Random evolution driven by Hamiltonian flows and Lax–Oleinik semigroup.

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The most relevant theoretical outputs are about asymptotic problems, more precisely singular perturbation results or central limit theorems in probabilistic terminology.
Roughly speaking, one multiply the evolution coefficients by a small parameter $\varepsilon$, speed up the related random trajectories multiplying by $\frac{1}{\varepsilon^\alpha}$ for a suitable $\alpha$, and then pass to the limit as $\varepsilon \to 0$. The interesting aspect in these results is that the limit problem can be of different nature with respect to the evolution operators. Typical example is evolutions governed by elliptic operators giving at the limit an hyperbolic equation. Another example can be found in Evans’ *Entropy and PDE* where he considers linear first order evolutions with switchings driven by a Markov chain and obtain at the limit a diffusion. The basic setting in is when evolution is associated with linear uniformly elliptic operators and the switching are governed by a continuous time Markov chain.
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admits (viscosity) solution on the whole space. There exists an Aubry set and other facts of weak KAM theory can be generalized to the system setting.
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We have used as base probability space the space $\mathcal{D}$ of paths

$$\omega : [0, +\infty) \rightarrow \{1, \cdots, M\}$$

taking as values the indices of the system, from 1 to $M$, and with finite jumps in any bounded time interval (cadlag).
The space $D$ is endowed with the minimal $\sigma$–algebra $\mathcal{F}$ making the evaluation maps

$$\omega \mapsto \omega(t)$$

measurable. $\mathcal{F}$ is the Borel $\sigma$–algebra related to a metric on $D$, named after Prohorov, making $D$ a Polish space, namely complete and separable.
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A natural filtration $\mathcal{F}_t$ is also considered. Roughly speaking, the sets of $\mathcal{F}_t$ are measurable sets whose trajectories are selected via conditions on the interval $[0, t]$. 
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In order to define a random Lax–Oleinik semigroup, we have adapted, with Andrea and Maxime, the probabilistic frame to the time–dependent case.
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$$(S(t)u^0)_i(x) = \inf_{\gamma(0,\omega)=x} \mathbb{E}_i \left[ u^0_{\omega(t)}(\gamma(t)) + \int_0^t L_{\omega(s)}(\gamma(s), -\dot{\gamma}(s)) \, ds \right].$$
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As we will see a main issue will be to show continuity of the function given by the formula
Admissible random curves

We call admissible curve a random variable \( \gamma : \mathbb{D} \to \mathbb{C} \) such that it is uniformly (in \( \omega \in \mathbb{D} \)) locally (in \( t \)) absolutely continuous, a time synchronization condition holds, namely \( \gamma \) is nonanticipating, i.e. for any \( t \geq 0 \) \( \omega_1 \equiv \omega_2 \) in \( [0, t] \) \( \implies \gamma (\cdot, \omega_1) \equiv \gamma (\cdot, \omega_2) \) in \( [0, t] \).

(1) The definition implies two crucial properties the set \( \{ (t, \omega) \in \mathbb{R}_+ \times \Omega : \gamma (\cdot, \omega) \text{ is not differentiable at } t \} \) belongs to the product \( \sigma \)-algebra \( B(\mathbb{R}_+) \otimes \mathcal{F} \) and has vanishing \( L^1 \times P \) measure; \( t \mapsto E_i [u_{\omega}(t)](t, \gamma(t)) \) is locally absolutely continuous in \( [0, +\infty) \).
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\omega_1 \equiv \omega_2 \text{ in } [0, t] \Rightarrow \gamma(\cdot, \omega_1) \equiv \gamma(\cdot, \omega_2) \text{ in } [0, t]. \tag{1}
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belongs to the product $\sigma$–algebra $\mathcal{B}({\mathbb{R}_+}) \otimes \mathcal{F}$ and has vanishing $\mathcal{L}^1 \times \mathbb{P}$ measure;
- $t \mapsto \mathbb{E}_i[u_{\omega(t)}(t, \gamma(t))]$ is locally absolutely continuous in $[0, +\infty)$. 
this in turn allows proving that time derivative of a locally Lipschitz continuous function on an admissible curve and expectations $\mathbb{E}_i$ commute, up to a term which, roughly speaking, records the indices jumps on the underlying paths and contains the coupling matrix.
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**Theorem**

*Let $u : [0, +\infty) \times \mathbb{R}^N \to \mathbb{R}^M$, $\gamma$, $i$ be a locally Lipschitz–continuous function, an admissible curve, and an index in \{1, \ldots, M\}, respectively. Then

$$\frac{d}{dt} \mathbb{E}_i[u_{\omega(t)}(t, \gamma(t))] \bigg|_{t=s} = \mathbb{E}_i \left[-(u(s, \gamma(s)) \omega(s)) + \frac{d}{dt}u_{\omega(s)}(t, \gamma(t)) \bigg|_{t=s}\right]$$

for a.e. $s \in [0, +\infty)$*
We say that a function $u : [0, +\infty) \times \mathbb{R}^N \rightarrow \mathbb{R}^M$ has dominated evolution if

$$u_i(s_0, \gamma(0)) - E_i[u_\omega(s_0 - t_0)(t_0, \gamma(s_0 - t_0))] \leq E_i \int_{s_0 - t_0}^0 L_{\omega}(s) (\gamma(s), -\dot{\gamma}(s)) \, ds$$

for any $s_0 \geq t_0 \geq 0$, $i = 1, \ldots, M$, any admissible curve $\gamma$. Exploiting this notion it will be proved continuity of Lax–Oleinik formula plus and it will be put in relation with (HJS).
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**Theorem**

*Let $u^0$ be bounded uniformly continuous then $u(t, x) = (S(t)u^0)(x)$ is the unique continuous solution to (HJS).*
Existence of minimizers

Theorem

Assume the initial datum $u^0$ to be locally Lipschitz–continuous, then for any $x$ there are random admissible curves starting at $x$ realizing the minimum in Lax–Oleinik formula.
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Theorem

- the minimizing curve $\eta$ is $C^1$ except at the switching times;
- the solution $u$ is $C^1$ on $\eta$ except at the switching times;
- for any minimizing curve $\eta$ there exists an adjoint random curve $P(\omega, t)$ with

$$P(\omega, t) \in \partial_v L_{\omega(t)}(\eta(\omega, t), -\dot{\eta}(\omega, t))$$

almost surely. Here $L_i$ are the Lagrangians associated to $H_i$. 