Example 1

Stochastic volatility with market completeness

$$(P)\quad \begin{cases} dS_t &= S_t a_t dt + S_t \sigma_t(Z_t) dw_t \\ dZ_t &= \mu(Z_t)dt + \tau(Z_t)dw_t \end{cases}$$

$$(Q)\quad \begin{cases} dS_t &= S_t r_t dt + S_t \sigma_t(Z_t) dw_t^Q \\ dZ_t &= \underbrace{\left[\mu(Z_t) - \sigma_t^{-1}(Z_t)\tau(Z_t)(a_t - r_t)\right]}_{\eta_t(Z_t)} dt + \tau(Z_t) dw_t^Q \end{cases}$$

where

$$dw_t^Q = dw_t + \sigma_t^{-1}(Z_t)(a_t - r_t)$$

It follows

$$(Q) \quad d\tilde{S}_t = \tilde{S}_t \sigma_t(Z_t) dw_t^Q$$

Given the claim

$$H_T = H(S_T, Z_T)$$

and, defining

$$\tilde{M}_t := E^Q\{\tilde{H}_T | \mathcal{F}_t\} = E^Q\{\tilde{H}_T | S_t, Z_t\} = \tilde{F}(t, S_t, Z_t)$$

one has

$$\begin{cases} d\tilde{V}_t &= \tilde{V}_t u_t \sigma_t(Z_t) dw_t^Q \\ \\ d\tilde{M}_t &= [\cdots]dt + \left[\tilde{F}_S(\cdot) S_t \sigma_t(Z_t) + \tilde{F}_Z(\cdot) \tau_t(Z_t)\right] dw_t^Q \end{cases}$$

It follows

$$u_t = \tilde{V}_t^{-1} \tilde{F}_S(\cdot) S_t + \tilde{V}_t^{-1} \tilde{F}_Z(\cdot) \sigma_t^{-1}(Z_t) \tau_t(Z_t)$$

and, respectively,

$$\phi_t = \frac{V_t u_t}{S_t} = \frac{\tilde{V}_t B_t u_t}{S_t} = F_S(t, S_t, Z_t) + F_Z(t, S_t, Z_t) \frac{\tau_t(Z_t)}{S_t \sigma_t(Z_t)}$$

Example 2

Stochastic volatility in incomplete market

$$(P) \quad \begin{cases} dS_t = S_t a_t dt + S_t \sigma_t(Z_t) dw_t \\ \\ dZ_t = \mu_t dt + \tau_t dv_t \end{cases} \qquad ((w_t) \perp (v_t))$$

Given a claim

$$H_T = H(S_T, Z_T)$$

for perfect hedging need an additional asset to complete the market; this asset has to be driven by v_t.

Introducing

$$d\bar{S}_t = \bar{S}_t \bar{a}_t dt + \bar{S}_t \sigma_t^1(Z_t) dw_t + \bar{S}_t \sigma_t^2(Z_t) dv_t$$

one obtains a complete market and a *unique mart.meas.* Q (translating the two Wieners to turn $\tilde{S}_t$ and $\tilde{\bar{S}}_t$ into martingales).

Defining

$$\tilde{M}_t := E^Q\{\tilde{H}_T | \mathcal{F}_t\} = E^Q\{\tilde{H}_T | S_t, Z_t\} = \tilde{F}(t, S_t, Z_t)$$

one has

$$\begin{cases} d\tilde{V}_t &= \tilde{V}_t \left[\left(u_t^S \sigma_t(Z_t) + u_t^{\bar{S}} \sigma_t^1(Z_t) \right) dw_t^Q + u_t^{\bar{S}} \sigma_t^2(Z_t) dv_t^Q \right] \\ d\tilde{M}_t &= \tilde{F}_S(t, S_t, Z_t) S_t \sigma_t(Z_t) dw_t^Q + \tilde{F}_Z(t, S_t, Z_t) \tau_t dv_t^Q \end{cases}$$

and it follows

$$\begin{cases} u_t^S \sigma_t(Z_t) + u_t^{\bar{S}} \sigma_t^1(Z_t) & = & \tilde{V}_t^{-1} \tilde{F}_S(t, S_t, Z_t) S_t \sigma_t(Z_t) \\ u_t^{\bar{S}} \sigma_t^2(Z_t) & = & \tilde{V}_t^{-1} \tilde{F}_Z(t, S_t, Z_t) \tau_t \end{cases}$$

from which

$$\begin{cases} u_t^{\bar{S}} = F_Z(t, S_t, Z_t) \frac{\tau_t}{V_t \sigma_t^2(Z_t)} \\ u_t^S = \frac{F_S(t,S_t,Z_t) S_t}{V_t} - \frac{F_Z(t,S_t,Z_t)\tau_t \sigma_t^1(Z_t)}{V_t \sigma_t^2(Z_t)} \end{cases} \qquad (V_t = F(t, S_t, Z_t))$$

and, respectively,

$$\begin{cases} \phi_t^{\bar{S}} = F_Z(t, S_t, Z_t) \frac{\tau_t}{S_t \sigma_t^2(Z_t)} \\ \phi_t^S = F_S(t, S_t, Z_t) - \frac{F_Z(t,S_t,Z_t)\tau_t \sigma_t^1(Z_t)}{S_t \sigma_t^2(Z_t)} \end{cases}$$

Assume the more general claim

$$H_T = H(S_T, \bar{S}_T, Z_T)$$

then

$$\tilde{M}_t = E^Q\{\tilde{H}_T | \, S_t, \bar{S}_t, Z_t\} = \tilde{F}(t, S_t, \bar{S}_t, Z_t)$$

and

$$\begin{cases} d\tilde{V}_t &= \tilde{V}_t \left[\left(u_t^S \sigma_t(Z_t) + u_t^{\bar{S}} \sigma_t^1(Z_t) \right) dw_t^Q + u_t^{\bar{S}} \sigma_t^2(Z_t) dv_t^Q \right] \\ \\ d\tilde{M}_t &= \left[\tilde{F}_S(t, S_t, \bar{S}_t, Z_t) S_t \sigma_t(Z_t) + \tilde{F}_{\bar{S}}(t, S_t, \bar{S}_t, Z_t) \bar{S}_t \sigma_t^1(Z_t) \right] dw_t^Q \\ &+ \left[\tilde{F}_{\bar{S}}(t, S_t, \bar{S}_t, Z_t) \bar{S}_t \sigma_t^2(Z_t) + \tilde{F}_Z(t, S_t, \bar{S}_t, Z_t) \tau_t \right] dv_t^Q \end{cases}$$

and it follows

$$\begin{cases} u_t^S \sigma_t(Z_t) + u_t^{\bar{S}} \sigma_t^1(Z_t) \\ \qquad = \tilde{V}_t^{-1} \left[\tilde{F}_S(t, S_t, \bar{S}_t, Z_t) S_t \sigma_t(Z_t) + \tilde{F}_{\bar{S}}(t, S_t, \bar{S}_t, Z_t) \bar{S}_t \sigma_t^1(Z_t) \right] \\ \\ u_t^{\bar{S}} \sigma_t^2(Z_t) = \tilde{V}_t^{-1} \left[\tilde{F}_{\bar{S}}(t, S_t, \bar{S}_t, Z_t) \bar{S}_t \sigma_t^2(Z_t) + \tilde{F}_Z(t, S_t, \bar{S}_t, Z_t) \tau_t \right] \end{cases}$$

from which

$$\begin{cases} u_t^{\bar{S}} = F_{\bar{S}}(t, S_t, \bar{S}_t, Z_t) \frac{\bar{S}_t}{V_t} + F_Z(t, S_t, \bar{S}_t, Z_t) \frac{\tau_t}{V_t \sigma_t^2(Z_t)} \\ \\ u_t^S = \frac{F_S(t, S_t, \bar{S}_t, Z_t) S_t}{V_t} + \frac{F_{\bar{S}}(t, S_t, \bar{S}_t, Z_t) \bar{S}_t \sigma_t^1(Z_t)}{V_t \sigma_t(Z_t)} \\ \qquad - \frac{F_{\bar{S}}(t, S_t, \bar{S}_t, Z_t) \bar{S}_t \sigma_t^1(Z_t)}{V_t \sigma_t(Z_t)} - \frac{F_Z(t, S_t, \bar{S}_t, Z_t) \tau_t \sigma_t^1(Z_t)}{V_t \sigma_t^2(Z_t) \sigma_t(Z_t)} \end{cases}$$

Example 3

Asian-type option

Let

$$(P) \quad dS_t = S_t a_t dt + S_t \sigma_t dw_t$$

and consider the claim

$$H_T = \left(\frac{1}{\varepsilon} \int_{T-\varepsilon}^{T} S_t dt - K \right)^+$$

Define the process (Z_t) as

$$Z_{t+\tau} = Z_t + \int_t^{t+\tau} g(u, S_u) du$$

where

$$g(t, s) = \begin{cases} 0 & \text{if } s \leq T - \varepsilon \\ \frac{s}{\varepsilon} & \text{if } s > T - \varepsilon \end{cases}$$

and the initial condition is

$$Z_t = \begin{cases} 0 & \text{if } t \leq T - \varepsilon \\ \frac{1}{\varepsilon} \int_{T-\varepsilon}^{t} S_u du & \text{if } T - \varepsilon < t \leq T \end{cases}$$

It then follows that

$$H_T = H(S_T, Z_T) \qquad \text{(depends actually only on } Z_T\text{)}$$

The market is complete and there exists a unique MM Q

Defining

$$\tilde{M}_t := E^Q\{\tilde{H}_T|\,\mathcal{F}_t\} = E^Q\{\tilde{H}_T|\,S_t, Z_t\} = \tilde{F}(t, S_t, Z_t)$$

one has

$$\begin{cases} d\tilde{V}_t &= \tilde{V}_t u_t \sigma_t dw_t^Q \\ \\ d\tilde{M}_t &= \underbrace{[\cdots\cdots]}_{=0} dt \quad + \tilde{F}_S(t, S_t, Z_t) S_t \sigma_t dw_t^Q \end{cases}$$

It follows

$$u_t = \frac{\tilde{F}_S(t, S_t, Z_t)S_t}{\tilde{V}_t} = \frac{F_S(t, S_t, Z_t)S_t}{V_t}$$

and, respectively,

$$\phi_t = \frac{u_t V_t}{S_t} = F_S(t, S_t, Z_t)$$

Example 4

More Wiener than assets

Let

$$dS_t = S_t \left[a_t dt + \sigma_t^1 dw_t^1 + \sigma_t^2 dw_t^2\right]$$

$\longrightarrow$ on $\mathcal{F}_t^{w^1} \otimes \mathcal{F}_t^{w^2}$: ∞ MM's

Equivalently one may write

$$dS_t = S_t \left[a_t dt + \sqrt{(\sigma_t^1)^2 + (\sigma_t^2)^2}\, dw_t\right]$$

and on $\mathcal{F}_t^w$ $(\mathcal{F}_t^S)$ $\exists!$ MM Q.

Attention in the case

$$\begin{cases} dS_t^1 & = & S_t^1\left[a_t^1 dt + \sigma_t^{11} dw_t^1 + \sigma_t^{12} dw_t^2\right] \\ dS_t^2 & = & S_t^2\left[a_t^2 dt + \sigma_t^{21} dw_t^1 + \sigma_t^{22} dw_t^2\right] \end{cases}$$

Rewriting it as

$$\begin{cases} dS_t^1 & = & S_t^1\left[a_t^1 dt + \sqrt{(\sigma_t^{11})^2 + (\sigma_t^{12})^2}\, dw_t\right] \\ dS_t^2 & = & S_t^2\left[a_t^2 dt + \sqrt{(\sigma_t^{21})^2 + (\sigma_t^{22})^2}\, dw_t\right] \end{cases}$$

the two Wiener w_t are not independent sources of randomness.

Instead one has to consider the marginals of the following transformation of $\mathbb{R}^2-$valued Wiener processes

$$\begin{cases} d\bar{w}_t^1 &= \sqrt{(\sigma_t^{11})^2 + (\sigma_t^{12})^2} \left[\sigma_t^{11} dw_t^1 + \sigma_t^{12} dw_t^2\right] \\ d\bar{w}_t^2 &= \sqrt{(\sigma_t^{21})^2 + (\sigma_t^{22})^2} \left[\sigma_t^{21} dw_t^1 + \sigma_t^{22} dw_t^2\right] \end{cases}$$

Example 5

Less Wiener than assets

- there does not exist a martingale measure;
- every claim can be hedged;
- it suffices to invest in as many assets as the number of independent Wiener;
- the investment strategy depends in general on the values of all the assets.

Arbitrage is possible in this case

Let $B_t \equiv 1$ and consider two assets with prices S_t^1, S_t^2 s.t.

$$\begin{cases} dS_t^1 & = & S_t^1 \sigma_t^1 dw_t \\ \\ dS_t^2 & = & S_t^2 \left[a_t dt + \sigma_t^2 dw_t \right] \end{cases}$$

The value of a self-financing portfolio satisfies

$$dV_t = V_t \left[u_t^1 \frac{dS_t^1}{S_t^1} + u_t^2 \frac{dS_t^2}{S_t^2} \right] = V_t \left[u_t^2 a_t dt + (u_t^1 \sigma_t^1 + u_t^2 \sigma_t^2) dw_t \right]$$

Choosing u_t^1, u_t^2 s.t.

$$\begin{cases} u_t^1 + u_t^2 = 1 \\ u_t^1 \sigma_t^1 + u_t^2 \sigma_t^2 = 0 \text{ with } u_t^2 > 0 \end{cases}$$

one has

$$dV_t = V_t u_t^2 a_t dt$$

so that, every time $u_t^2 a_t \neq 0$, there is arbitrage.