Ruin probabilities in a finite-horizon risk model with investment and reinsurance

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Abstract

A finite horizon insurance model is studied where the risk/reserve process can be controlled by reinsurance and investment in the financial market. Our setting is innovative in the sense that we describe in a unified way the timing of the events, that is the arrivals of claims and the changes of the prices in the financial market, by means of a continuous-time Semi-Markov process (SMP) which appears to be more realistic than, say, classical diffusion-based models. Obtaining explicit optimal solutions for the minimizing ruin probability is a difficult task. Therefore we derive a specific methodology, based on recursive relations for the ruin probability, to obtain a reinsurance and investment policy that minimizes an exponential bound (Lundberg-type bound) on the ruin probability. We also obtain an explicit analytic solution for a specific case of the underlying SMP model, namely for exponential intra-event times, which allows us furthermore to obtain some qualitative insight into the impact that investment in the financial market may have on the ruin probability.

Keywords: Risk process, Semi-Markov processes, optimal reinsurance and investment, Lundberg-type bounds.

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1 Introduction

We consider a discrete-time insurance risk/reserve process which can be controlled by reinsurance and investment in the financial market, and we study the ruin probability problem in the finite horizon case. Although controlling a risk/reserve process is a very active area of research (see Chen et al. (2000), Wang et al. (2004), Schmidli (2008), Huang et al. (2009) and references therein), obtaining explicit optimal solutions minimizing the ruin probability is in general a difficult task even for the classical Cramér-Lundberg risk process. Thus, an alternative method commonly used in ruin theory is to derive inequalities for ruin probabilities. The inequalities can be used to obtain upper bounds for the ruin probabilities (see Willmot & Lin (2001), Grandell (1991), Schmidli (2002)), and this is the approach followed in the present paper. The basis of this approach is the well known fact that in the classical Cramér-Lundberg model, if the claim sizes have finite exponential moments, then the ruin probability decays exponentially as the initial surplus increases (see for instance the book by Asmussen (2000)). For the heavy-tailed claims' case it is also shown to decay with a rate depending on the distribution of the claim size; see e.g., Gaier et al. (2003). Paulsen (1998) reviews general processes for the ruin problem when the insurance company invests in a risky asset. Xiong & Yang (2011) give conditions for the ruin probability to be equal to 1 for any initial endowment and without any assumption on the distribution of the claim size as long as it is not identically zero.

Control problems for risk/reserve processes are commonly formulated in continuous time. Schäl (2004) introduces a formulation of the problem where events (arrivals of claims and asset price changes) occur at discrete points in time that may be deterministic or random, but their total number is fixed. Diasparra & Romera (2009) consider a similar formulation in discrete time. Having a fixed total number of events implies that in the case of random time points the horizon is random as well. In the present paper we follow an approach inspired by Edoli & Runggaldier (2010) who claim that a more natural way to formulate the problem in case of random time points is to consider a given fixed time horizon so that also the number of event times becomes random and this makes the problem nonstandard. Accordingly it is reasonable to assume that also the control decisions (level of reinsurance and amount invested) correspond to these random time points. Notice that this formulation can be seen equivalently in discrete or continuous time.

The stochastic elements that affect the evolution of the risk/reserve process are thus the timing and size of the claims as well as the dynamics of the prices of the assets in which the insurer is investing. This evolution is controlled by the sequential choice of the reinsurance and investment levels.

Claims occur at random points in time and also their sizes are random, while asset price evolutions are usually modeled as continuous time processes. On small time scales, prices actually change at discrete random time points and vary by tick size. In the proposed model we let also asset prices change only at discrete random time points with their sizes being random as well. This will allow us to consider the timing of the events, namely the arrivals of claims and the changes of the asset prices, to be triggered by a same continuous-time semi-Markov process, i.e. a stochastic process where the embedded jump chain (the discrete process registering what values the process takes) is a Markov Chain, and where the holding times (time between jumps) are random variables, whose distribution function may depend on the two states between which the move is made. Since between event times the situation for the insurer does not change, we shall consider controls only at event times.

The rest of the paper is organized as follows. In section 2 we describe the model and, in particular, the risk process. Section 3 is devoted to derive recursive relations for the ruin probability. On the basis of these recursive relations, in section 4 we obtain exponential (Lundberg-type) bounds on the ruin probability. In section 5 we then discuss a general approach to determine the insurance and investment levels in order to minimize these bounds. In subsection 5.2.1 an analytic solution is obtained in the case when the intra-event times are exponentially distributed, a case that is widely considered in the literature. Finally, section 6 contains concluding remarks.

2 The model

We consider a finite time horizon $T > 0$. More precisely, to model the timing of the events (arrival of claims and asset price changes), inspired by Schäl (2005) we introduce the process $\{K_t\}_{t>0}$ for $t \leq T$, a continuous time semi-Markov process (SMP) on $\{0, 1\}$, where $K_t = 0$ holds for the *arrival of a claim*, and $K_t = 1$ for a *change in the asset price.* The embedded Markov chain, i.e.,the jump chain associated to the SMP $\{K_t\}_{t>0}$, evolves according to a transition probability matrix $P = \|p_{ij}\|_{i,j\in\{0,1\}}$ that is supposed to be given, and the holding times (time between jumps) are random variables whose probability distribution function may depend on the two states between which the move is made. We come back to this point in the next subsection 2.1.

Let T_n be the random time of the $n-th$ event, $n \geq 1$, and let the counting process N_t denote the number of events having occurred up to time t, defined as follows

$$N_t = \sum_{j=1}^{\infty} 1_{\{T_j \leq t\}} \tag{1}$$

and so

$$T_n = \min\{t \geq 0 \mid N_t = n\}. \tag{2}$$

2.1 Risk process

In this section we introduce the dynamics of the controlled risk process X_t for $t \in [0, T]$ with T a given fixed horizon. For this purpose let Y_n be the $n-th$ $(n \geq 1)$ claim payment represented by a sequence of independent and identically distributed (i.i.d.) random variables with common probability distribution function (p.d.f.) $F(y)$ having support in the positive half line. Let Z_n be the random variable denoting the time between the occurrence of the $n-1$st and nth $(n \geq 1)$ jumps of the SMP $\{K_t\}_{t>0}$. We assume that $\{Z_n\}$ is a sequence of i.i.d. random variables with p.d.f. $G(\cdot)$. From this we may consider that the transition probabilities of the SMP $\{K_t\}_{t>0}$ are

$$P\{K_{T_{n+1}} = j, Z_{n+1} \leq s \mid K_{T_n} = i\} = p_{ij} G(s)$$

Notice that for a full SMP model, the distribution function $G(\cdot)$ depends also on i and j. While the results derived below go through in the same way also for a $G_{ij}(\cdot)$ depending on i, j, for the specific setting of section 5.2 we shall have to restrict ourselves to independent $G(\cdot)$. We shall therefore do so already from the beginning.

Example 1 *A specific form of SMP, which we shall refer to also later, arises for example as follows: let* N_t^1 *and* N_t^2 *be independent Poisson processes with intensities* λ^1 *and* λ^2 *respectively. We may think of* N_t^1 *as counting the number of claims and* N_t^2 *that of price changes and we have that* $N_t = N_t^1 + N_t^2$ *is again a Poisson process with intensity* $\lambda = \lambda^1 + \lambda^2$*. We may then put*

$$K_{T_n} = \begin{cases} 0 & \text{if at } T_n \text{ a jump of } N_t^1 \text{ has occurred (claim)} \\ \\ 1 & \text{if at } T_n \text{ a jump of } N_t^2 \text{ has occurred (price change)} \end{cases}$$

It then follows easily that

$$\begin{aligned}
&P\{K_{T_{n+1}} = j, Z_{n+1} \leq s \mid K_{T_n} = i\} \\
&= P\{K_{T_{n+1}} = j \mid Z_{n+1} \leq s,\, K_{T_n} = i\}\, P\{Z_{n+1} \leq s \mid K_{T_n} = i\} \\
&= p_{ij} P\{Z_1 \leq s\} = \tfrac{\lambda^j}{\lambda} \left[1 - e^{-\lambda s}\right]
\end{aligned}$$

so that, in this case

$$\begin{cases} p_{ij} & = \tfrac{\lambda^j}{\lambda} := p_j\ , \quad \forall i \\ \\ G(s) & = \left[1 - e^{-\lambda s}\right] \end{cases}$$

Remark 2 *Our model, in particular the specific case of Example 1, does not allow for simultaneous jumps (claims and price changes may however occur very close to one another). On the other hand we could include simultaneous jumps into our model by extending K_t to take three possible values, namely*

$$K_{T_n} = \begin{cases} 0 & \textit{for a claim} \\ 1 & \textit{for a price change} \\ 2 & \textit{for simultaneous claim and price change} \end{cases}$$

The risk process is controlled by reinsurance and investment. In general this means that we may choose adaptively at the event times T_{N_t} (they correspond to the jump times of N_t) the retention level (or proportionality factor or risk exposure) b_{N_t} of a reinsurance contract as well as the amount δ_{N_t} to be invested in the risky asset, namely in S_{N_t} with S_t denoting discounted prices. For the values b that the various b_{N_t} may take we assume that $b \in (b_{\min}, 1] \subset (0, 1]$, where $b_{\min}$ will be introduced below and for the values of δ for the various δ_{N_t} we assume $\delta \in [\underline{\delta}, \overline{\delta}]$ with $\underline{\delta} \leq 0$ and $\overline{\delta} > 0$ exogenously given. Notice that this condition allows also for negative values of δ meaning that, see also Schäl (2004), short selling of stocks is allowed. On the other hand, with an exogenously given upper bound $\bar{\delta}$, it might occasionally happen that $\delta_{N_t} > X_{N_t}$ implying a temporary debt of the agent beyond his/her current wealth in order to invest optimally in the financial market. By choosing a policy that minimizes the ruin probability, this debt is however only instantaneous and with high probability leads to a positive wealth already at the next event time.

Assume that prices change only according to

$$\frac{S_{N_t+1} - S_{N_t}}{S_{N_t}} = \left(e^{W_{N_t+1}} - 1\right) K_{T_{N_t+1}}, \tag{3}$$

where W_n is a sequence of i.i.d. random variables taking values in $[\underline{w}, \bar{w}]$ with $\underline{w} < 0 < \bar{w}$, where one may also have $\underline{w} = -\infty$, $\bar{w} = +\infty$ and with p.d.f. $H(w)$. For simplicity and without loss of generality we consider only one asset to invest in. An immediate generalization would be to allow for investment also in the money market account.

Let c be the premium rate (income) paid by the customer to the company, fixed in the contract. Since the insurer pays to the reinsurer a premium rate, which depends on the retention level b_{N_t} chosen at the various event times T_{N_t}, we denote by $C(b_{N_t})$ the net income rate of the insurer at time $t \in [0, T]$. For $b \in (b_{\min}, 1]$ we let $h(b, Y)$ represent the part of the generic claim Y paid by the insurer and in what follows we take the function $h(b, Y)$ to be of the form $h(b, Y) = b \cdot Y$ (proportional reinsurance). We shall call *policy* a sequence $\pi = (b_n, \delta_n)$ of *control actions*. Control actions over a single period will be denoted by $\phi_n = (b_n, \delta_n)$. According to the expected value principle with safety loading θ of the reinsurer, for a given starting time $t < T$, the function $C(b)$ can be chosen as follows:

$$C(b) := c - (1+\theta)\frac{E\{Y_1 - h(b, Y_1)\}}{E\{Z_1 \wedge (T-t)\}}, \;\; 0 < t < T \tag{4}$$

Note that $C(b)$ depends on the starting and the terminal times t and T only via $T - t$ in the denominator of (4); the value of $T - t$ can however be considered as given so that $C(b)$ can indeed be considered as depending only on the current value b of the retention level.

We use Z_1 and Y_1 in the above formula since, by our i.i.d. assumption, the various Z_n and Y_n are all independent copies of Z_1 and Y_1. Notice also that, in order to keep formula (4) simple and possibly similar to standard usage, in the denominator of the right hand side we have considered the random time Z_1 between to successive events, while more correctly we should have taken the random time between two successive claims, which is larger. For this we can however play with the safety loading factor. In fact, if we denote by $\bar{Z}$ the average time between successive claims before T and, for a given θ put $\bar{\theta} = (1+\theta)\frac{\bar{Z}}{E\{Z_1 \wedge (T-t)\}} - 1$ we have that $\frac{(1+\theta)}{E\{Z_1 \wedge (T-t)\}} = \frac{(1+\bar{\theta})}{\bar{Z}}$. Since in this way $1 + \bar{\theta} = (1+\theta)\frac{\bar{Z}}{E\{Z_1 \wedge (T-t)\}}$ and $\bar{Z} > E\{Z_1 \wedge (T-t)\}$, we are assured that $(1+\bar{\theta}) > 1$. We can now define $b_{\min} := \min\{b \in (0,1] \mid c \geq C(b) \geq c^*\}$, where $c^* \geqslant 0$ denotes the minimal value of the premium considered by the insurer. We make the

Assumption 3 *Let*

i) *The random variables* $(Z_n, Y_n, W_n)_{n\geq 1}$ *are, conditionally on* K_t*, mutually independent.*

ii) $E\left\{e^{rY_1}\right\} < +\infty$ *for* $r \in (0,\bar{r})$ *with* $\bar{r} \in (0,\infty)$.

iii) $c - (1+\theta)\frac{E\{Y_1\}}{E\{Z_1 \wedge T\}} \geq 0$.

Remark 4 *Notice that*

i) *Since* $b \leq 1$*, point ii) in Assumption 3 implies that also* $E\left\{e^{rbY_1}\right\} < +\infty$ *for* $r \in (0,\bar{r})$ *and all* $b \in (b_{min}, 1]$.

ii) *For* $h(b,Y) = bY$*, point iii) in Assumption 3 implies that* $c \geqslant C(b) \geqslant c^* \geqslant 0$, $\forall\, b \in (b_{min}, 1]$ *and that, furthermore,* $c \geqslant 0$.

iii) *Since the support of* Y_1 *is in the positive half line, we have* $\lim_{r\uparrow\bar{r}} E\left\{e^{rY_1}\right\} = +\infty$*. Notice that* $\bar{r}$ *may be equal to* $+\infty$*, e.g. if the support of* Y_1 *is bounded.*

In the given setting we obtain now for the insurance risk process (surplus) X the following one-step transition dynamics between the generic random times T_n and T_{n+1} when at T_n a control action $\phi = (b,\delta)$ is taken for a certain $b \in (b_{\min}, 1] \subset (0,1]$, and $\delta \in [\underline{\delta}, \bar{\delta}]$,

$$X_{T_{n+1}} = X_{T_n} + C(b)Z_{n+1} - (1-K_{T_{n+1}})h(b, Y_{n+1}) + K_{T_{n+1}}\delta(e^{W_{n+1}} - 1) \quad (5)$$

Definition 5 *Letting* $U := [b_{\min}, 1] \times [\underline{\delta}, \overline{\delta}]$*, we shall say that a control action* $\phi = (b,\delta)$ *is* ***admissible*** *if* $(b,\delta) \in U$*. Notice that* U *is compact.*

We want now to express the one-step dynamics in (5) when starting from a generic time instant $t < T$ with a capital x. For this purpose note that if, for a given $t < T$ one has $N_t = n$, the time T_{N_t} is the random time of the $n-th$ event and $T_n \leq t \leq T_{n+1}$. Since, when standing at time t, we observe the time that has elapsed since the last event in T_{N_t}, it is not restrictive to assume that $t = T_{N_t}$ (see the comment below after (6)). A further justification for letting $t = T_{N_t}$ can be given in the case of Example 1: the random variable there has a negative exponential distribution and this distribution is memoryless. Furthermore, since Z_n, Y_n, W_n are i.i.d., in the one-step random dynamics for the risk process X_t we may replace the generic $(Z_{n+1}, Y_{n+1}, W_{n+1})$ by (Z_1, Y_1, W_1). We may thus write

$$X_{N_t+1} = x + C(b)Z_1 - (1 - K_{T_{N_t+1}})h(b, Y_1) + K_{T_{N_t+1}}\delta(e^{W_1} - 1) \qquad (6)$$

for $0 < t < T$, $T > 0$ and with $X_t = x \geq 0$ (recall that we assumed $t = T_{N_t}$). Notice that, if we had $t \neq T_{N_t}$ and therefore $t > T_{N_t}$, the second term on the right in (6) would become $C(b)[Z_1 - (t - T_{N_t})]$ and (6) could then be rewritten as

$$X_{N_t+1} = [x - C(b)(t - T_{N_t})] + C(b)Z_1 - (1 - K_{T_{N_t+1}})h(b, Y_1) + K_{T_{N_t+1}}\delta(e^{W_1} - 1)$$

with the quantity $[x - C(b)(t - T_{N_t})]$, which is known at time t, replacing x. This is the sense in which above we mentioned that it is not restrictive to assume that $t = T_{N_t}$. In what follows we shall work with the risk process X_t, (or X_{N_t}) as defined by (6). For convenience we shall denote by (b_n, δ_n) the values of $\phi = (b, \delta)$ at $t = T_{N_t}$. Accordingly, we shall also write (b_{N_t}, δ_{N_t}) for $\left(b_{T_{N_t}}, \delta_{T_{N_t}}\right)$.

Following Schmidli (2008) we shall finally introduce an absorbing (cemetery) state $\varkappa$, such that if $X_{N_t} < 0$ or $X_{N_t} = \varkappa$, then $X_{N_t+1} = \varkappa$, $\forall t \leq T$. The state space is denoted by $\mathfrak{X} = \mathbb{R} \cup \{\varkappa\}$.

3 Recursions

We start this section by specifying some notation and introducing the basic definitions concerning our ruin probabilities.

3.1 Notation and Definitions

Note that (6) may be rewritten as

$$X_{N_t+1} = x - [(1 - K_{T_{N_t+1}})h(b, Y_1) - C(b)Z_1 - K_{T_{N_t+1}}\delta(e^{W_1} - 1)] \qquad (7)$$

From this expression one can see that ruin occurs if the difference between the amount $(1 - K_{T_{N_t+1}})h(b, Y_1)$ of the claim and that $[C(b)Z_1 + K_{T_{N_t+1}}\delta(e^{W_1} -$

1)] of the reserve increase due to premiums and investment is greater than the initial reserve x. On the other hand, in terms of the reserve increase due to premiums, the ruin occurs if

$$C(b)Z_1 \leq (1 - K_{T_{N_t+1}})h(b, Y_1) - K_{T_{N_t+1}}\delta(e^{W_1} - 1) - x.$$

Thus, given a policy π (of which in the definitions below we need just to consider the generic individual control action $\phi = (b, \delta)$), we introduce the following functions

$$\begin{aligned} u^\pi(y, z, w, k) &: = (1-k)by - C(b)z - k\delta(e^w - 1) \\ \tau^\pi(y, w, k, x) &: = \frac{(1-k)by - k\delta(e^w - 1) - x}{C(b)} \end{aligned} \tag{8}$$

so that $u^\pi(y, z, w, k) < x \iff z > \tau^\pi(y, w, k, x)$, as well as the disjoint sets

$$\begin{aligned} A^+_{x,\pi} &:= \{(y, z, w, k) | u^\pi(y, z, w, k) < x\} = \{(y, z, w, k) | \tau^\pi(y, w, k, x) < z\}, \\ A^-_{x,\pi} &:= \{(y, z, w, k) | u^\pi(y, z, w, k) \geq x\} = \{(y, z, w, k) | \tau^\pi(y, w, k, x) \geq z\}. \end{aligned}$$

and ruin occurs for an event in $A^-_{x,\pi}$.

For simplicity of notation, in what follows we shall often write X_k for X_{T_k}. We give now the following

Definition 6 *Assume we are standing at time t with a surplus value of $X_t = X_{T_{N_t}} = x$ and with $K_t = K_{T_{N_t}} = k \in \{0, 1\}$. We shall denote by $\psi^\pi_n(t, x; k)$ the ruin probability when at most n events are considered in the interval $[t, T]$ and a policy π is adopted. More formally, we have*

$$\begin{aligned} \psi^\pi_n(t, x; k) &: = P^\pi \left\{ \bigcup_{k=N_t+1}^{(N_t+n)\wedge N_T} \{X_k < 0\} \,|\, X_{N_t} = x, K_{T_{N_t}} = k \right\} \\ &: = P^\pi_{x,k} \left\{ \bigcup_{k=N_t+1}^{(N_t+n)\wedge N_T} \{X_k < 0\} \right\} \end{aligned} \tag{9}$$

Remark 7 *Notice that for $t_1 < t_2$ we have*

$$\psi^\pi_n(t_1, x; k) \geq \psi^\pi_n(t_2, x; k) \tag{10}$$

Our first purpose in the next section is to obtain a recursive relation for $\psi^\pi_n(t, x; k)$.

3.2 Recursive relations

In view of obtaining a recursive relation for $\psi_n^\pi(t,x,k)$, in addition to the sets $A^+_{x,\pi}$, $A^-_{x,\pi}$ define, for a given $t < T$, the events

$$B := \{X_{N_t+1} < 0\} \ ; \quad C_n := \bigcup_{h=N_t+2}^{(N_t+n)\wedge N_T} \{X_h < 0\} \tag{11}$$

and notice that $B \cap C_n = \emptyset$ and $C_n \cap \{N_T - N_t \leq 1\} = \emptyset$. Furthermore, given (x,k), the event B is equivalent to an event happening in the set $A^-_{x,\pi}$.

The main result of this section is the recursive relation in the following

Proposition 8 *For an initial surplus x at a given time $t \in [0,T]$, as well as an initial event $K_{T_{N_t}} = k$ and a given policy π, the ruin probability according to Definition 6 (see (9)) admits the following recursive relation*

$$\begin{array}{l} \psi_n^\pi(t,x,k) \\ \\ = P\{N_T - N_t > 0\} \sum_{h=0}^1 p_{k,h} \int_{\underline{w}}^{\bar{w}} \int_0^\infty G(\tau^\pi(y,w,h,x) \wedge (T-t)) dF(y) dH(w) \\ \quad + P\{N_T - N_t > 1\} \sum_{h=0}^1 p_{k,h} \cdot \\ \quad \cdot \int_{\underline{w}}^{\bar{w}} \int_0^\infty \int_{\tau^\pi(y,w,h,x)}^{T-t} \psi_{n-1}^\pi(t+z, x-u^\pi(y,z,w,h), h) dG(z) dF(y) dH(w) \end{array} \tag{12}$$

from which it immediately also follows that

$$\begin{array}{l} \psi_1^\pi(t,x,k) = P\{N_T - N_t = 1\} \\ \quad \sum_{h=0}^1 p_{k,h} \int_{\underline{w}}^{\bar{w}} \int_0^\infty G(\tau^\pi(y,w,h,x) \wedge (T-t)) dF(y) dH(w) \end{array} \tag{13}$$

since in the case of at most one jump one has that $P\{N_T - N_t > 0\} = P\{N_T - N_t = 1\}$ and $P\{N_T - N_t > 1\} = 0$.

Proof: With the definitions in (11), from (9) we can write

$$\psi_n^\pi(t,x,k) = P_{x,k}^\pi\{B \cup C_n\} = P_{x,k}^\pi\{B\} + P_{x,k}^\pi\{C_n\} =$$

$$= P_{x,k}^\pi\{A_{x,\pi}^- \cap \{N_T - N_t > 0\}\} + P_{x,k}^\pi\{C_n \cap A_{x,\pi}^+ \cap \{N_T - N_t > 1\}\} =$$

$$\begin{aligned}&= P\{N_T - N_t > 0\} P_{x,k}^\pi\{\tau^\pi(Y_1, W_1, K_{T_{N_t+1}}, x) \geq Z_1 \text{ with } t + Z_1 \leq T\} + \\ &\quad + P\{N_T - N_t > 1\} \cdot \\ &\quad \cdot P_{x,k}^\pi\left\{\{(\tau^\pi(Y_1, W_1, K_{T_{N_t+1}}, x) < Z_1 < T - t)\} \cap \left(\bigcup_{h=N_t+2}^{(N_t+n)\wedge N_T}\{X_h < 0\}\right)\right\} =\end{aligned}$$

$$\begin{aligned}&= P\{N_T - N_t > 0\} \sum_{h=0}^1 p_{k,h} \int_{\underline{w}}^{\bar{w}} \int_0^\infty \int_0^{\tau^\pi(y,w,h,x)\wedge(T-t)} dG(z)dF(y)dH(w) + \\ &\quad + P\{N_T - N_t > 1\} \sum_{h=0}^1 p_{k,h} \cdot \\ &\quad \cdot \int_{\underline{w}}^{\bar{w}} \int_0^\infty \int_{\tau^\pi(y,w,h,x)}^{T-t} \psi_{n-1}^\pi(t+z, x - u^\pi(y,z,w,h), h) dG(z)dF(y)dH(w) =\end{aligned}$$

$$\begin{aligned}&= P\{N_T - N_t > 0\} \sum_{h=0}^1 p_{k,h} \int_{\underline{w}}^{\bar{w}} \int_0^\infty G(\tau^\pi(y,w,h,x) \wedge (T-t)) dF(y)dH(w) + \\ &\quad + P\{N_T - N_t > 1\} \sum_{h=0}^1 p_{k,h} \cdot \\ &\quad \cdot \int_{\underline{w}}^{\bar{w}} \int_0^\infty \int_{\tau^\pi(y,w,h,x)}^{T-t} \psi_{n-1}^\pi(t+z, x - u^\pi(y,z,w,h), h) dG(z)dF(y)dH(w)\end{aligned}$$

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4 Bounds

We derive here bounds on the ruin probability in a general setting and, in the next section 5, we then minimize them with respect to the reinsurance and investment policy. We base ourselves on results in Diasparra & Romera (2009) and Diasparra & Romera (2010) that are here extended to the general setup of the present paper. To stress the fact that the process X defined in (5) corresponds to the choice of a specific policy π, in what follows we shall use the notation X^π.

Given a policy $\pi_t = (b_t, \delta_t)$ and defining for $t \in [0,T]$ the random variable

$$V_t^\pi := C(b)(Z_1 \wedge (T-t)) - \mathbf{1}_{\{Z_1 \leq T-t\}}[(1 - K_{T_{N_t}+1}) b Y_1 + K_{T_{N_t}+1} \delta (e^{W_1} - 1)] \quad (14)$$

where $b = b_t$ and $\delta = \delta_t$ let, for $r \in (0, \bar{r})$ and $k \in \{0,1\}$

$$l_r^\pi(t,k) := E_{t,k}\{e^{-rV_t^\pi}\} - 1 \quad (15)$$

where, for reasons that should become clear below, we distinguish the dependence of l^π on r from that on (t,k).

Remark 9 *Notice that, by its definition,* $l_r^\pi(t,k)$ *is, for given* π *and* $r \in (0,\bar{r})$ *with* $\bar{r}$ *such that Assumption 3, ii) is satisfied, a bounded function of* $(t,k) \in [0,T] \times \{0,1\}$. *Given its continuity in* r, *it is uniformly bounded on any compact subset of* $(0,\bar{r})$, *e.g. on* $[\eta, \bar{r}-\eta]$ *for* $\eta \in (0,\bar{r})$. *Having fixed* η, *denote this bound by* L, *i.e.*

$$\sup_{(t,k)\in[0,T]\times\{0,1\},\, r\in[\eta,\bar{r}-\eta]} |l_r^\pi(t,k)| \leq L\,, \quad \pi \in \mathcal{A}\, given \tag{16}$$

Remark 10 *Notice that, since (see Remark 4 (iii))* $\lim_{r\uparrow\bar{r}} l_r^\pi(t,k) = +\infty$, *one can choose* η *such that* $l_{\bar{r}-\eta}^\pi(t,k) > 0$.

Definition 11 *We shall call a policy* π ***admissible*** *and denote their set by* $\mathcal{A}$ *if at each* $t \in [0,T]$ *the corresponding control action* $(b_t, \delta_t) \in U$ *and, for any* $(t,k) \in [0,T] \times \{0,1\}$, *it holds that* $E_{t,k}^\pi\{V_t^\pi\} > 0$.

Notice that $\mathcal{A}$ is non-empty since, see Assumption 3, iii), it contains at least the stationary policy $(b_{N_t}, \delta_{N_t}) \equiv (b_{\min}, 0)$.

Proposition 12 *For each* $(t,k) \in [0,T] \times \{0,1\}$ *and each* $\pi \in \mathcal{A}$ *we have that*

- **i)** *As a function of* $r \in (0,\bar{r})$ *with* $\bar{r}$ *such that Assumption 3, ii) is satisfied,* $l_r^\pi(t,k)$ *is convex with a negative slope at* $r = 0$;
- **ii)** *the equation* $l_r^\pi(t,k) = 0$, *seen as an equation in* r, *has a unique positive root in* $(0,\bar{r})$ *that we denote by* $R^\pi(t,k)$, *so that the defining relation for* $R^\pi(t,k)$ *is*

$$l_{R^\pi(t,k)}^\pi(t,k) = 0 \quad \forall t \in [0,T],\ k \in \{0,1\} \tag{17}$$

Proof: Differentiating with respect to r under the expectation sign leads to

$$\begin{cases} \frac{\partial}{\partial r}\left(l_r^\pi(t,k)\right)_{|r=0} & = & E_{t,k}\{-V_t^\pi\} < 0 \\ \\ \frac{\partial^2}{\partial r^2}\left(l_r^\pi(t,k)\right) & = & E_{t,k}\left\{\left(V_t^\pi\right)^2 \cdot e^{-rV_t^\pi}\right\} > 0 \end{cases} \tag{18}$$

where the first inequality follows from the admissibility of π (see Definition 11), from which statement i) follows immediately. In view of ii) notice that from Assumption 3, ii) (see also Remark 4 (iii)) one obtains $\lim_{r\uparrow\bar{r}} l_r^\pi(t,k) = +\infty$. This fact, combined with i) leads to ii). ∎

Definition 13 *For given $\pi \in \mathcal{A}$ and $t \in [0,T]$ let*

$$R_t^\pi := \min[R^\pi(t,0),\ R^\pi(t,1)] \tag{19}$$

Remark 14 *Notice that by Remark 10 we can always assume that $R^\pi(t,k) < \bar{r} - \eta$, $\forall (t,k) \in [0,T] \times \{0,1\}$ and thus also $R_t^\pi < \bar{r} - \eta$.*

We come now to deriving the bounds on the ruin probabilities. Inspired by Diasparra & Romera (2009) and Diasparra & Romera (2010), notice that for any $\theta > 0$ we may write

$$G^\pi(\theta) := \left(\frac{\int_0^\theta e^{-R^\pi C(b) z} dG(z)}{e^{-R^\pi C(b)\theta} G(\theta)} \right)^{-1} \cdot e^{R^\pi C(b)\theta} \int_0^\theta e^{-R^\pi C(b) z} dG(z) \tag{20}$$

Since, uniformly in $\pi \in \mathcal{A}$ and $\theta \in [0,T]$, we have for $z \leq \theta$

$$\frac{\int_0^\theta e^{-R^\pi C(b) z} dG(z)}{e^{-R^\pi C(b)\theta} G(\theta)} \geq \frac{\int_0^\theta e^{-R^\pi C(b)\theta} dG(z)}{e^{-R^\pi C(b)\theta} G(\theta)} = \frac{e^{-R^\pi C(b)\theta} G(\theta)}{e^{-R^\pi C(b)\theta} G(\theta)} = 1$$

it follows that

$$G(\theta) \leq e^{R^\pi C(b)\theta} \int_0^\theta e^{-R^\pi C(b) z} dG(z) \leq e^{R^\pi C(b)\theta} E^\pi \left\{ e^{-R^\pi C(b)(Z \wedge \theta)} \right\} \tag{21}$$

In view of the main result of this section, Theorem 18 below, we first prove

Lemma 15 *Given an initial time $t \in [0,T]$ and an initial event $k \in \{0,1\}$, we have*

$$\psi_1^\pi(t,x,k) \leq e^{-R_t^\pi x}, \quad \forall x > 0,\ \pi \in \mathcal{A} \tag{22}$$

where R_t^π is as in Definition 13 (see (19)).

Proof: Notice first that, whenever an event in $A^-_{x,\pi}$ occurs, then $P\{N_T - N_t > 0\} = P\{N_T - N_t = 1\}$. Using (21), where we put $\theta = \tau^\pi(y,w,h,x) \wedge (T-t)$, and the definition of $\tau^\pi(\cdot)$ in (8), from (13) we have((b,δ) corresponds to π in the first period)

$$\begin{aligned}
&\psi^\pi_1(t,x,k) \\
&= P\{N_T - N_t = 1\} \textstyle\sum_{h=0}^1 p_{k,h} \int_{\underline{w}}^{\bar{w}} \int_0^\infty G(\tau^\pi(y,w,h,x) \wedge (T-t)) dF(y) dH(w) \\
&\leq P\{N_T - N_t = 1\} \textstyle\sum_{h=0}^1 p_{k,h} \\
&\quad \cdot \int_0^\theta \int_{\underline{w}}^{\bar{w}} \int_0^\infty e^{-R^\pi_t C(b) z + \frac{R^\pi_t C(b)}{C(b)}[(1-h)by - h\delta(e^w-1) - x]} dG(z) dF(y) dH(w) \\
&\leq P\{N_T - N_t = 1\} \textstyle\sum_{h=0}^1 p_{k,h} \\
&\quad \int_0^{T-t} \int_{\underline{w}}^{\bar{w}} \int_0^\infty e^{-R^\pi_t [C(b)z - (1-h)by + h\delta(e^w-1) + x]} dG(z) dF(y) dH(w) \\
&= P\{N_T - N_t = 1\} \textstyle\sum_{h=0}^1 p_{k,h} \\
&\quad \int_0^\infty \int_{\underline{w}}^{\bar{w}} \int_0^\infty e^{-R^\pi_t [C(b)z - (1-h)by + h\delta(e^w-1) + x] \mathbf{1}_{\{z \leq T-t\}}} dG(z) dF(y) dH(w) \\
&= P\{N_T - N_t = 1\} \\
&\quad E^\pi_{t,k} \left\{ e^{-R^\pi_t C(b)(Z_1 \wedge (T-t))} e^{R^\pi_t \left[(1-K_{T_{N_t+1}})bY_1 - K_{T_{N_t+1}} \delta\left(e^W - 1\right) - x\right] \mathbf{1}_{\{Z \leq T-t\}}} \right\} \\
&= P\{N_T - N_t = 1\} e^{-R^\pi_t x} E^\pi_{t,k} \left\{ e^{-R^\pi_t V^\pi_t} \right\} \\
&\leq P\{N_T - N_t = 1\} e^{-R^\pi_t x}
\end{aligned} \tag{23}$$

where in the last relation we have used the fact that $l^\pi_{R^\pi}(t,k) \leq 0$, being (see (19)) $R^\pi_t := \min[R^\pi(t,0),\, R^\pi(t,1)]$ and, for $k \in \{0,1\}$, it holds that $l^\pi_{R^\pi(t,k)}(t,k) = 0$. ∎

Lemma 16 *For given (t,k) we have*

$$\psi^\pi_n(t,x,k) \leq \gamma_n e^{-R^\pi_t x}\,, \quad \forall n \in \mathbb{N},\ x > 0,\ \pi \in \mathcal{A} \tag{24}$$

where R^π_t is defined in (19) with $R^\pi(t,k)$ as in (17) and γ_n is defined recursively by

$$\begin{cases} \gamma_1 = 1 \\ \gamma_n = \gamma_{n-1} P\{N_T - N_t > 1\} + P\{N_T - N_t = 1\} \end{cases} \tag{25}$$

Remark 17 *Due to the defining relations (25) it follows immediately that* $\gamma_n \leq 1$ *for all* $n \in \mathbb{N}$*. In fact, using forward induction, we see that the inequality is true for* $n = 1$ *and, assuming it true for* $n-1$*, we have*

$$\gamma_n = \gamma_{n-1} P\{N_T - N_t > 1\} + P\{N_T - N_t = 1\} \leq P\{N_T - N_t > 0\} \leq 1 \quad (26)$$

Proof: Before proving (24) let us recall that, for given t and a given $k = K_{T_{N_t}} \in \{0, 1\}$,

$$E_{t,k}\left\{e^{-r\,V_t^\pi}\right\} = l_r^\pi(t,k) + 1 \quad (27)$$

On the other hand (see the end of the proof of Lemma 15) $l^\pi_{R_t^\pi}(t,k) \leq 0$ so that

$$E_{t,k}\left\{e^{-R_t^\pi\,V_t^\pi}\right\} \leq 1. \quad (28)$$

The proof of (24) now proceeds by induction. By Lemma 15 the statement is true for $n = 1$. Assume it holds true for $n-1$. From Proposition 8, whereby for the first term on the right hand side of (12) one uses Lemma 15 (see also its proof), noticing that on the event $z \leq \tau^\pi(y, w, k, x)$ (immediate ruin) one has $P\{N_T - N_t > 0\} = P\{N_T - N_t = 1\}$, we obtain the following sequence of inequalities where we use the monotonicity of $\psi_n^\pi(t, x, k)$ in t as pointed out in Remark 7,

$$\begin{aligned}
\psi_n^\pi(t,x,k) &= P\{N_T - N_t = 1\}e^{-R_t^\pi x} + P\{N_T - N_t > 1\}\textstyle\sum_{h=0}^{1} p_{k,h}\cdot \\
&\quad \cdot \textstyle\int_{\underline{w}}^{\bar{w}} \int_0^\infty \int_{\tau^\pi(y,w,h,x)}^{T-t} \psi_{n-1}^\pi(t+z, x-u^\pi(y,z,w,h), h)dG(z)dF(y)dH(w)
\end{aligned}$$

$$\begin{aligned}
&\le P\{N_T - N_t = 1\}e^{-R_t^\pi x} + P\{N_T - N_t > 1\}\textstyle\sum_{h=0}^{1} p_{k,h}\cdot \\
&\quad \cdot \textstyle\int_{\underline{w}}^{\bar{w}} \int_0^\infty \int_{\tau^\pi(y,w,h,x)}^{T-t} \psi_{n-1}^\pi(t, x-u^\pi(y,z,w,h), h)dG(z)dF(y)dH(w)
\end{aligned}$$

$$\begin{aligned}
&\le P\{N_T - N_t = 1\}e^{-R_t^\pi x} + P\{N_T - N_t > 1\} \\
&\quad \gamma_{n-1}\textstyle\sum_{h=0}^{1} p_{k,h} \int_{\underline{w}}^{\bar{w}} \int_0^\infty \int_0^{T-t} e^{-R_t^\pi[x-u^\pi(y,z,w,h)]}dG(z)dF(y)dH(w)
\end{aligned}$$

$$\begin{aligned}
&\le P\{N_T - N_t = 1\}e^{-R_t^\pi x} + P\{N_T - N_t > 1\} \\
&\quad \gamma_{n-1}e^{-R_t^\pi x}E_{t,k}\left\{e^{-R_t^\pi\left[C(b)Z_1-(1-K_{T_{N_t+1}})bY_1+K_{T_{N_t+1}}\delta\left(e^{W_1}-1\right)\right]\mathbf{1}_{\{Z_1\le T-t\}}}\right\}
\end{aligned}$$

$$\begin{aligned}
&\le P\{N_T - N_t = 1\}e^{-R_t^\pi x} + P\{N_T - N_t > 1\} \\
&\quad \gamma_{n-1}e^{-R_t^\pi x}E_{t,k}\left\{e^{-R_t^\pi C(b)(Z_1\wedge(T-t))}e^{R_t^\pi\left[(1-K_{T_{N_t+1}})bY_1+K_{T_{N_t+1}}\delta\left(e^{W_1}-1\right)\right]\mathbf{1}_{\{Z_1\le T-t\}}}\right\}
\end{aligned}$$

$$= P\{N_T - N_t = 1\}e^{-R_t^\pi x} + P\{N_T - N_t > 1\}\gamma_{n-1}e^{-R_t^\pi x}E_{t,k}\left\{e^{-R_t^\pi V_t^\pi}\right\}$$

$$\le (P\{N_T - N_t = 1\} + \gamma_{n-1}P\{N_T - N_t > 1\})e^{-R_t^\pi x} \le \gamma_n e^{-R_t^\pi x}$$

where in the next-to-last two expression we have used (28). ∎

We come now to our main result in this section, namely Theorem 18 whose proof follows immediately from Lemma 16 noticing that, see Remark 17, one has $\gamma_n \le 1$.

Theorem 18 *Given an initial surplus $x > 0$ at a given time $t \in [0,T]$, we have, for all $n \in \mathbb{N}$ and any initial event $k \in \{0,1\}$ and for all $\pi \in \mathcal{A}$*

$$\psi_n^\pi(t,x,k) \le e^{-R_t^\pi x}$$

5 Optimizing the bounds

As mentioned previously, it is in general a difficult task to obtain an explicit solution to the reinsurance-investment problem in order to minimize the ruin probability even for the classical risk process. We shall thus choose the reinsurance level and the investment in order to minimize the bounds that we have derived. By Theorem 18 this amounts to choosing a strategy $\pi \in \mathcal{A}$ such that for each $t \in [0,T]$ the value of R_t^π is as large as possible. In order to achieve this goal notice that, by Proposition 12 the function $l_r^\pi(t,k)$ is, as a function of $r \in (0,\bar{r})$ (for every fixed $(t,k) \in [0,T] \times \{0,1\}$ and $\pi \in \mathcal{A}$), convex with a zero in $r=0$ and (see Remark 14) a unique positive one in $R^\pi(t,k) \in [0,\bar{r}-\eta]$. To obtain, for a given $t \in [0,T]$ the largest value of $R_t^\pi = \min[R^\pi(t,0), R^\pi(t,1)]$ it thus suffices to choose $\pi \in \mathcal{A}$ that minimizes $l_r^\pi(t,k)$ at $r = R_t^\pi$. This appeals also to intuition in the sense that, by its definition in (15), minimizing $l_r^\pi(t,k)$ amounts to penalizing negative values of V_t^π and thus also of X_t^π thereby minimizing the possibility of ruin.

5.1 Policy improvement

Concerning the minimization of $l_r^\pi(t,k)$ at $r = R_t^\pi$ notice that decisions concerning the control actions $\phi = (b,\delta)$ have to be made only at the event times T_n. The minimization of $l_r^\pi(t,k)$ with respect to $\pi \in \mathcal{A}$ has thus to be performed only for pairs (t,k) of the form (T_n, K_{T_n}) thus leading to a policy π with individual control actions $\phi_{T_n} = (b_{T_n}, \delta_{T_n})$.

Our problem to determine an investment and insurance policy to minimize the bounds on the ruin probability may thus be solved by solving the following subproblems:

i) for a given policy $\bar{\pi} \in \mathcal{A}$ determine $l_r^{\bar{\pi}}(t,k)$ for pairs (t,k) of the form (T_n, K_{T_n});

ii) determine $R^{\bar{\pi}}(T_n, K_{T_n})$ that is a solution with respect to r of $l_r^{\bar{\pi}}(T_n, K_{T_n}) = 0$ and put $R_t^{\bar{\pi}} = \min[R^{\bar{\pi}}(t,0), R^{\bar{\pi}}(t,1)]$;

iii) improve the policy $\bar{\pi}$ by minimizing $l_{R^{\bar{\pi}}}^{\pi}(T_n, K_{T_n})$ with respect to $\pi \in \mathcal{A}$.

This leads to a policy improvement-type approach, more precisely, one may proceed as follows:

a) start from a given policy π^0 (e.g. the one requiring minimal reinsurance and no investment in the financial market);

b) determine R^{π^0} corresponding to π^0 for the various (T_n, K_{T_n});

c) determine $\pi^1 \in \mathcal{A}$ that minimizes $l^{\pi}_{R^{\pi^0}}(T_n, K_{T_n})$ with respect to $\pi \in \mathcal{A}$ for the various (T_n, K_{T_n}); repeat the procedure until a stopping criterion is met (notice that by the above procedure $R^{\pi^n} > R^{\pi^{n-1}}$).

A practical way to implement steps b) and c) above is to discretize the time interval $[0, T]$ and then register an event only at the end of the interval in which it occurred (multiple events in a same subinterval may be recorded at the end as a single event for each of the two categories: claim, price change). The function $l^{\pi}_r(t, k)$ has then to be determined only for t corresponding to an end point of the various subintervals for each of the two possible values of k. Analogously for R^{π} as well as for the policy.

One crucial step in this procedure is to determine the function $l^{\pi}_r(t, k)$ corresponding to a given $\pi \in \mathcal{A}$ and this will be discussed in the next subsection.

5.2 Computing the value function for a given policy (value iteration)

Recall that Z_n are i.i.d. with probability distribution function $G(\cdot)$ and that (see Remark 9), for given $\pi \in \mathcal{A}$ and $r \in [\eta, \bar{r} - \eta]$, the functions $l^{\pi}_r(t, k)$ are bounded by some $L > 0$. We start now with the following

Definition 19 *For given $\pi \in A$ define T^{π} as the operator acting on bounded functions $v(t, k)$ of (t, k), in the following way*

$$\begin{aligned} T^{\pi}(v)(t,k) &= \mathbf{1}_{\{t \le T\}} E^{\pi}_{t,k} \big\{ \mathbf{1}_{\{t+Z_1 \le T\}} v\left(t + Z_1, K_{t+Z_1}\right) \\ &\qquad + \mathbf{1}_{\{t \le T \le t+Z_1\}} \left[e^{-rC(b)(T-t)} - 1 \right] \big\} \qquad (29) \\ &= \textstyle\sum_{h=0}^{1} p_{k,h} \int_0^{T-t} v(t+z, h) dG(z) + \bar{G}(T-t) \left[e^{-rC(b)(T-t)} - 1 \right] \end{aligned}$$

with $\bar{G}(z) = 1 - G(z)$ and where, given $\pi_t = (b_t, \delta_t)$, the value of b is $b = b_t$.

Remark 20 *When $t = T_n$ so that the next jump time of N_t is T_{n+1}, (29)* *can more specifically be rewritten as*

$$\begin{aligned} T^{\pi}(v)(T_n,k) &= \mathbf{1}_{\{T_n\le T\}}E^{\pi}_{T_n,k}\big\{\mathbf{1}_{\{T_n+Z_1\le T\}}v\left(T_n+Z_1,K_{T_n+Z_1}\right) \\ &\qquad +\mathbf{1}_{\{T_n\le T\le T_n+Z_1\}}\left[e^{-rC(b)(T-T_n)}-1\right]\big\} \\ &= \textstyle\sum_{h=0}^{1}p_{k,h}\int_0^{T-T_n}v(T_n+z,h)dG(z)+\bar{G}(T-T_n)\left[e^{-rC(b)(T-T_n)}-1\right] \end{aligned} \tag{30}$$

Remark 21 *Notice that the dependence of* T^{π} *on* π *is only through its values* $\phi=(b,\delta)$ *at* $t=T_{N_t}$*. We may thus equivalently write*

$$T^{\pi}(v)(\cdot)=T^{\phi}(v)(\cdot) \tag{31}$$

For the function $l^{\pi}_r(.)$ defined in (15) we now obtain

Proposition 22 *For a given* $\pi\in\mathcal{A}$ *and any value of the parameter* $r\in(\eta,\bar{r}-\eta)$*, the function* $l^{\pi}_r(\cdot)$ *is a fixed point of* T^{π}*, i.e.*

$$l^{\pi}_r(t,k)=T^{\pi}(l^{\pi}_r)(t,k) \tag{32}$$

Proof: For the proof notice first that, from (5) and its definition in (14), the random variable V^{π}_t represents the increment of the risk process X_t between two successive event times, provided these event times occur within the interval $[t,T]$. By the independence of (Z_n,Y_n,W_n), the random variable V^{π}_t is, for given $\pi\in\mathcal{A}$ and conditionally on K_t, independent of the past history of (Z_n,Y_n,W_n) and depends only on t and the value k of K_t at time t. Consequently, by a double conditioning argument, we may write

$$E^{\pi}_{t,k}\left\{\mathbf{1}_{\{t+Z_1\le T\}}\left[e^{-rV^{\pi}_t}-1\right]\right\}=E^{\pi}_{t,k}\left\{\mathbf{1}_{\{t+Z_1\le T\}}E^{\pi}_{t+Z_1,K_{t+Z_1}}\left[e^{-rV^{\pi}_{t+Z_1}}-1\right]\right\} \tag{33}$$

where, conditionally on Z_1, (see the definition in (14))

$$\begin{aligned} V^{\pi}_{t+Z_1} &= C(b)\left(Z_2\wedge(T-t-Z_1)\right) \\ &\quad -\mathbf{1}_{\{Z_2\le T-t-Z_1\}}\left[\left(1-K_{T_{N_t}+Z_1+Z_2}\right)bY_2+K_{T_{N_t}+Z_1+Z_2}\delta\left(E^{W_2}-1\right)\right] \end{aligned}$$

From the defining relation (15), including the obvious factor $\mathbf{1}_{\{t \leq T\}}$, as well as from (29) in Definition 19 we then have

$$
\begin{aligned}
l_r^\pi(t,k) &= \mathbf{1}_{\{t\leq T\}} E_{t,k}^\pi \big\{ \mathbf{1}_{\{t+Z_1 \leq T\}} \left[e^{-rV_t^\pi} - 1 \right] + \mathbf{1}_{\{t \leq T \leq t+Z_1\}} \left[e^{-rC(b)(T-t)} - 1 \right] \big\} \\
&= \mathbf{1}_{\{t\leq T\}} E_{t,k}^\pi \big\{ \mathbf{1}_{\{t+Z_1 \leq T\}} E_{t+Z_1, K_{t+Z_1}} \left\{ e^{-rV_{t+Z_1}^\pi} - 1 \right\} \\
&\qquad + \mathbf{1}_{\{t \leq T \leq t+Z_1\}} \left[e^{-rC(b)(T-t)} - 1 \right] \big\} \\
&= \mathbf{1}_{\{t\leq T\}} E_{t,k}^\pi \big\{ \mathbf{1}_{\{t+Z_1 \leq T\}} l_r^\pi \left(t+Z_1, K_{t+Z_1} \right) \\
&\qquad + \mathbf{1}_{\{t \leq T \leq t+Z_1\}} \left[e^{-rC(b)(T-t)} - 1 \right] \big\} \\
&= T^\pi \left(l_r^\pi \right) (t,k)
\end{aligned}
\tag{34}
$$

■

In view of the recursive relations (35) below, we make the following

Remark 23 *Assuming $l_r^\pi(t,k)$ to be given for $t \geq T_{n+1}$, one can then obtain $l_r^\pi(T_n,k)$ by applying T^π to the given $l_r^\pi(\cdot)$ and evaluating it at (T_n,k).*

Based on this Remark 23 we may now consider the following recursive relations

$$
\left\{
\begin{array}{lcl}
l_r^{\pi,0}(T_n,k) &=& \bar{G}(T-T_n) \left[e^{-rC(b_n)(T-T_n)} - 1 \right] \\
l_r^{\pi,m}(T_n,k) &=& T^\pi(l_r^{\pi,m-1})(T_n,k), \quad \text{for} \quad m = 1,2,\cdots
\end{array}
\right.
\tag{35}
$$

that we may view as *Value iteration* algorithm.

To obtain $l_r^\pi(\cdot)$, namely the fixed point of T^π, the recursions in (35) may have to be iterated infinitely many times. If however the mappings T^π are contracting in the sense that

$$
\| T^\pi(v_1) - T^\pi(v_2) \} \leq \gamma \| v_1 - v_2 \| \tag{36}
$$

for bounded functions $v_1(\cdot)$ and $v_2(\cdot)$ and with $\gamma < 1$, then $l_r^{\pi,m}(T_n,k)$ approximates $l_r^\pi(T_n,k)$ arbitrarily well in the sup-norm, provided m is sufficiently large.

For this purpose we now make the following assumption (in the next subsection 5.2.1 in Proposition 29 we shall present a sufficient condition for this assumption to hold).

Assumption 24 *Given $\pi \in \mathcal{A}$, the mapping T^{π} is a contraction mapping with contraction constant $\gamma < 1$, independent of π, i.e. for two functions $v_1(t,k)$ and $v_2(t,k)$ that are bounded by L (recall from Remark 9 that, for $\pi \in \mathcal{A}$ and $r \in [\eta, \bar{r} - \eta]$, the functions $l_r^{\pi}(\cdot)$ are bounded by L) one has*

$$\|T^{\phi}(v_1) - T^{\phi}(v_2)\| \leq \gamma \|v_1 - v_2\| \tag{37}$$

where $\|\cdot\|$ is the sup-norm in the space of bounded functions and where we use the equivalence $T^{\pi} = T^{\phi}$ given in (31).

Under this assumption we can now obtain the minimal number of iterations required in (35) in order that $l_r^{\pi,m}(T_n, k)$ comes uniformly close to $l_r^{\pi}(T_n, k)$ within a given ε. We have in fact

Corollary 25 *Let $l_r^{\pi,m}(\cdot)$ be the function obtained from iterating (35) m times. Then, with L the upper bound for the functions $l_r^{\pi}(\cdot)$, for*

$$m^{\varepsilon} > \frac{-\log\left(\frac{\varepsilon(1-\gamma)}{2L+1}\right)}{\log \gamma} \tag{38}$$

we have

$$\|l_r^{\pi} - l_r^{\pi,m}\| < \varepsilon \tag{39}$$

uniformly in $r \in [\eta, \bar{r} - \eta]$ (recall from Remark 14 that $\forall t \in [0,T]$ we have $R_t^{\pi} \in [\eta, \bar{r} - \eta]$).

Proof: Noticing that from the definition of T^{π} in (29) we have $\|T^{\pi} l_r(\cdot)\| \leq L + 1$, we obtain the following chain of inequalities

$$\begin{aligned}
&\|l_r^{\pi} - l_r^{\pi,m}\| = \|\textstyle\sum_{k=m}^{\infty}(l_r^{\pi,k+1} - l_r^{\pi,k})\| \\
&\leq \textstyle\sum_{k=m}^{\infty} \|(T^{\pi})^k (l_r^{\pi,1} - l_r^{\pi,0})\| \leq \sum_{k=m}^{\infty} \gamma^k \|T^{\pi} l_r^{\pi,0} - l_r^{\pi,0}\| \\
&\leq \textstyle\sum_{k=m}^{\infty} \gamma^k \left(\|T^{\pi} l_r^{\pi,0}\| + \|l_r^{\pi,0}\|\right) \leq \sum_{k=m}^{\infty} \gamma^k (2L+1) \\
&= (2L+1)\gamma^m \textstyle\sum_{k=1}^{\infty} \gamma^k = (2L+1)\,\frac{\gamma^m}{1-\gamma}
\end{aligned}$$

Imposing that the last expression is less than ε, we obtain the result.

5.2.1 A specific case with an analytic solution

We consider now the specific case described in Example 1 in which Z_n are i.i.d. having a negative exponential distribution with parameter $\lambda = \lambda^1 + \lambda^2$. This is in fact one of the cases most discussed in the literature under different settings. In this case we have that, independently of $k \in \{0, 1\}$,

$$p_{k,h} = \frac{\lambda^1}{\lambda} \text{ for } h = 0 \quad ; \quad p_{k,h} = \frac{\lambda^2}{\lambda} \text{ for } h = 1$$

By the independence of $p_{k,h}$ on k, from (29) in Definition 19 it follows that T^π maps bounded functions $v(t)$ of t only into bounded functions again of t only and this in turn implies that in the present specific case we may, instead of considering functions $v(t, k)$, consider more simply functions $v(t)$. In particular, the mapping T^π in (29) then becomes

$$\begin{aligned} T^\pi(v)(t) = \mathbf{1}_{\{t \le T\}} \left[\tfrac{\lambda^1}{\lambda} \int_0^\infty e^{-rby} dF(y) + \tfrac{\lambda^2}{\lambda} \int_{\underline{w}}^{\overline{w}} e^{-r\delta(e^w - 1)} dH(w) \right] \\ \cdot \left[\int_t^T e^{-rC(b)(\xi - t)} v(\xi) \lambda e^{-\lambda(\xi - t)} d\xi \right] + e^{-\lambda(T-t)} \left[e^{-rC(b)(T-t)} - 1 \right] \end{aligned} \tag{40}$$

Let us now make the following additional assumption on the distributions of Z_n, Y_n, W_n as well as on the range $[\underline{\delta}, \overline{\delta}]$ of the amounts invested in the risky asset that will be used to obtain the analytic solution below.

Assumption 26 *Let*

i) *Z_n are i.i.d. negative exponential with parameter $\lambda = \lambda^1 + \lambda^2$ (see description above).*

ii) *Y_n are i.i.d. negative exponential with parameter $\mu > 0$.*

iii) *$[\underline{\delta}, \overline{\delta}]$ and the p.d.f. $H(w)$ are such that (recall that $\underline{w} < 0 < \overline{w}$)*

$$\int_{\underline{w}}^0 e^{-r\overline{\delta}(e^w - 1)} dH(w) + \int_0^{\overline{w}} e^{-r\underline{\delta}(e^w - 1)} dH(w) < 1$$

Remark 27 *Depending on the distribution $H(w)$, the requirement iii) in the above Assumption 26 puts a rather strong restriction on the magnitude of $\underline{\delta}$ and $\overline{\delta}$ allowing thus for little flexibility to invest in the financial market. This is partly in line with general findings (for a recent account see e.g. Hult &*

Lindskog (2010)) pointing at the fact that since, after many or large negative returns on investments, a few claims of moderate size are sufficient to cause ruin, it is advisable to select a sufficiently conservative policy. The latter would in turn imply that the choice of the investment policy has little impact on the ruin probability unless, as we do here, one allows also to select a reinsurance policy.

Lemma 28 *Under the assumption 26 we have that*

$$\frac{\lambda^1}{\lambda}\int_0^\infty e^{-rb_n y}dF(y)+\frac{\lambda^2}{\lambda}\int_{\underline{w}}^{\overline{w}} e^{-r\delta_n(e^w-1)}dH(w)\le 1 \tag{41}$$

Proof: For the first term on the left in (41) notice that

$$\int_0^\infty e^{-rb_n y}\mu e^{-\mu y}dy=\frac{\mu}{rb_n+\mu}\int_0^\infty (rb_n+\mu)e^{-(rb_n+\mu)y}dy<1$$

For the second term on the left in (41) we have by Assumption 26

$$\begin{aligned}&\textstyle\int_{\underline{w}}^{\overline{w}} e^{-r\delta_n(e^w-1)}dH(w)\\ &\quad\le \textstyle\int_{\underline{w}}^{0} e^{-r\overline{\delta}(e^w-1)}dH(w)+\int_0^{\overline{w}} e^{-r\underline{\delta}(e^w-1)}dH(w)<1\end{aligned}$$

Putting the two inequalities together we obtain

$$\frac{\lambda^1}{\lambda}\int_0^\infty e^{-rb_n y}dF(y)+\frac{\lambda^2}{\lambda}\int_{\underline{w}}^{\overline{w}} e^{-r\delta_n(e^w-1)}dH(w)<\frac{\lambda^1}{\lambda}+\frac{\lambda^2}{\lambda}=1$$

∎

From Lemma 28 we now obtain the following proposition

Proposition 29 *For the model of Example 1 and with the assumptions of the previous Lemma 28 we have that for $\pi\in\mathcal{A}$ the operator $T^\pi(\cdot)$ is, on the space of bounded functions $v(t)$ of t, a contraction operator with contraction constant $(1-e^{-\lambda T})<1$, more precisely for two bounded functions $v_1(\cdot)$ and $v_2(\cdot)$ we have*

$$\|T^\pi(v_1)-T^\pi(v_2)\|\le(1-e^{-\lambda T})\|v_1-v_2\|$$

Proof:

$$
\begin{aligned}
&\|T^\pi(v_1) - T^\pi(v_2)\| \\
&\le \left[\tfrac{\lambda^1}{\lambda}\int_0^\infty e^{-rb_n y}dF(y) + \tfrac{\lambda^2}{\lambda}\int_{\underline{w}}^{\overline{w}} e^{-r\delta(e^w-1)}dH(w)\right] \\
&\qquad\qquad \int_t^T e^{-rC(b)(\xi-t)}\|v_1 - v_2\|\lambda e^{-\lambda(\xi-t)}d\xi \\
&\le \|v_1 - v_2\| \int_t^T \lambda e^{-\lambda(\xi-t)}d\xi = \|v_1 - v_2\|\left(1 - e^{-\lambda(T-t)}\right) \\
&\le (1 - e^{-\lambda T})\|v_1 - v_2\|
\end{aligned}
$$

Notice that the setting of this subsection leads thus to an example where Assumption 24 is satisfied.

In Proposition 22 we had seen that the function $l_r^\pi(t,k)$ is a fixed point of the operator T^π. In the next theorem we show that, in the setting of this subsection, the fixed point of T^π does not depend on k, so that we have $l_r^\pi(t,k) = l_r^\pi(t)$ independent of k. Furthermore, the next Theorem 30 gives an analytic expression for $l_r^\pi(t)$. We have in fact

Theorem 30 *In the setting of this subsection and under the assumptions of Lemma 28 we have that the fixed point of T^π for $\pi \in \mathcal{A}$ is*

$$
\begin{aligned}
&l_r^\pi(t) = e^{-\lambda(T-t)}\left[e^{-rC(b)(T-t)} - 1\right] \\
&+ \left[\lambda^1\int_0^\infty e^{-rby}dF(y) + \lambda^2\int_{\underline{w}}^{\overline{w}} e^{-r\delta(e^w-1)}dH(w)\right] \\
&\cdot \int_t^T \exp\left[rC(b) + \lambda + \lambda^1\int_0^\infty e^{-rby}dF(y) + \lambda^2\int_{\underline{w}}^{\overline{w}} e^{-r\delta(e^w-1)}dH(w)\right] \\
&\qquad\qquad \cdot e^{-\lambda(T-\xi)}\left[e^{-rC(b)(T-\xi)} - 1\right]d\xi
\end{aligned}
\tag{42}
$$

Proof: For the proof we first recall that a Volterra integral equation of the form

$$f(t) = g(t) + A\int_a^t e^{\lambda(t-\xi)}f(\xi)d\xi \tag{43}$$

has as solution

$$f(t) = g(t) + A\int_a^t e^{(\lambda+A)(t-\xi)}g(\xi)d\xi \tag{44}$$

Notice next that by the fixed point property of $l_t^\pi(t)$ (see Proposition 22)

$$l_r^\pi(t) = T^\pi(l_r^\pi)(t)$$

We thus have

$$\begin{aligned} l_r^\pi(t) = {} & e^{-\lambda(T-t)}\left[e^{-rC(b)(T-t)}-1\right] \\ & +\lambda\left[\frac{\lambda^1}{\lambda}\int_0^\infty e^{-rby}dF(y)+\frac{\lambda^2}{\lambda}\int_{\underline{w}}^{\overline{w}} e^{-r\delta(e^w-1)}dH(w)\right] \\ & \cdot\int_t^T e^{[rC(b)+\lambda](t-\xi)}l_r^\pi(\xi)d\xi \end{aligned} \tag{45}$$

which is of the form of equation (43). The statement then follows by just writing the solution (44) for $g(t), A$ and λ as resulting from (45). ∎

6 Conclusions

We have considered the problem of minimizing the ruin probability in an insurance model that allows to dynamically choose the level of reinsurance and investment in the financial market. It is a general innovative model that describes in a unifying way the timing of the events, that is the arrivals of claims and the changes of the prices in the market. It is based on a continuous-time Semi-Markov process model (SMP) and it is believed that this model represents reality more faithfully than, say, classical diffusion-based models. It leads also to some advantages when estimating the parameters in the model in the sense that it allows one to separate between the information coming from observing the frequency of the individual events and that of the duration between successive events. Our insurance model is also general enough to contain particular cases ranging from the classical risk process to models with reinsurance, investment in the financial market and dividends. It could possibly be generalized to include also the recent general risk model with reinsurance in Eisenberg & Schmidli (2010) where, to prevent a negative surplus, the insurer may inject additional capital.

We developed a specific methodology to obtain a policy that minimizes the exponential bound on the ruin probability in Theorem 18. For a specific case of the underlying SMP model, in subsection 5.2.1 we obtain an explicit analytic solution. The bound minimizing policy that is obtained may of course be of interest in itself, but the most important aspect is that it leads to an "optimal" bound on the ruin probability with respect to which other standard policies may then be evaluated. It also allows to obtain some qualitative insight into the impact that investment in the financial market may have on the ruin probability.

Contrary to many asymptotic (in time and in the initial surplus) approaches (see e.g. Gaier et al. (2003), Paulsen (1998), Hult & Lindskog (2010)) we obtain our results for a fixed, but arbitrary finite horizon and for any given positive initial surplus.

The policy iteration procedure to obtain the strategy that minimizes the bounds has been described in Section 5.1 where, in order to obtain $l_t^\pi(t,k)$ and thus R_t^π for a generic $t \in [0,T]$ and $\pi \in \mathcal{A}$, one may use the value iteration procedure described in Section 5.2. In the specific case of subsection 5.2.1 one can even obtain an analytic solution for $l_t^\pi(t,k)$ which, in that case, does not depend on k.

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