^T(t, r_t) [\alpha^T(t, r_t)dt + \sigma^T(t, r_t)dw_t]$$

with

$$\begin{cases} \alpha^T(\cdot) &= \frac{F_t^T(\cdot) + \mu(\cdot)F_r^T(\cdot) + \frac{1}{2}\sigma^2(\cdot)F_{rr}^T(\cdot)}{F^T(\cdot)} \\ \sigma^T(\cdot) &= \frac{\sigma(\cdot)F_r^T(\cdot)}{F^T(\cdot)} \end{cases}$$

- Doing the same for  $p(t, S) = F^S(t, r_t)$  it is possible to form a self financing portfolio for which

$$dV_t = V_t \left[ \frac{\alpha^S(\cdot)\sigma^T(\cdot) - \alpha^T(\cdot)\sigma^S(\cdot)}{\sigma^T(\cdot) - \sigma^S(\cdot)} \right] dt$$

$$\rightarrow \frac{\alpha^S(\cdot)\sigma^T(\cdot) - \alpha^T(\cdot)\sigma^S(\cdot)}{\sigma^T(\cdot) - \sigma^S(\cdot)} = r_t$$

$$\rightarrow \frac{\alpha^S(t, r_t) - r_t}{\sigma^S(t, r_t)} = \frac{\sigma^T(t, r_t) - r_t}{\sigma^T(t, r_t)} := \lambda(t, r_t)$$

- For any  $T$  we thus obtain for the corresponding  $F^T(t, r)$  the following *Term structure equation*

$$\left\{ \begin{array}{l} F_t^T(\cdot) + (\mu(\cdot) - \lambda(\cdot)\sigma(\cdot))F_r^T(\cdot) + \frac{1}{2}\sigma^2(\cdot)F_{rr}^T(\cdot) - rF^T(\cdot) = 0 \\ F^T(T, r) = 1 \end{array} \right.$$

- $\lambda(t, r)$  : *market price of* (interest rate) *risk* : not *determined* endogenously by the model, but *by the market*.

- One thus has

$$\lambda(t, r) \longrightarrow F^T(t, r; \lambda)$$

- Let  $\bar{p}(t, T_i)$ ,  $i = 1, \dots, K$  be prices observed on the market.

→ *Calibration* : choose  $\lambda^*$  s.t., possibly

$$F^{T_i}(t, r; \lambda^*) \approx \bar{p}(t, T_i), \quad i = 1, \dots, K$$

- *By Feynman-Kac*

$$F^T(t, r; \lambda) = E^Q \left\{ \exp \left[ - \int_t^T r_s ds \right] \mid r_t = r \right\}$$

$$(Q) \quad dr_t = (\mu(t, r_t) - \lambda(t, r_t)\sigma(t, r_t))dt + \sigma(t, r_t)dw_t^Q$$

→ *What matters is*

$$a(t, r) := \mu(t, r) - \lambda(t, r)\sigma(t, r)$$

## Martingale models for the short rate

(Dynamics of  $r$  directly under the martingale measure)

$$dr_t = a(t, r_t; \theta)dt + \sigma(t, r_t; \theta) dw_t^Q$$

$\theta$  : an (infinite-dimensional) parameter to be *calibrated to the market*.

- *Calibration* (with respect to the “derivative” prices) then becomes

$$\theta \longrightarrow F^T(t, r; \theta)$$

→ Choose  $\theta^*$  s.t., possibly,

$$F^{T_i}(t, r; \theta^*) \approx \bar{p}(t, T_i), \quad i = 1, \dots, K$$

→ *Interest in having models for which  $F^T(t, r; \theta)$  can be more easily determined.*

## (Exponentially) affine term structure models

These are models for which

$$p(t, T) = \exp[A(t, T) - B(t, T)r_t]$$

*One has that, if*

$$dr_t = a(t, r_t; \theta)dt + \sigma(t, r_t; \theta)dw_t^Q$$

$$\text{with } \begin{cases} a(t, r; \theta) & = \alpha_t r + \beta_t \\ \sigma^2(t, r; \theta) & = \gamma_t r + \delta_t \end{cases}$$

*then the term structure (TS) is affine and*

$$\theta = [\alpha_t, \beta_t, \gamma_t, \delta_t]$$

- If the term structure is affine, then the solution of the term structure equation reduces to :

- i) the solution of a *Riccati-type equation* to obtain  $B(t, T)$ , i.e.

$$\begin{cases} B_t(t, T) + \alpha_t B(t, T) - \frac{1}{2} \gamma_t B^2(t, T) = -1 \\ B(T, T) = 0 \end{cases}$$

- ii) *an integration* to obtain  $A(t, T)$ , i.e.

$$\begin{cases} A_t(t, T) = \beta_t B(t, T) - \frac{1}{2} \delta_t B^2(t, T) \\ A(T, T) = 0 \end{cases}$$

*Example* : Ho-Lee model

$$dr_t = \Phi_t dt + \sigma dw_t^Q \quad (\sigma \text{ supposed known})$$

$$\begin{cases} B_t(t, T) = -1 \\ B(T, T) = 0 \end{cases} \Rightarrow B(t, T) = T - t$$

$$\begin{cases} A_t(t, T) = \Phi_t B(t, T) - \frac{1}{2} \sigma^2 B^2(t, T) \\ A(T, T) = 0 \end{cases}$$

$$\Rightarrow A(t, T) = \int_t^T \Phi_s \cdot (s - T) ds + \frac{\sigma^2 (T - t)^3}{6}$$

Since

$$f(t, T) = -\frac{\partial}{\partial T} \log p(t, T) = -A_T(t, T) + B_T(t, T)r_t$$

$$\Rightarrow f^*(0, T) = \int_0^T \Phi_s ds - \frac{\sigma^2 T^2}{2} + r_0$$

and so

$$\Phi_t = \frac{\partial f^*(0, t)}{\partial T} + \sigma^2 t$$

## Calibration (Issues)

*Purpose* : Obtaining a possibly good fit

$$F^{T_i}(t, r; \theta^*) \approx \bar{p}(t, T_i), \quad i = 1, \dots, K$$

→ *for finite-dimensional  $\theta$  procedure simpler, but fit poorer;*

→ *for  $\infty$ -dimensional  $\theta$  procedure more complicated, but fit better.*

## *Alternative possibilities*

- i) Instead of a scalar “factor”  $r_t$ , a multivariate “factor”  $x_t$ . Furthermore, instead of an exponentially affine, an exponentially quadratic term structure.
- ii) Take the entire forward curve as an  $\infty$ –dimensional “factor” .

## Heath-Jarrow-Morton (HJM) framework

- Take as “factor” the entire forward rate curve

$$df(t, T) = \alpha(t, T)dt + \sigma(t, T)dw_t$$

→ *The initially observed forward-rate curve enters directly into the model as initial condition (thus reducing the burden of model calibration), i.e.*

$$f(t, T) = f(0, T) + \int_0^t \alpha(s, T)ds + \int_0^t \sigma(s, T)dw_s$$

- One (vector) Wiener and many (infinite) assets, namely  $T$ -bonds for various  $T \Rightarrow$  *possibility of arbitrage*

## Absence of arbitrage for HJM

- Impose conditions so that  $\exists$  at least one martingale measure (*under which the discounted prices of all  $T$ -bonds are martingales*).
- Define the model directly under a martingale measure.

→ The condition becomes the *HJM “drift condition”*

$$\alpha(t, T) = \sigma(t, T) \int_t^T \sigma(t, u) du$$

- Recall in fact that from

$$df(t, T) = \alpha(t, T) + \sigma(t, T)dw_t$$

we obtained

$$\begin{aligned} dp(t, T) &= \\ &= p(t, T) \left[ r_t + A(t, T) + \frac{1}{2}S^2(t, T) \right] dt + S(t, T)dw_t \end{aligned}$$

with

$$\begin{cases} A(t, T) = - \int_t^T \alpha(t, s) ds \\ S(t, T) = - \int_t^T \sigma(t, s) ds \end{cases}$$

- In order that the discounted value of  $p(t, T)$ , i.e.  $B_t^{-1}p(t, T)$  is a martingale, one needs that

$$dp(t, T) = p(t, T) [r_t dt + S(t, T)] dw_t$$

i.e. that, for all  $T$ ,

$$A(t, T) + \frac{1}{2}S^2(t, T) = 0$$

Differentiating with respect to  $T$  leads to the required condition (HJM drift condition).

## *Inputs to the HJM model*

**i)** volatility structure  $\sigma(t, T)$ ;

**ii)** initially observed forward rate.

→ *Calibration reduces to determining possible parameters in  $\sigma(t, T)$  (which is a priori simpler than in the finite dimensional factor models).*

## Procedure

1. Choose  $\sigma(t, T)$ ;

2. put  $\alpha(t, T) = \sigma(t, T) \int_t^T \sigma(t, u) du$ ;

3. from the market obtain  $\{f^*(0, T); \forall T > 0\}$   
(*reconstruction and interpolation*);

4.  $f(t, T) = f^*(0, T) + \int_0^t \alpha(s, T) ds + \int_0^t \sigma(s, T) dw_s$ ;

5.  $p(t, T) = \exp \left[ - \int_t^T f(t, u) du \right]$ .

*Example* (leading to Ho-Lee)

Choose  $\sigma(t, T) \equiv \sigma$

$$\Rightarrow \alpha(t, T) = \sigma \int_t^T \sigma ds = \sigma^2(T - t)$$

It follows

$$\begin{aligned} f(t, T) &= f^*(0, T) + \int_0^t \sigma^2(T - s) ds + \int_0^t \sigma dw_s \\ &= f^*(0, T) + \sigma^2 t \left(T - \frac{t}{2}\right) + \sigma w_t \end{aligned}$$

and

$$r_t = f^*(0, t) + \sigma^2 \frac{t^2}{2} + \sigma w_t$$

From the last relation one obtains

$$dr_t = [f_T^*(0, t) + \sigma^2 t] dt + \sigma dw_t = \Phi_t dt + \sigma dw_t$$

with

$$\Phi_t = f_T^*(0, t) + \sigma^2 t$$

→ *One reobtains thus the Ho-Lee model. Notice the perfect “fitting” without the need of “inverting the yield curve” as in the classical model.*

## Change of numeraire (forward measures)

- By default one takes  $B_t$  as *numeraire*, i.e. for  $Q$  a martingale measure one has that

$$\frac{S_s}{B_s} = E^Q \left\{ \frac{S_t}{B_t} \mid \mathcal{F}_s \right\}, \quad \forall s < t \text{ and all asset prices } S_t$$

- In particular, for the price  $\Pi_t$  of a derivative asset with claim  $H_T$  one has (recall  $B_t = B_0 \exp \left[ \int_0^t r_s ds \right]$ )

$$\frac{\Pi_t}{B_t} = E^Q \left\{ \frac{H_T}{B_T} \mid \mathcal{F}_t \right\} \Rightarrow \begin{cases} \Pi_t = B_t E^Q \left\{ \frac{H_T}{B_T} \mid \mathcal{F}_t \right\} \\ \Pi_t = E^Q \left\{ e^{-\int_t^T r_s ds} H_T \mid \mathcal{F}_t \right\} \end{cases}$$

→ *A change of numeraire induces a change in the MM, which may result in certain cases convenient for the calculation of derivative prices.*

- Since for a generic numeraire  $S_t^0$  with associated MM  $Q^0$  one has

$$\frac{\Pi_t}{S_t^0} = E^{Q^0} \left\{ \frac{H_T}{S_T^0} \mid \mathcal{F}_t \right\} \quad \leftrightarrow \quad \Pi_t = S_t^0 E^{Q^0} \left\{ \frac{H_T}{S_T^0} \mid \mathcal{F}_t \right\}$$

the following two examples illustrate possible advantages resulting from a change of numeraire.

1. Assume a *claim can be factored* as  $H_T = S_T^0 \bar{H}_T$ .
- With  $B_t$  as numeraire one would have to calculate

$$\Pi_t = B_t E^Q \left\{ \frac{S_T^0 \bar{H}_T}{B_T} \mid \mathcal{F}_t \right\}$$

- with  $S_t^0$  as numeraire, and  $Q^0$  the corresponding MM, the same  $\Pi_t$  can be obtained as

$$\Pi_t = S_t^0 E^{Q^0} \{ \bar{H}_T \mid \mathcal{F}_t \}$$

2. Let the *short rate be stochastic*.

- With  $B_t$  as numeraire one would have to calculate

$$\Pi_t = E^Q \left\{ e^{-\int_t^T r_s ds} H_T \mid \mathcal{F}_t \right\}$$

*i.e. the conditional expectation of the product of the (correlated) random variables  $\exp \left[ -\int_t^T r_s ds \right]$  and  $H_T$ .*

- With  $p(t, T)$  as numeraire, and denoting by  $Q^T$  the corresponding MM (*forward measure*)

$$\Pi_t = p(t, T) E^{Q^T} \left\{ \frac{H_T}{p(T, T)} \mid \mathcal{F}_t \right\} = p(t, T) E^{Q^T} \{ H_T \mid \mathcal{F}_t \}$$

## Theorem

- Let  $Q$  be a MM for the filtered probability space  $(\Omega, \mathcal{F}, \mathcal{F}_t, Q)$  ( $t \leq T$  and  $\mathcal{F}_T = \mathcal{F}$ ).
- Let  $S_t$  be the price process, under AOA, of any given asset on the market, i.e.  $\frac{S_t}{B_t}$  is a  $(Q, \mathcal{F}_t)$ –martingale.
- For any fixed price process  $S_t^0 > 0$  there exists then a probability measure  $Q^0 \sim Q$  such that  $\frac{S_t}{S_t^0}$  is a  $(Q^0, \mathcal{F}_t)$ –martingale. The R.-N. derivative is

$$L = \frac{dQ^0}{dQ} = \frac{S_T^0}{B_T S_0^0} \quad (\text{assuming } B_0 = 1)$$

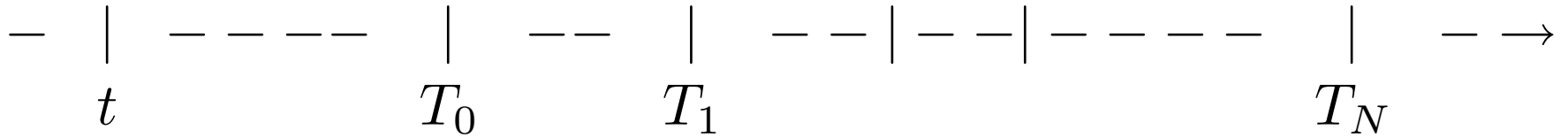
from which one obtains the mean-1  $(Q, \mathcal{F}_t)$ –martingale

$$L_t := E^Q\{L \mid \mathcal{F}_t\} = \frac{S_t^0}{B_t S_0^0}$$

*Proof :*

$$\begin{aligned} E^{Q^0} \left\{ \frac{S_t}{S_t^0} \mid \mathcal{F}_s \right\} &= \frac{E^Q \left\{ L \frac{S_t}{S_t^0} \mid \mathcal{F}_s \right\}}{E^Q \{L \mid \mathcal{F}_s\}} \\ &= \frac{E^Q \left\{ \frac{S_t}{S_t^0} E^Q \{L \mid \mathcal{F}_t\} \mid \mathcal{F}_s \right\}}{L_s} = \frac{E^Q \left\{ \frac{S_t}{S_t^0} L_t \mid \mathcal{F}_s \right\}}{L_s} \\ &= \frac{S_0^0 B_s}{S_s^0} E^Q \left\{ \frac{S_t}{S_t^0} \frac{S_t^0}{B_t S_0^0} \mid \mathcal{F}_s \right\} = \frac{S_0^0 B_s}{S_s^0} \frac{S_s}{B_s S_0^0} = \frac{S_s}{S_s^0} \end{aligned}$$

## Caps, Floors, Swaptions



$$\alpha_i := T_i - T_{i-1}, \quad i = 1, \dots, N \quad (\text{"tenors"})$$

$T_i$  : "resettlement dates"

$$p_i(t) := p(t, T_i)$$

$$L_i(t) := L(t; T_{i-1}, T_i)$$

$$L_i(t) = \frac{1}{\alpha_i} \frac{p_{i-1}(t) - p_i(t)}{p_i(t)} = \frac{1}{\alpha_i} \left( \frac{p_{i-1}(t)}{p_i(t)} - 1 \right)$$

# Caps

- Assume one has to make a series of interest rate payments on a (unitary) notional/principal at the current spot LIBOR rate. The rate is set at  $T_{i-1}$ , payment is made at  $T_i$ ;  $i = 1, \dots, N$  (“settlement in arrears”)
- *Cap* =  $\sum$  *Caplets* : a contract, established at  $t < T_0$ , that puts a “cap” on the interest rate; denote it by  $R$ .
- *i-th Caplet* : an option with expiration  $T_i$  and claim

$$X_i = \alpha_i \max[L_i(T_{i-1}) - R, 0] = \alpha_i (L_i(T_{i-1}) - R)^+$$

→ One always has to pay at the rate  $L_i(T_{i-1})$ , the exceeding part  $(L_i(T_{i-1}) - R)^+$  is however reimbursed.

→ The rate at which one actually pays is therefore

$$L_i(T_{i-1}) - (L_i(T_{i-1}) - R)^+ = \min\{L_i(T_{i-1}), R\}$$

## *LIBOR market model*

- Since a Caplet is of the form of an European call option, it would be convenient to price it with the Black and Scholes formula, which requires the underlying - here  $L_i(t)$  - to be lognormal.
  - *Despite of the fact that the B.&S. formula considers a deterministic  $r_t$ , while here  $L_i(t)$  and thus also  $r_t$  are stochastic, this is in fact market practice and to justify it one introduces a model guaranteeing lognormality*
    - *“market model”*.
  - Market models also turn out to be convenient for calibration.

- The  $i$ -th Caplet can be evaluated as follows ( $E^i$  denotes expectation w.r. to  $Q^i$ , i.e. the MM corresponding to  $p(t, T_i)$  as numeraire)

$$Capl_i(t) = \begin{cases} \alpha_i E^Q \left\{ e^{-\int_t^{T_i} r_s ds} (L_i(T_{i-1}) - R)^+ \mid \mathcal{F}_t \right\} \\ \alpha_i p_i(t) E^i \left\{ (L_i(T_{i-1}) - R)^+ \mid \mathcal{F}_t \right\} \end{cases}$$

**Lemma :**  $L_i(t)$  is, on  $[0, T_{i-1}]$ , a martingale w.r. to  $Q^i$ .

**Proof :** Since  $\alpha_i L_i(t) = \frac{p_{i-1}(t)}{p_i(t)} - 1$ , it suffices to show that  $\frac{p_{i-1}(t)}{p_i(t)}$  is a martingale, which follows from the fact that the price  $p_{i-1}(t)$  is here expressed in units of  $p_i(t)$ .

- For the “LIBOR market model” one would now like to define the LIBOR dynamics such that, for each  $i = 1, \dots, N$ ,  $L_i(t)$  is a lognormal martingale in the measure  $Q^i$ .

→ *This reduces to putting*

$$dL_i(t) = L_i(t)\sigma_i(t)dw_t^i, \quad i = 1, \dots, N$$

*with  $\sigma_i(t)$  deterministic and  $w_t^i$  Wiener in the measure  $Q^i$ .*

→ Existence of the Wiener processes  $w_t^i$  can be shown.

Under  $Q^i$  one then has

$$\log L_i(t) \sim \mathcal{N} \left( -\frac{1}{2} \int_0^t \sigma_i^2(s) ds, \int_0^t \sigma_i^2(s) ds \right)$$

and so the price of the  $i$ -th Caplet can be computed by the following Black& Scholes formula (setting  $r = 0$ )

$$Capl_i(t) = \alpha_i p_i(t) [L_i(t) \mathcal{N}(d_1) - R \mathcal{N}(d_2)]$$

$$\text{with } \begin{cases} d_1 = \frac{1}{\Sigma_i(t, T_{i-1})} \left[ \log \left( \frac{L_i(t)}{R} \right) + \frac{1}{2} \Sigma_i^2(t, T_{i-1}) \right] \\ d_2 = d_1 - \Sigma_i(t, T_{i-1}) \end{cases}$$

where  $\Sigma_i^2(t, T_{i-1}) = \int_t^{T_{i-1}} \sigma_i^2(s) ds$

→ Need to determine  $\sigma_i(t)$  (*implied volatility*)

## Pricing of Caps via the standard MM Q

$$\begin{aligned}
 Capl_i(t) &= \alpha_i E^Q \left\{ e^{-\int_t^{T_i} r_s ds} (L_i(T_{i-1}) - R)^+ \mid \mathcal{F}_t \right\} \\
 &= \alpha_i E^Q \left\{ E^Q \left\{ e^{-\int_t^{T_i} r_s ds} (L_i(T_{i-1}) - R)^+ \mid \mathcal{F}_{T_{i-1}} \right\} \mid \mathcal{F}_t \right\} \\
 &= \alpha_i E^Q \left\{ e^{-\int_t^{T_{i-1}} r_s ds} p(T_{i-1}, T_i) (L_i(T_{i-1}) - R)^+ \mid \mathcal{F}_t \right\} \\
 &= E^Q \left\{ e^{-\int_t^{T_{i-1}} r_s ds} p(T_{i-1}, T_i) \left( \frac{1}{p(T_{i-1}, T_i)} - 1 - R\alpha_i \right)^+ \mid \mathcal{F}_t \right\} \\
 &= E^Q \left\{ e^{-\int_t^{T_{i-1}} r_s ds} [1 - (1 + R\alpha_i)p(T_{i-1}, T_i)]^+ \mid \mathcal{F}_t \right\} \\
 &= (1 + R\alpha_i) E^Q \left\{ e^{-\int_t^{T_{i-1}} r_s ds} \left( \frac{1}{1 + R\alpha_i} - p(T_{i-1}, T_i) \right)^+ \mid \mathcal{F}_t \right\}
 \end{aligned}$$

which can be computed as a European-type put option with maturity  $T_{i-1}$ , exercise price  $K_i = \frac{1}{1 + R\alpha_i}$  and having as underlying the bond with maturity  $T_i$  ( $r_t$  may be stochastic).

## Floors

- *Floor* =  $\sum$  *Floorlets* : a contract that puts a “floor” on the interest rate; denote it by  $R$ .
- *i*-th *Floorlet* : an option with expiration  $T_i$  and claim

$$X_i = \alpha_i \max[R - L_i(T_{i-1}), 0] = \alpha_i (R - L_i(T_{i-1}))^+$$

→ One always receives at the rate  $L_i(T_{i-1})$ , the missing part  $(R - L_i(T_{i-1}))^+$  is however reimbursed.

→ The rate at which one actually receives payments is therefore

$$L_i(T_{i-1}) + (R - L_i(T_{i-1}))^+ = \max\{L_i(T_{i-1}), R\}$$

- While a Cap protects those who have to make payments against rising rates, Floors protect those who receive payments against falling rates.
- *Floors can be priced in perfect duality with Caps. Since the former correspond to a put option and the latter to a call option, a put-call-parity-type relation can be exploited here as well.*

## Interest Rate Swaps (IRS)

(forward swaps settled in arrears)

- Exchanging a sequence of payments at a “floating rate” (“floating leg”) against a sequence of payments at a fixed rate “fixed leg” (the sequence concerns the same resettlement dates  $T_0, \dots, T_n$  as before).
- Referring always to the fixed leg, one distinguishes :
  - i) *receiver swap* : one receives at the dates  $T_1, \dots, T_N$  the fixed leg and pays the floating leg.
  - ii) *payer swap* : one receives the floating and pays the fixed leg.

More precisely : *Payer IRS*

- Payments occur at dates  $T_1, \dots, T_n$ ;
- *for each period  $[T_i, T_{i+1}]$ ;  $i = 0, \dots, N - 1$  the LIBOR rate  $L_{i+1}(T_i)$  is set at time  $T_i$  and the payment of  $\alpha_{i+1}L_{i+1}(T_i)$  (for unitary nominal/principal) is received at time  $T_{i+1}$ ;*
- the fixed amount  $\alpha_{i+1}K$  is paid at  $T_{i+1}$ .

→ *Notice that, due to the equality*

$$(L - R)^+ - (R - L)^+ = L - R$$

*the difference between a Cap and a Floor is equivalent to a forward payer IRS. (For IRS we use  $K$  for the fixed rate, the symbol  $R$  will be reserved for the Swap rate to be introduced below).*

## Arbitrage free value of a payer IRS

### 1. Value in $t < T_0$ of the floating leg ?

- Recalling that  $L_{i+1}(t)$  is, on  $[0, T_i]$  a  $Q^{i+1}$ -martingale, the value in  $t$  of the floating payment made in  $T_{i+1}$  is

$$\Pi^{i+1}(t) = p_{i+1}(t) E^{i+1} \{ \alpha_{i+1} L_{i+1}(T_i) \mid \mathcal{F}_t \}$$

$$= p_{i+1}(t) \alpha_{i+1} L_{i+1}(t) = p_{i+1}(t) \frac{p_i(t) - p_{i+1}(t)}{p_{i+1}(t)} = p_i(t) - p_{i+1}(t)$$

- The total value in  $t < T_0$  of the floating leg is then

$$\sum_{i=0}^{N-1} [p_i(t) - p_{i+1}(t)] = p_0(t) - p_N(t)$$

## 2. *Value in $t < T_0$ of the fixed leg ?*

The total value is just the sum of the discounted individual amounts, i.e.

$$\sum_{i=0}^{N-1} p_{i+1}(t) \alpha_{i+1} K = K \sum_{i=1}^N \alpha_i p_i(t)$$

## 3. *Total value $PS_0^N(t; K)$ in $t < T_0$ of the payer IRS*

It is given by the difference of the values of the two legs, i.e.

$$PS_0^N(t; K) = p_0(t) - p_N(t) - K \sum_{i=1}^N \alpha_i p_i(t)$$

## Forward **swap rate** (payer IRS)

- The swap rate for a forward payer IRS is that value of the fixed amount  $K$  for which  $PS_0^N(t; K) = 0$ , i.e. it is the value  $R_0^N(t)$  given by

$$R_0^N(t) := \frac{p_0(t) - p_N(t)}{\sum_{i=1}^N \alpha_i p_i(t)}$$

→ For  $N = 1$  the swap rate coincides with the LIBOR rate.

→ The denominator  $S_0^N(t) := \sum_{i=1}^N \alpha_i p_i(t)$  (also referred to as *accrual factor*) can be interpreted as the value of a traded asset, namely a buy-and-hold portfolio consisting, for each  $i$ , of  $\alpha_i$  units of the  $T_i$ -bond.

→ 
$$R_0^N(t) := \frac{p_0(t) - p_N(t)}{S_0^N(t)}$$

is, on  $[0, T_0]$ , a martingale in the measure corresponding to  $S_0^N(t)$  as numeraire.

## Swaption (Payer IRS)

- It is a contract, established at  $t < T_0$ , that gives the holder the right but not the obligation to enter, at  $t = T_0$ , a payer IRS with given fixed rate  $K$ .

→ A Swaption (swap-option) for a payer IRS and a fixed rate  $K$  is thus an option with maturity  $T_0$  and claim

$$X_0^N = (PS_0^N(T_0; K))^+$$

→ Noticing that in terms of  $R_0^N(t)$  (*swap rate*) and  $S_0^N(t)$  (*accrual factor*) one may rewrite

$$PS_0^N(t; K) = (R_0^N(t) - K) S_0^N(t)$$

the Swaption claim also takes the form

$$X_0^N = (PS_0^N(t; K))^+ = (R_0^N(T_0) - K)^+ S_0^N(T_0)$$

which reminds one of a call option with underlying the swap rate  $R_0^N$  and strike  $K$ .

## Arbitrage-free value in $t < T_0$ of a Swaption

- Given the factor  $S_0^N(T_0)$  in the claim, one may choose as numeraire  $S_0^N(t)$  so that for the value  $PSW_0^N(t)$  in  $t < T_0$  of a payer IRS swaption one has

$$PSW_0^N(t) = S_0^N(t) E^{0,N} \{ (R_0^N(T_0) - K)^+ \mid \mathcal{F}_t \}$$

with  $E^{0,N}$  denoting the expectation in the martingale measure  $Q^{0,N}$  that corresponds to  $S_0^N(t)$  as numeraire.

→ *Convenient to have lognormality of  $R_0^N(t)$  in the measure  $Q^{0,N}$ .*

## *Swap market model*

- Analogously to the LIBOR market model one would now like to define the swap rate dynamics such that  $R_0^N(t)$  is a lognormal martingale in the measure  $Q^{0,N}$ .

→ *This reduces to putting*

$$dR_0^N(t) = R_0^N(t) \sigma_t^{0,N} dw_t^{0,N}$$

*with  $\sigma_t^{0,N}$  deterministic and  $w_t^{0,N}$  a Wiener in the measure  $Q^{0,N}$ .*

→ One can then use the *B.& S. formula* obtaining

$$PSW_0^N(t) = S_0^N(t) [R_0^N(t)\mathcal{N}(d_1) - K\mathcal{N}(d_2)]$$

$$\text{with } \begin{cases} d_1 = \frac{1}{\Sigma_{0,N}(t)} \left[ \log \left( \frac{R_0^N(t)}{K} \right) + \frac{1}{2} \Sigma_{0,N}^2(t) \right] \\ d_2 = d_1 - \Sigma_{0,N}(t) \end{cases}$$

$$\text{where } \Sigma_{0,N}^2(t) = \int_t^{T_0} \sigma_s^{0,N} ds$$

→  $\sigma_t^{0,N}$  has to be calibrated to the Swaption market analogously to what happens for the LIBOR market models.

- *Remark 1* : The *Swap market model is incompatible with the LIBOR market model* in the sense that lognormality in one model does not imply lognormality in the other (*this incompatibility is however more theoretical than practical*).
- *Remark 2* : The preceding development concerned payer IRS and corresponding swaptions. For a receiver IRS and corresponding swaption the development can be obtained analogously by inverting the floating and fixed legs (*like passing from a call to a put option*). Notice furthermore that, always due to the equality

$$(R - K)^+ - (K - R)^+ = R - K$$

one has that *the difference between a payer swaption and the corresponding receiver swaption is equivalent to a forward payer IRS*.

## A1. Option pricing when interest rate is stochastic (an application of the change of numeraire technique)

- Consider an *European call option* on an underlying with price  $S_t$ . The claim is

$$H_T = (S_T - K)^+ = (S_T - K)\mathbf{1}_{\{S_T > K\}}$$

- Considering besides  $B_t$  also  $S_t$  and  $p(t, T)$  as numeraires, one obtains the following expressions for the same arbitrage free price  $\Pi_0$  at time  $t=0$ , namely (put  $B_0 = 1$ )

$$\Pi_0 = E^Q \left\{ \frac{H_T}{B_T} \right\} = \begin{cases} S_0 E^S \left\{ \frac{H_T}{S_T} \right\} \\ p(0, T) E^T \{ H_T \} \end{cases}$$

with  $E^Q, E^S, E^T$  denoting expectation w.r. to the martingale measures corresponding to the above three numeraires respectively.

- From the foregoing it follows that one can write

$$\begin{aligned}\Pi_0 &= E^Q \{ B_T^{-1} S_T \mathbf{1}_{\{S_T > K\}} \} - K E^Q \{ B_T^{-1} \mathbf{1}_{\{S_T > K\}} \} \\ &= S_0 Q^S \{ S_T > K \} - K p(0, T) Q^T \{ S_T > K \}\end{aligned}$$

i.e.  $\Pi_0$  can be computed by computing the probability of the event  $S_T > K$  under the two measures  $Q^S$  and  $Q^T$ .

- *For actual computation* assume that  $S_t$  is such that  $Z_t := \frac{S_t}{p(t,T)}$  satisfies (for any measure under which  $w_t$  is a Wiener process)

$$dZ_t = Z_t [m_t dt + \sigma_t^Z dw_t]$$

with  $\sigma_t^Z$  a deterministic time function.

→  $Z_t := \frac{S_t}{p(t,T)}$  is a  $Q^T$ -martingale so that, under  $Q^T$ ,

$$dZ_t = Z_t \sigma_t^Z dw_t^T$$

with  $w_t^T$  a  $Q^T$ -Wiener and, consequently, under  $Q^T$

$$Z_T = \frac{S_0}{p(0,T)} \exp \left[ -\frac{1}{2} \int_0^T (\sigma_t^Z)^2 dt + \int_0^T \sigma_t^Z dw_t^T \right]$$

- Putting  $\Sigma_T^2 := \int_0^T (\sigma_t^Z)^2 dt$  one then has

$$Q^T \{S_T > K\} = Q^T \left\{ \frac{S_T}{p(T,T)} > K \right\} = Q^T \{Z_T > K\}$$

$$= P \left\{ \log \frac{S_0}{p(0,T)} + \mathcal{N} \left( -\frac{1}{2} \Sigma_T^2, \Sigma_T^2 \right) > \log K \right\}$$

$$P \left\{ \mathcal{N}(0, 1) > \frac{\log \frac{K p(0,T)}{S_0} + \frac{1}{2} \Sigma_T^2}{\Sigma_T} \right\} = \mathcal{N}(d_2)$$

where

$$d_2 = \frac{\log \frac{S_0}{K p(0,T)} - \frac{1}{2} \Sigma_T^2}{\Sigma_T}$$

- Next, putting

$$Y_t := \frac{1}{Z_t} = \frac{p(t, T)}{S_t}$$

which is a  $Q^S$ -martingale, under  $Q^S$  one obtains

$$dY_t = -Y_t \sigma_t^Z dw_t^S$$

with  $w_t^S$  a  $Q^S$ -Wiener and, consequently, under  $Q^S$

$$Y_T = \frac{p(0, T)}{S_0} \exp \left[ -\frac{1}{2} \int_0^T (\sigma_t^Z)^2 dt - \int_0^T \sigma_t^Z dw_t^S \right]$$

- It follows that

$$Q^S \{S_T > K\} = Q^S \left\{ \frac{p(T,T)}{S_T} < \frac{1}{K} \right\} = Q^S \left\{ Y_T < \frac{1}{K} \right\}$$

$$= P \left\{ \log \frac{p(0,T)}{S_0} + \mathcal{N} \left( -\frac{1}{2}\Sigma_T^2, \Sigma_T^2 \right) < -\log K \right\}$$

$$P \left\{ \mathcal{N}(0, 1) < \frac{\log \frac{S_0}{K p(0,T)} + \frac{1}{2}\Sigma_T^2}{\Sigma_T} \right\} = \mathcal{N}(d_1)$$

where

$$d_1 = d_2 + \Sigma_T$$

- *Summarizing*, one has that, if  $Z_t := \frac{S_t}{p(t,T)}$  satisfies

$$dZ_t = Z_t [m_t dt + \sigma_t^Z dw_t]$$

with  $\sigma_t$  a deterministic time function, then the price  $\Pi_0$  at time  $t = 0$  of the claim  $H_T = (S_T - K)^+$  is given by

$$\Pi_0 = S_0 \mathcal{N}(d_1) - K p(0, T) \mathcal{N}(d_2)$$

where  $\mathcal{N}(\cdot)$  denotes the cumulative Gaussian distribution function and

$$\begin{cases} d_1 &= \frac{\log \frac{S_0}{K p(0,T)} + \frac{1}{2} \Sigma_T^2}{\Sigma_T} \\ d_2 &= d_1 - \Sigma_T \end{cases}$$

with  $\Sigma_T^2 := \int_0^T (\sigma_t^Z)^2 dt$ .

## Application

*(Pricing a call option on a bond)*

- Consider the *Hull-White model* for the spot rate, i.e.

$$dr_t = [\Phi_t - ar_t]dt + \sigma dw_t^Q$$

- *This leads to an affine term structure, i.e.*

$$p(t, T) = \exp[A(t, T) - B(t, T)r_t]$$

*with*

$$B(t, T) = \frac{1}{a} \left[ 1 - e^{-a(T-t)} \right]$$

- Putting  $T_1 = T$  with  $T$  the maturity of the option, take  $S_t = p(t, T_2)$  with  $T_2 > T_1$ .
- For this case one has

$$\begin{aligned} Z_t &= \frac{p(t, T_2)}{p(t, T_1)} \\ &= \exp [(A(t, T_2) - A(t, T_1)) - (B(t, T_2) - B(t, T_1))r_t] \end{aligned}$$

and therefore (*one has to verify only the volatility*)

$$dZ_t = Z_t[\dots\dots]dt + Z_t\sigma_t^Z dw_t$$

with  $\sigma_t^Z = -\sigma[B(t, T_2) - B(t, T_1)] = \frac{\sigma}{a} e^{at} [e^{-aT_1} - e^{-aT_2}]$   
 which is indeed deterministic.

- *Summarizing*, for  $H_{T_1} = (p(T_1, T_2) - K)^+$  one then has

$$\Pi_0 = p(0, T_2)\mathcal{N}(d_1) - Kp(0, T_1)\mathcal{N}(d_2)$$

with  $d_1 = \frac{\log \frac{p(0, T_2)}{K p(0, T_1)} + \frac{1}{2}\Sigma_{T_1}^2}{\Sigma_{T_1}}$  ;  $d_2 = d_1 - \Sigma_{T_1}$  and where

$$\begin{aligned} \Sigma_{T_1}^2 &= \frac{\sigma^2}{a^2} [e^{-aT_1} - e^{-aT_2}]^2 \int_0^{T_1} e^{2at} dt \\ &= \frac{\sigma^2}{2a^3} (e^{2aT_1} - 1) [e^{-aT_1} - e^{-aT_2}]^2 \\ &= \frac{\sigma^2}{2a^3} (1 - e^{-2aT_1}) e^{2aT_1} [e^{-aT_1} - e^{-aT_2}]^2 \\ &= \frac{\sigma^2}{2a^3} (1 - e^{-2aT_1}) (1 - e^{-a(T_2 - T_1)})^2 \end{aligned}$$

→ Can be used to *calibrate the parameter  $a$* .

## A2. Solved problems/additional topics

### *Problem 1.*

**The problem :** Consider the following dynamics, under a martingale measure  $Q$ , of the spot rate  $r_t$ ,

$$dr_t = ar_t dt + \sigma dw_t$$

and assume that the term structure has the form

$$\begin{aligned} p(t, T) &= F^T(t, r_t) \\ &= \exp [A(t, T) + B(t, T)r_t + C(t, T)r_t^2 + D(t, T)r_t^3] \end{aligned}$$

- a) Imposing on  $F^T(t, r)$  to satisfy the term structure equation, show that this implies  $D(t, T) \equiv 0$ . What is the resulting system of equations for  $A(t, T), B(t, T), C(t, T)$  ?
- b) *What changes if  $r_t$  satisfies lognormal dynamics, i.e.*

$$dr_t = r_t[adt + \sigma dw_t] \quad ?$$

*Knowing that for a deterministic spot rate  $r_t$  one has  $p(t, T) = \exp\left[-\int_t^T r_s ds\right]$ , verify the correctness of the result in the case of  $\sigma = 0$ .*

## Solution :

a) The term structure equation is

$$F_t + arF_r + \frac{1}{2}\sigma^2 F_{rr} - rF = 0$$

Since

$$F_t = F[A_t + B_t r + C_t r^2 + D_t r^3]$$

$$F_r = F[B + 2Cr + 3Dr^2]$$

$$F_{rr} = F\{[B + 2Cr + 3Dr^2]^2 + [2C + 6Dr]\}$$

and since the term structure equation has to hold for any value of  $r$ , the coefficients of the various powers of  $r$  have to vanish and this implies the following :

$$\left\{ \begin{array}{l} A_t + \frac{1}{2}\sigma^2[B^2 + 2C] = 0 \\ B_t + aB + 2BC\sigma^2 + 3D\sigma^2 - 1 = 0 \\ C_t + 2Ca + 2C^2\sigma^2 + 3BD\sigma^2 = 0 \\ D_t + 3Da + 3CD\sigma^2 = 0 \\ \frac{9}{2}D^2\sigma^2 = 0 \end{array} \right. , \quad \begin{array}{l} A(T, T) = 0 \\ B(T, T) = 0 \\ C(T, T) = 0 \\ D(T, T) = 0 \end{array}$$

As a consequence  $D(t, T) = 0$  and one has the remaining system

$$\left\{ \begin{array}{l} A_t + \frac{1}{2}\sigma^2[B^2 + 2C] = 0 \\ B_t + aB + 2BC\sigma^2 - 1 = 0 \\ C_t + 2Ca + 2C^2\sigma^2 = 0 \end{array} \right.$$

from which it follows that also  $C(t, T) = 0$ .

**b)** In this case the system becomes (with the terminal conditions always equal to zero)

$$\left\{ \begin{array}{l} A_t = 0 \\ B_t + aB + \frac{1}{2}\sigma^2[B^2 + 2C] - 1 = 0 \\ C_t + 2Ca + 2\sigma^2BC + 3\sigma^2D = 0 \\ D_t + 3Da + 2\sigma^2C^2 + 3\sigma^2BD = 0 \\ 3CD\sigma^2 = 0 \\ \frac{9}{2}\sigma^2D^2 = 0 \end{array} \right.$$

The system thus reduces to

$$\begin{cases} A_t = 0 \\ B_t + aB + \frac{1}{2}\sigma^2 B^2 - 1 = 0 \end{cases}$$

which implies  $A(t, T) = 0$  and thus one finds

$$F(t, r) = \exp[B(t, T)r]$$

with

$$\begin{cases} B_t + aB + \frac{1}{2}\sigma^2 B^2 = 1 \\ B(T, T) = 0 \end{cases}$$

- In the case when  $\sigma = 0$  the equation for  $B$  becomes  $B_t + aB = -1$  with solution  $B(t, T) = \frac{e^{a(T-t)} - 1}{a}$  and thus

$$p(t, T) = \exp \left[ \frac{r_t}{a} \left( e^{a(T-t)} - 1 \right) \right]$$

On the other hand, for  $\sigma = 0$  one has as solution of the equation for  $r_t$  the following

$$r_s = r_t \exp[a(s - t)]$$

but

$$p(t, T) = \exp \left[ - \int_t^T r_s ds \right] = \exp \left[ \frac{r_t}{a} \left( e^{a(T-t)} - 1 \right) \right]$$

and this coincides in fact with the above expression.

## Problem 2.

**The problem :** Assume that, under a martingale measure  $Q$ , the spot rate  $r_t$  satisfies the following model (CIR)

$$dr_t = (a - br_t)dt + \sigma\sqrt{r_t}dw_t$$

a) On the basis of the term structure equation show that, for  $\beta, \lambda > 0$ , one has

$$E_{t,r} \left\{ e^{-\beta r_T} \exp \left[ - \int_t^T r_s ds \right] \right\} = \exp [A(t, T) - B(t, T)r]$$

where  $B(t, T)$  and  $A(t, T)$  satisfy

$$\begin{cases} B_t(t, T) = b B(t, T) + \frac{1}{2}\sigma^2 B^2(t, T) - 1, & B(T, T) = \beta \\ A_t(t, T) = a B(t, T), & A(T, T) = 0 \end{cases}$$

**b)** Taking as known that the solution of previous system is

$$\left\{ \begin{array}{l} B(t, T) = \frac{\beta[\gamma+b+e^{\gamma(T-t)}(\gamma-b)]+2(e^{\gamma(T-t)}-1)}{\beta\sigma^2(e^{\gamma(T-t)}-1)+\gamma-b+e^{\gamma(T-t)}(\gamma+b)} \\ A(t, T) = \frac{2a}{\sigma^2} \log \left( \frac{2\gamma e^{\frac{(T-t)(\gamma+b)}{2}}}{\beta\sigma^2(e^{\gamma(T-t)}-1)+\gamma-b+e^{\gamma(T-t)}(\gamma+b)} \right) \end{array} \right.$$

with  $\gamma := \sqrt{b^2 + 2\sigma^2}$ , obtain an explicit expression for the price  $p(t, T)$  of a  $T$ -bond.

## Solution :

a) Putting

$$F(t, r) := E_{t,r} \left\{ e^{-\beta r T} \exp \left[ - \int_t^T r_s ds \right] \right\}$$

the function  $F(t, r)$  gives the price, at  $t$ , of a claim  $\exp[-\beta r_T]$  with maturity  $T$ . It therefore satisfies the term structure equation

$$\begin{cases} F_t + (a - br)F_r + \frac{1}{2}\sigma^2 r F_{rr} - rF = 0 \\ F(T, r) = \exp[-\beta r] \end{cases}$$

Since

$$\begin{aligned}F_t(t, r) &= F(t, r)[A_t(t, T) - B_t(t, T)r] \\F_r(t, r) &= -F(t, r)B(t, T) \\F_{rr}(t, r) &= F(t, r)B^2(t, T)\end{aligned}$$

one thus obtains

$$A_t - B_t r - aB + bBr + \frac{1}{2}\sigma^2 B^2 r - r = 0$$

which, having to hold for all values of  $r$ , leads to

$$\begin{cases} A_t = aB & A(T, T) = 0 \\ B_t = bB + \frac{1}{2}\sigma^2 B^2 - 1 & B(T, T) = \beta \end{cases}$$

**b)** One has  $p(t, T) = F(t, r)$  where  $A(t, T), B(t, T)$  satisfy the system in point a) of the problem statement for  $\beta = 0$ . Consequently

$$\begin{aligned}
 p(t, T) &= \\
 &= \exp \left[ \frac{2a}{\sigma^2} \log \left( \frac{2\gamma e^{\frac{(T-t)(\gamma+b)}{2}}}{\gamma - b + e^{\gamma(T-t)}(\gamma+b)} \right) - \frac{2(e^{\gamma(T-t)} - 1)}{\gamma - b + e^{\gamma(T-t)}(\gamma+b)} r_t \right] \\
 &= \left( \frac{2\gamma e^{\frac{(T-t)(\gamma+b)}{2}}}{\gamma - b + e^{\gamma(T-t)}(\gamma+b)} \right)^{\frac{2a}{\sigma^2}} \exp \left[ \frac{\gamma - b + e^{\gamma(T-t)}(\gamma+b)}{2(e^{\gamma(T-t)} - 1)} r_t \right]
 \end{aligned}$$

### Problem 3.

**The problem :** Consider the following multifactor model for the term structure

$$p(t, T) = F^T(t, x_t) = \exp [A(t, T) + B(t, T)x_t]$$

where  $x_t \in \mathbb{R}^p$  is a multivariate Gaussian process which, under a martingale measure  $Q$ , satisfies

$$dx_t = Gx_t dt + Hdw_t$$

with  $w_t$  a multivariate Wiener process and  $G$  and  $H$  are deterministic matrices (to be calibrated to market data). Put  $x_t^1 = r_t$  namely the spot rate and let

$$a(t, T) = -A_T(t, T) \quad , \quad b(t, T) = -B_T(t, T)$$

a) Show that

$$df(t, T) = [a_t(t, T) + b_t(t, T)x_t + b(t, T)Gx_t] dt + b(t, T)Hdw_t$$

b) Imposing the *HJM drift condition* (which justifies the dynamics of  $x_t$  under a martingale measure) show that under absence of arbitrage the following has to hold

$$\begin{cases} b_t(t, T) &= -b(t, T)G \\ a_t(t, T) &= b(t, T)HH' \int_t^T b'(t, u)du \end{cases}$$

with terminal conditions

$$b(T, T) = (1, \underline{0}) \quad (\underline{0} \in \mathbb{R}^{p-1}); \quad a(T, T) = 0$$

(to show the latter, determine the expression for  $r_t$  in  $t = T$  and recall that we had put  $x_t^1 = r_t$ ).



## Solution :

a) One has

$$\begin{aligned} f(t, T) &= -\frac{\partial}{\partial T} \log p(t, T) = -A_T(t, T) - B_T(t, T)x_t \\ &= a(t, T) + b(t, T)x_t \end{aligned}$$

from which

$$df(t, T) = [a_t(t, T) + b_t(t, T)x_t + b(t, T)Gx_t]dt + b(t, T)Hdw_t$$

**b)** The *HJM drift condition* is

$$a_t(t, T) + (b_t(t, T) + b(t, T)G) x_t = b(t, T)HH' \int_t^T b'(t, u)du$$

and it has to hold for all  $x_t$ . Therefore

$$\begin{cases} b_t(t, T) &= -b(t, T)G \\ a_t(t, T) &= b(t, T)HH' \int_t^T b'(t, u)du \end{cases}$$

From the first relation in point a) above one obtains  $r_t = f(t, t) = a(t, t) + b(t, t)x_t$  which, for  $t = T$  (*one initial or terminal condition for the above system suffices*) and recalling that  $x_t^1 = r_t$ , gives the desired terminal conditions.

c) From  $F^T(t, x) = \exp[A(t, T) + B(t, T)x]$  one has

$$\begin{aligned}F_t^T(t, x) &= F^T(t, x) (A_t(t, T) + B_t(t, T)x) ; \\F_x^T(t, x) &= F^T(t, x) B(t, T) \\F_{xx}^T(t, x) &= F^T(t, x) B'(t, T) B(t, T)\end{aligned}$$

so that the PDE becomes

$$\begin{aligned}0 &= A_t(\cdot) + B_t(\cdot)x + B(\cdot)Gx + \frac{1}{2}tr(HH'B'(\cdot)B(\cdot)) - r \\ &= A_t(\cdot) + B_t(\cdot)x + B(\cdot)Gx + \frac{1}{2}B'(\cdot)(HH')B(\cdot) - x^1\end{aligned}$$

Since the equation has to hold for all values of  $x \in \mathbb{R}^p$ , one obtains

$$\begin{cases} B_t^1(t, T) + (B(t, T)G)^1 - 1 = 0 \\ B_t^k(t, T) + (B(t, T)G)^k = 0 \quad \text{for } k = 2, \dots, p \\ A_t(t, T) + \frac{1}{2}B(t, T)(HH')B'(t, T) = 0 \end{cases}$$

These equations in turn have to hold for all  $T \geq t$  and so, being  $A(\cdot)$  and  $B(\cdot)$  differentiable, one finally obtains

$$\begin{cases} b_t(t, T) + b(t, T)G = 0 \\ a_t(t, T) + b(t, T)(HH')B'(t, T) = 0 \end{cases}$$

where (see the definition of  $b(t, T)$  in the problem statement)  $B'(t, T) = -\int_t^T b'(t, u)du$ . The terminal conditions result from the same considerations as in the previous point b) using the fact that  $r_t = f(t, t) = a(t, t) + b(t, t)x_t$ .

### Problem 4.

**The problem :** Following the HJM approach suppose that  $\sigma(t, T) \equiv \sigma$  with  $\sigma$  a given parameter. Let the initially observed forward rate, namely  $f^*(0, T)$ , be given.

- a) Determine the explicit expression of  $f(t, T)$  and  $r_t$  (in terms of  $w_t^Q$ )
- b) Show that this setup leads to an affine term structure. Using this fact and following the classical approach, determine  $A(t, T)$  and  $B(t, T)$  for which

$$p(t, T) = \exp [A(t, T) - B(t, T)r_t]$$

Determine the explicit expression of  $p(t, T)$  in terms of  $r_t$ .

**c)** Using the two explicit expressions in terms of  $w_t^Q$  obtained under point a) for  $f(t, T)$  and  $r_t$ , determine the explicit expression of  $p(t, T)$  on the basis of  $p(t, T) = \exp \left[ - \int_t^T f(t, s) ds \right]$ , first in terms of  $w_t^Q$  and then of  $r_t$ .

**d)** Using the expression of the moment generating function of a Gaussian, in particular

$$E \left\{ \exp \left[ \gamma \mathcal{N}(0, \sigma^2) \right] \right\} = \exp \left[ \frac{\gamma^2 \sigma^2}{2} \right],$$

on the basis of the explicit expression of  $p(t, T)$  in terms of  $w_t^Q$ , obtained under point c), determine the expected value, in  $t = 0$ , of  $p(t, T)$ .

## Solution :

a) By the HJM drift condition, having  $\sigma(t, T) \equiv \sigma$ , one obtains  $a(t, T) = \sigma^2(T - t)$ , from which

$$\begin{aligned} f(t, T) &= f^*(0, T) + \int_0^t \sigma^2(T - s) ds + \int_0^t \sigma dw_s \\ &= f^*(0, T) + \sigma^2 t \left(T - \frac{t}{2}\right) + \sigma w_t \end{aligned}$$

and

$$r_t = f^*(0, t) + \sigma^2 \frac{t^2}{2} + \sigma w_t$$

**b)** Differentiating the last relation one finds

$$dr_t = [f_T^*(0, t) + \sigma^2 t] dt + \sigma dw_t$$

and so one ends up with a Ho-Lee model, i.e. we obtain an affine term structure. Therefore

$$p(t, T) = F^T(t, r_t) = \exp[A(t, T) - B(t, T)r_t]$$

with  $A(t, T)$  and  $B(t, T)$  satisfying the known ordinary differential equations which leads to

$$\begin{cases} B(t, T) = T - t \\ A(t, T) = \int_t^T (f_T^*(0, s) + \sigma^2 s) (s - T) ds + \frac{\sigma^2}{2} \int_t^T (T - s)^2 ds \end{cases}$$

Taking into account that  $f(t, T) = -\frac{\partial}{\partial T} \log p(t, T)$ , and performing an integration by parts, one obtains for  $A(t, T)$  the expression

$$A(t, T) = f^*(0, t)(T - t) + \log \frac{p^*(0, T)}{p^*(0, t)} - \frac{\sigma^2 t}{2} (T - t)^2$$

and so, finally,

$$\begin{aligned} p(t, T) &= \\ &= \frac{p^*(0, T)}{p^*(0, t)} \exp \left[ (T - t) f^*(0, t) - \frac{\sigma^2 t}{2} (T - t)^2 - (T - t) r_t \right] \end{aligned}$$

c) One has

$$\begin{aligned} p(t, T) &= \exp \left[ - \int_t^T f^*(0, s) ds - \frac{\sigma^2}{2} t T (T - t) - \sigma (T - t) w_t^Q \right] \\ &= \frac{p^*(0, T)}{p^*(0, t)} \exp \left[ - \frac{1}{2} \sigma^2 t T (T - t) - \sigma (T - t) w_t^Q \right] \end{aligned}$$

From the expression of  $r_t$  in terms of  $w_t^Q$  one obtains

$$\sigma w_t^Q = r_t - f^*(0, t) - \frac{\sigma^2 t^2}{2}$$

Replacing this in the first relation one ends up with

$$p(t, T) = \frac{p^*(0, T)}{p^*(0, t)} \exp \left[ (T - t) f^*(0, t) - \frac{\sigma^2}{2} t (T - t)^2 - (T - t) r_t \right]$$

**d)** One has

$$\begin{aligned} E\{p(t, T)\} &= \\ &= \frac{p^*(0, T)}{p^*(0, t)} \exp\left[-\frac{1}{2}\sigma^2 t T (T - t)\right] E\left\{e^{-\sigma(T-t)w_t^Q}\right\} \\ &= \frac{p^*(0, T)}{p^*(0, t)} \exp\left[-\frac{1}{2}\sigma^2 t T (T - t)\right] \exp\left[\sigma^2 (T - t)^2 t\right] \\ &= \frac{p^*(0, T)}{p^*(0, t)} \exp\left[\sigma^2 (T - t) t \left(\frac{T}{2} - t\right)\right] \end{aligned}$$

### Problem 5.

**The problem :** Assume that, under a martingale measure  $Q$ , the prices  $S_t, S_t^0$  of two risky assets satisfy

$$dS_t = S_t \left[ r_t dt + \sigma_t dw_t^Q \right] \quad , \quad dS_t^0 = S_t^0 \left[ r_t dt + v_t dw_t^Q \right]$$

a) Denoting by  $Q^0$  the martingale measure equivalent to  $Q$  that corresponds to having  $S_t^0$  as numeraire, show that

$$L_t := E^Q \left\{ \frac{dQ^0}{dQ} \mid \mathcal{F}_t \right\} = \exp \left[ \int_0^t v_s dw_s^Q - \frac{1}{2} \int_0^t v_s^2 ds \right]$$

b) What are the dynamics of  $S_t$  under  $Q^0$  ?

## Solution :

a) It is known that  $L_t = \frac{S_t^0}{S_0^0 B_t}$  so that, under  $Q$ ,

$$\begin{aligned} dL_t &= \frac{1}{S_0^0} \left[ B_t^{-1} S_t^0 r_t dt + S_t^0 B_t^{-1} v_t dw_t^Q - r_t B_t^{-1} S_t^0 dt \right] \\ &= L_t v_t dw_t^Q \end{aligned}$$

The solution of this equation leads immediately to the desired formula.

**b)** Given the expression for  $L_t$  obtained in a), one has that the process  $w_t^{Q^0}$  obtained from  $w_t^Q$  by translating it by  $v_t$ , i.e.

$$dw_t^{Q^0} = dw_t^Q - v_t dt$$

is, under  $Q^0$ , a Wiener process and this implies that, under  $Q^0$ ,

$$dS_t = S_t \left[ (r_t + \sigma v_t) dt + \sigma dw_t^{Q^0} \right]$$

## Problem 6.

**The problem :** Suppose that, under a martingale measure  $Q$ ,

$$dr_t = ar_t dt + \sigma dw_t$$

It follows that one has an affine term structure, i.e.

$$p(t, T) = \exp [A(t, T) - B(t, T)r_t]$$

Assuming  $T_i - T_{i-1} = 1$  it is known that

$$L(t, T_i) = \frac{p(t, T_{i-1})}{p(t, T_i)} - 1$$

- a)** Deriving first the expression for  $\frac{p(t, T_{i-1})}{p(t, T_i)}$  as it results from the affine term structure, determine the dynamics, for  $t \leq T_{i-1}$ , of  $L(t, T_i)$  under the martingale measure  $Q^i$  corresponding to  $p(t, T_i)$  as numeraire.
- b)** Verify whether, under  $Q^i$ , one has lognormality of  $L(t, T_i)$ .

**Solution :** Under  $Q^i$  the process  $L(t, T_i)$  is a martingale and therefore its drift vanishes. It thus suffices to consider the term in  $dw^i$  (where  $w^i$  is a Wiener under  $Q^i$ ).

**a)** One has

$$\begin{aligned}
 dL(t, T_i) &= \\
 &= d(\exp [(A(t, T_{i-1}) - A(t, T_i)) - (B(t, T_{i-1}) - B(t, T_i))r_t]) \\
 &= \frac{p(t, T_{i-1})}{p(t, T_i)} [\dots dt - (B(t, T_{i-1}) - B(t, T_i))\sigma dw_t^i] \\
 &= -[L(t, T_i) + 1] [(B(t, T_{i-1}) - B(t, T_i))\sigma dw_t^i]
 \end{aligned}$$

**b)** Due to the term  $+1$  in the first factor of the rightmost expression, one cannot have lognormality for  $L(t, T_i)$  under  $Q^i$ .

## Problem 7.

**The problem :** Assume that, under a martingale measure  $Q$ , the spot rate  $r_t$  satisfies the following Ho-Lee model

$$dr_t = \Phi_t dt + \sigma dw_t$$

so that

$$p(t, T) = \exp [A(t, T) - B(t, T)r_t]$$

with  $B(t, T) = T - t$ . Given two time points  $T_1, T_2$  with  $t < T_1 < T_2$ , consider an European call option with maturity  $T_1$  and  $p(t, T_2)$  as underlying. Its value in  $T_1$  is thus

$$H_{T_1}^c = (p(T_1, T_2) - K)^+$$

a) Show that the arbitrage free price in  $t = 0$  of this option is

$$\Pi_0^c = p(0, T_2)\Phi(d_1) - Kp(0, T_1)\Phi(d_2)$$

with

$$\begin{cases} d_1 = \frac{\log\left(\frac{p(0, T_2)}{Kp(0, T_1)}\right) + \frac{1}{2}\sigma^2(T_2 - T_1)^2 T_1}{\sigma(T_2 - T_1)\sqrt{T_1}} \\ d_2 = d_1 - \sigma(T_2 - T_1)\sqrt{T_1} \end{cases}$$

**b)** Show that for the corresponding put option

$$H_{T_1}^p = (K - p(T_1, T_2))^+$$

the price in  $t = 0$  is given by

$$\Pi_0^p = p(0, T_2)[\Phi(d_1) - 1] + Kp(0, T_1)[1 - \Phi(d_2)]$$

To this effect notice that one has

$$E^Q \left\{ e^{-\int_0^{T_1} r_s ds} p(T_1, T_2) \right\} = p(0, T_2)$$

c) Assume as known that the price in  $t = 0$  of a Caplet for the generic interval  $[T_{i-1}, T_i]$  and with cap rate  $R$  is, putting  $\alpha_i := (T_i - T_{i-1})$ ,

$$\begin{aligned}
 \text{Capl}_i(0) &= \\
 &= \alpha_i E^Q \left\{ e^{-\int_0^{T_i} r_s ds} (L(T_{i-1}, T_i) - R)^+ \right\} \\
 &= (1 - R\alpha_i) E^Q \left\{ e^{-\int_0^{T_i} r_s ds} \left( \frac{1}{1 - R\alpha_i} - p(T_{i-1}, T_i) \right)^+ \right\}
 \end{aligned}$$

What does this last expression become in the context of the given problem ?

## Solution :

- a) Use the formula for the pricing of options when the short rate is stochastic (section A1). It can be applied whenever the process

$$Z_t := \frac{p(t, T_2)}{p(t, T_1)}$$

has a deterministic volatility.

Since

$$Z_t = \exp[(A(t, T_2) - A(t, T_1)) - (T_2 - T_1)r_t]$$

taking into account the dynamics of  $r_t$ , one obtains

$$dZ_t = Z_t \{[\dots]dt - \sigma \cdot (T_2 - T_1)dw_t\}$$

The result now follows immediately from the formula that was derived in the subsection “Applications” of section A1. for the case of the Hull and White model. Since Ho-Lee corresponds to Hull and White for  $a = 0$ , it suffices to let  $a \rightarrow 0$ .

**b)** Apply the *put-call-parity* that here is based on

$$(K - p(T_1, T_2))^+ = K + (p(T_1, T_2) - K)^+ - p(T_1, T_2)$$

One then has

$$\begin{aligned} & E^Q \left\{ e^{-\int_0^{T_1} r_s ds} (K - p(T_1, T_2))^+ \right\} \\ &= Kp(0, T_1) + p(0, T_2)\Phi(d_1) - Kp(0, T_1)\Phi(d_2) - p(0, T_2) \\ &= p(0, T_2)[\Phi(d_1) - 1] + Kp(0, T_1)[1 - \Phi(d_2)] \end{aligned}$$

The first equality in the previous formula follows from

$$\begin{aligned}
 & E^Q \left\{ e^{-\int_0^{T_1} r_s ds} p(T_1, T_2) \right\} \\
 &= E^Q \left\{ e^{-\int_0^{T_1} r_s ds} E^Q \left\{ e^{-\int_{T_1}^{T_2} r_s ds} \mid \mathcal{F}_{T_1} \right\} \right\} \\
 &= E^Q \left\{ e^{-\int_0^{T_2} r_s ds} \right\} = p(0, T_2)
 \end{aligned}$$

*Notice that the difference  $\Pi_0^p - \Pi_0^c$  of the two prices in  $t = 0$  is  $Kp(0, T_1) - p(0, T_2)$  in accordance with the fact that the value of this difference in  $t = T_1$ , i.e.  $K - p(T_1, T_2)$  has to be discounted to  $t = 0$ .*

**c)** Except for the factor  $(1 - R\alpha_i)$  we are in the context of point b) with  $K = \frac{1}{1 - R\alpha_i}$ ,  $T_1 = T_{i-1}$  and  $T_2 = T_i$ .