Well-posedness for some LWR models on a junction

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Statement of the problem

We consider a junction where $m$ incoming and $n$ outgoing roads meet.

- Incoming roads: $x \in \Omega_i = \mathbb{R}_-, i = 1, \ldots, m$;
- Outgoing roads: $x \in \Omega_j = \mathbb{R}_+, j = m + 1, \ldots, m + n$;
- The junction is located at $x = 0$. 
Statement of the problem

On each road the evolution of traffic is described by

\[ \partial_t \rho_h + \partial_x f_h(\rho_h) = 0, \quad h = 1, \ldots, m + n, \]

- \( \rho_h \) density of vehicles, \([0, R] \)-valued for all \( h \)
- \( f_h \) bell-shaped, non linearly non degenerate, Lipschitz fluxes

\[ \forall h \quad f_h(0) = 0 = f_h(R) \]

Moreover, we postulate conservativity at the junction:

\[ \sum_{i=1}^{m} f_i(\rho_i(t, 0^-)) = \sum_{j=m+1}^{m+n} f_j(\rho_j(t, 0^+)). \]
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$$\sum_{i=1}^{m} f_i(\rho_i(t, 0^-)) = \sum_{j=m+1}^{m+n} f_j(\rho_j(t, 0^+)).$$
Fix $\tilde{\rho}_0 = (\rho_0^1, \ldots, \rho_0^{m+n})$ s.t. $\rho_0^h \in L^\infty(\Omega_h, [0, R])$, $\forall h \in \{1, \ldots, m + n\}$.
We call “solution” a $(m + n)$-uple $\tilde{\rho} = (\rho_1, \ldots, \rho_{m+n})$ s.t.

- $\forall h \rho_h$ is a Kruzhkov entropy solution in $\mathbb{R}^+ \times \{\Omega_h \setminus \partial\Omega_h\}$.
  Namely $\forall k \in [0, R]$ and $\forall \xi \in C^1_c(\mathbb{R}^+ \times \Omega_h)$, $\xi \geq 0$

$$\int_{\mathbb{R}^+} \int_{\Omega_h} |\rho_h - k| \xi_t + q_h(\rho_h, k) \xi_x \, dx \, dt \geq 0$$

(with $q_h(\rho_h, k) = \text{sign}(\rho_h - k)(f_h(\rho_h) - f_h(k))$ the Kruzhkov entropy flux)

**Idea:** $k$ is an obvious solution...
the above inequalities are “Kato inequalities” between $\rho_h$ and $k$

- conservation at the junction holds.

There is no hope to prove well-posedness for “solutions”.

**Analogy:**
junction coupling conditions (JCC) play the role of boundary conditions (BC). Imposing mere conservativity condition as JCC leaves the Cauchy problem underdetermined! [A. ESAIM Proc.’15].
“Solutions”

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Many different approaches to single out “suitable” solutions

For the Riemann problem at the junction (road-wise constant initial conditions):

- [Holden, Risebro SIMA’95] maximize a concave “entropy” function at the junction;
- [Coclite, Piccoli SIMA’02], [Coclite, Garavello, Piccoli SIMA’05] traffic distribution matrix + optimization;
- [Lebacque ’96], [Lebacque, Khoshyaran ’02] Supply-Demand model;
- ...

We prove well-posedness for solutions to the Cauchy problem which are limits of vanishing viscosity (VV) approximations.

Essential ingredient: intrinsic characterization of VV limits (a notion of solution, expressed e.g. via some “entropy inequalities”)

VV limits obey rather artificial JCC... but the study is instructive!
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Fix $\varepsilon > 0$. Consider convection-$\varepsilon$-diffusion + JunctionCouplingCondition:

\[
\begin{aligned}
& \rho_{h,t}^\varepsilon + f_h(\rho_h^\varepsilon)_x = \varepsilon \rho_{h,xx}^\varepsilon, \\
& \sum_{i=1}^{m} \left( f_i(\rho_i^\varepsilon(t,0)) - \varepsilon \rho_{i,x}^\varepsilon(t,0) \right) = \sum_{j=m+1}^{m+n} \left( f_j(\rho_j^\varepsilon(t,0)) - \varepsilon \rho_{j,x}^\varepsilon(t,0) \right), \\
& \rho_{h}^\varepsilon(t,0) = \rho_{h'}^\varepsilon(t,0), \\
& \rho_{h}^\varepsilon(0,x) = \rho_{h,\varepsilon}^0(x),
\end{aligned}
\]

where the approximated initial conditions $\tilde{\rho}_{0,\varepsilon}$ satisfy

\[
\begin{aligned}
\rho_{h,\varepsilon}^0 &\in W^{2,1}(\Omega_h) \cap C^\infty(\Omega_h), \quad [0, R]\text{-valued,} \\
\rho_{h,\varepsilon}^0 &\longrightarrow \rho_h^0, \quad \text{a.e. and in } L^p(\Omega_h), \quad 1 \leq p < \infty, \quad \text{as } \varepsilon \to 0, \\
\|\rho_{h,\varepsilon}^0\|_{L^1(\Omega_h)} &\leq \|\rho_h^0\|_{L^1(\Omega_h)}, \quad \|(\rho_{h,\varepsilon}^0)_x\|_{L^1(\Omega_h)} \leq TV(\rho_h^0), \\
\varepsilon \|(\rho_{h,\varepsilon}^0)_{xx}\|_{L^1(\Omega_h)} &\leq C_0,
\end{aligned}
\]

with $C_0 > 0$ independent from $\varepsilon, h$. 
Coclite and Garavello, 2010

Theory of semigroups \( \Rightarrow \forall \varepsilon > 0 \) there exists a unique \( \bar{\rho}^\varepsilon \) s.t.

\[
\rho^\varepsilon_h \in C([0, \infty); L^2(\Omega_h)) \cap L^1_{\text{loc}}((0, \infty); W^{2,1}(\Omega_h)), \quad h \in \{1, \ldots, m + n\},
\]

\[
0 \leq \rho^\varepsilon_h \leq R, \quad \sum_{h=1}^{m+n} \|\rho^\varepsilon_h(t, \cdot)\|_{L^1(\Omega_h)} \leq \sum_{h=1}^{m+n} \|\rho^0_h\|_{L^1(\Omega_h)}, \quad \forall t \geq 0,
\]

+ additional a priori estimates like \( \|\sqrt{\varepsilon \rho^\varepsilon_x}\|_{L^2} \leq C \).

Compensated compactness \( \Rightarrow \) existence of a sequence \( \{\varepsilon_\ell\}_{\ell \in \mathbb{N}} \), \( \varepsilon_\ell \to 0 \) and a solution \( \bar{\rho} \) of the inviscid Cauchy problem such that

\[
\forall h \quad \rho^{\varepsilon_\ell}_h \rightharpoonup \rho_h, \quad \text{a.e. and in } L^p_{\text{loc}}(\mathbb{R}_+ \times \Omega_h), \quad 1 \leq p < \infty. \tag{1}
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Uniqueness of VV solutions for the inviscid problem?
It is proved [Coclite, Garavello SIMA’10] in the case $m = n$, $f_h \equiv f$
based on comparison with the obvious solution $\vec{k} = (k, \ldots, k)$.

“Local objective”: extend these results to general junctions
“Global objective”: better understand solvers for different JCC
Monotonicity of the $VV$ solver

Not only the viscous problem of Coclite-Garavello is well posed.

The key property, independent of $\varepsilon$, can be expressed as:

(i) monotonicity (order-preservation) of the solver:

$\tilde{\rho}_0 \geq \hat{\rho}_0$ (componentwise) $\Rightarrow \forall t > 0 \, \tilde{\rho}^\varepsilon(t, \cdot) \geq \hat{\rho}^\varepsilon(t, \cdot)$;

(ii) $L^1$-contractivity of the solver:

$$
\sum_{h=1}^{m+n} \| \rho_h^\varepsilon(t, \cdot) - \hat{\rho}_h^\varepsilon(t, \cdot) \|_{L^1(\Omega_h)} \leq \sum_{h=1}^{m+n} \| \rho_{h,\varepsilon} - \hat{\rho}_{h,\varepsilon} \|_{L^1(\Omega_h)};
$$

(iii) Kato inequality: for all test function $\xi \in \mathcal{D}((0, +\infty) \times \mathbb{R})$, $\xi \geq 0$

$$
- \int_{\mathbb{R}^+} \int_{\Omega_h} (|\rho_h^\varepsilon - \hat{\rho}_h^\varepsilon| \xi_t + q_h(\rho_h^\varepsilon, \hat{\rho}_h^\varepsilon) \xi_x + \varepsilon |\rho_h^\varepsilon - \hat{\rho}_h^\varepsilon| x \xi_x) \leq 0,
$$

Links: (iii) $\Rightarrow$ (ii) (with $\xi \sim 1_{[0,t]}$); (ii) $\Leftrightarrow$ (i) (Crandall-Tartar lemma)

These properties are inherited by VV admissible solutions.

Kato inequality guarantees uniqueness of VV solutions

provides their intrinsic characterization
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\[ \tilde{\rho}_0 \geq \hat{\rho}_0 \, \text{(componentwise)} \implies \forall t > 0 \, \tilde{\rho}\varepsilon(t, \cdot) \geq \hat{\rho}\varepsilon(t, \cdot); \]

(ii) $L^1$-contractivity of the solver:
\[ \sum_{h=1}^{m+n} \| \rho\varepsilon_h(t, \cdot) - \hat{\rho}\varepsilon_h(t, \cdot) \|_{L^1(\Omega_h)} \leq \sum_{h=1}^{m+n} \| \rho^0_h,\varepsilon - \hat{\rho}^0_h,\varepsilon \|_{L^1(\Omega_h)}; \]

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These properties are inherited by VV admissible solutions.

Kato inequality guarantees uniqueness of VV solutions and provides their intrinsic characterization.
At least heuristically, \( \text{JCC} \Leftrightarrow \text{Riemann solver at junction.} \) Different solution semigroups for LWR on networks originate from different Riemann solvers at junction (Garavello, Piccoli,…) 

Byproduct of our analysis:

A subclass of these semigroups shares key features of VV solutions

Required property: monotonicity of the junction Riemann solver (larger data on a road lead to larger solutions on the whole network)

General principle:

Monotone, Lipschitz Riemann solver at junction \( \Rightarrow \) an intrinsic notion of solution + well-posedness.

Notion of solution and uniqueness:
Mimic tools developed for discontinuous-flux scalar conservation laws [A., Karlsen, Risebro ARMA’11]: admissibility germ, adapted entropies

Construction of solutions:
Approximations, e.g. by the Godunov finite volume scheme
Monotonicity of the Riemann solver at junction

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Godunov’s numerical flux

Consider the Riemann problem for pure SCL:

\[
\begin{aligned}
  u_t + f(u)_x &= 0, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R} \\
  u_0(x) &= \begin{cases}
    a, & \text{if } x < 0, \\
    b, & \text{if } x > 0.
  \end{cases}
\end{aligned}
\]

Denote by \( \mathcal{R}[a, b] \) its Kruzhkov entropy solution. The Godunov flux is the function \( (a, b) \mapsto f(\mathcal{R}[a, b])|_{x=0} \pm \). Analytically

\[
G(a, b) = \begin{cases}
  \min_{s \in [a, b]} f(s) & \text{if } a \leq b, \\
  \max_{s \in [b, a]} f(s) & \text{if } a \geq b.
\end{cases}
\]

Key properties of the Godunov flux:

- **Consistency**: for all \( a \in [0, R] \), \( G(a, a) = f(a) \);
- **Monotonicity and Lipschitz continuity**: \( \exists L > 0 : \forall (a, b) \in [0, R]^2 \) there holds

\[
0 \leq \partial_a G(a, b) \leq L, \quad -L \leq \partial_b G(a, b) \leq 0.
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&u_t + f(u)_x = 0, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R} \\
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Denote by \( R[a, b] \) its Kruzhkov entropy solution.
The Godunov flux is the function \((a, b) \mapsto f(R[a, b])|_{x=0^\pm}\).

Analytically

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\[0 \leq \partial_a G(a, b) \leq L, \quad -L \leq \partial_b G(a, b) \leq 0.\]
Bardos-LeRoux-Nédélec boundary conditions

Consider the initial and boundary value problem

\[
\begin{cases}
  u_t + f(u)_x = 0, & \text{for } t > 0, \ x < 0 \\
  u(t, 0^-) \approx u_b(t), \\
  u(0, x) = u_0(x),
\end{cases}
\]

\(u\) is an entropy solution for the IBVP if

- \(u\) is a Kruzhkov entropy solution in the interior of \(\mathbb{R}_+ \times \mathbb{R}_-\),
- \(u\) satisfies the boundary condition in the (BLN) sense

\[q(u(t, 0^-), k) \equiv \text{sign}(u(t, 0^-) - k) \left( f(u(t, 0^-)) - f(k) \right) \geq 0\] (\(\ast\))

A reformulation of BLN:

\(u\) satisfies BLN (\(\ast\)) \iff \(f(u(t, 0^-)) = G(u(t, 0^-), u_b(t)).\)

(known since [Dubois, LeFloch JDE’88])
The junction as a family of IBVPs

Fix $\vec{\rho}_0 = (\rho_0^1, \ldots, \rho_0^{m+n})$ s.t. $\rho_0^h \in L^\infty(\Omega_h, [0, R])$, $\forall h \in \{1, \ldots, m+n\}$. We look for $\vec{\rho} = (\rho_1, \ldots, \rho_{m+n})$ s.t.

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where $\vec{v} = (v_1, \ldots, v_{m+n}) : \mathbb{R}_+ \to [0, R]^{m+n}$ is to be chosen

- depending on the JCC one wants to express,
- in particular, ensuring conservation at the junction.

The case of VV limits: require that $v_h$ be the same on all roads

[A., Cancès JHDE’15], [A., Mitrović AnnIHP’15]: motivations in the “discontinuous-flux” case $m = n = 1$.

[A., Cancès CompGS’13, JHDE’15]: examples of different admissibility criteria (e.g. vanishing capillarity).
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Admissibility at the junction

Continuity of $\rho$ at the junction - up to boundary layers! - is desired.

Definition I (Formalized from heuristics of the model)

Given $\vec{\rho}_0$ i.c., we say that $\vec{\rho} = (\rho_1, \ldots, \rho_{m+n})$ is an admissible solution for the Cauchy problem on the network, if $\exists p$ in $L^\infty(\mathbb{R}_+, [0, R])$ s.t.

- each component $\rho_h$ is entropy solution for the IBVP

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\rho_h(0, x) &= \rho^h_0(x), & \text{on } \Omega_h,
\end{align*}
\]

(this includes the “density-at-junction” condition $\forall h \, v_h(t) = p(t)$)

- and “flux-at-junction” condition (conservativity) holds, i.e.

\[
\sum_{i=1}^{m} G(\rho_i(t, 0^-), p(t)) = \sum_{j=m+1}^{m+n} G(p(t), \rho_j(t, 0^+)), \quad \text{for a.e. } t.
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Drawback: with this definition \{ uniqueness is not obvious... existence is not obvious.\}
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\[
\begin{align*}
\rho_h, t + f_h(\rho_h) x &= 0, & \text{on } \mathbb{R}_+ \times \Omega_h, \\
\rho_h(t, 0) &\approx p(t), & \text{on } \mathbb{R}_+, \\
\rho_h(0, x) &= \rho^h_0(x), & \text{on } \Omega_h,
\end{align*}
\]

(this includes the “density-at-junction” condition $\forall h \nu_h(t) = p(t)$)

- and “flux-at-junction” condition (conservativity) holds, i.e.

\[
\sum_{i=1}^{m} G(\rho_i(t, 0^-), p(t)) = \sum_{j=m+1}^{m+n} G(p(t), \rho_j(t, 0^+)), \quad \text{for a.e. } t.
\]

**Drawback:** with this definition

- uniqueness is not obvious...
- existence is not obvious.
The vanishing viscosity germ

JCC ⇔ “Germ” ∼ \{ stationary road-wise constant admissible sol. \}

The germ underlying the above description of admissibility is

\[ G_{VV} = \left\{ \vec{u} = (u_1, \ldots, u_{m+n}) : \exists p \in [0, R] \text{ s.t. :} \right\} \]

\[ \sum_{i=1}^{m} G_i(u_i, p) = \sum_{j=m+1}^{m+n} G_j(p, u_j) \]

\[ G_i(u_i, p) = f_i(u_i), \quad G_j(p, u_j) = f_j(u_j) \forall i, j \]

Proposition (the Riemann solver \( R_{VV} \) at junction)

Given any \( \vec{u} = (u_1, \ldots, u_{m+n}) \in [0, R]^{m+n} \) the corresponding Riemann problem at the junction has an admissible solution \( \vec{\rho} = R_{VV}[\vec{u}] \).

The vector of traces \( \vec{\gamma}\rho = (\rho_1(0^-), \ldots, \rho_{m+n}(0^+)) \) belongs to \( G_{VV} \).

Idea of proof: given Riemann data \( \vec{u} \), construct \( p \)

Find \( p_{\vec{u}} \) s.t.

\[ \sum_{i=1}^{m} G_i(u_i, p_{\vec{u}}) = \sum_{j=m+1}^{m+n} G_j(p_{\vec{u}}, u_j) \]
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Introduction Monotone junction solvers BLN conditions via Godunov flux Admissibility conditions. Germ $G_{VV}$. Well-posedness

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Example

Consider a junction consisting of two incoming and one outgoing roads.

\[ f_1(x) = -x^2 + 1, \quad f_2(x) = -2x^2 + 2, \quad f_3(x) = -3x^2 + 3. \]

Given the initial condition \( (\rho^1_0 = -\sqrt{1/2}, \rho^2_0 = 1/4, \rho^3_0 = \sqrt{1/6}) \) one can trace \( p \mapsto G_i(\rho^i_0, p), \ i = 1, 2, \ p \mapsto G_3(p, \rho^3_0) \)

For all \( p \in [-\sqrt{1/6}, 0], \)

\[ \sum_{i=1}^{2} G_i(\rho^i_0, p) = G_3(p, \rho^3_0) = 2.5. \]

The fluxes \( G_{1,2}(\rho^i_0, p), \ G_3(p, \rho^3_0) \) are independent of \( p \in [-\sqrt{1/6}, 0] \).

\textbf{NB:} In practice, \( \rho_{\bar{u}} \) can be found by \textit{regula falsi} method.
Germ-based equivalent definitions of admissibility

A function $\vec{\rho} = (\rho_1, \ldots, \rho_{m+n})$ is an admissible solution if and only if

**Definition II (trace-based: used to prove uniqueness)**

- $\forall h \in \{1, \ldots, m + n\}$, $\rho_h$ is a Kruzhkov solution on the road $\Omega_h$;
- traces-in-germ condition holds:

  for a.e. $t \in \mathbb{R}_+$, $\vec{\gamma}_\rho(t) := (\rho_1(t, 0^-), \ldots, \rho_{m+n}(t, 0^+)) \in G_{VV}$.

cf. [Garavello, Natalini, Piccoli, Terracina ’07]
(admissibility in terms of Riemann solver at junction)

**Definition III (integral formulation: used to prove existence)**

- $\forall h \in \{1, \ldots, m + n\}$, $\rho_h$ is a Kruzhkov solution on the road $\Omega_h$;
- adapted entropy inequalities hold: $\forall \xi \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R})$, $\xi \geq 0$

$$\forall \vec{k} \in G_{VV} \sum_{h=1}^{m+n} \left( \int_{\mathbb{R}_+} \int_{\Omega_h} \{|\rho_h - k_h|\xi_t + q_h(\rho_h, k_h)\xi_x\} \, dx \, dt \right) \geq 0.$$  

cf. [Baiti, Jenssen’97],[Audusse, Perthame’05]. These are Kato ineq.!
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Crucial properties of $G_{VV}$

- The germ $G_{VV}$ is "complete": namely, the Riemann solver $R_{VV}$ is defined for all data.
- The germ $G_{VV}$ is "dissipative": namely, for all $\vec{k}, \vec{c} \in G_{VV}$
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  \sum_{i=1}^{m} q_i(k_i, c_i) - \sum_{j=m+1}^{m+n} q_j(k_j, c_j) \geq 0. \ (#) 
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**Lemma (Oleinik-like condition)**

$G_{VV}$ is characterized by a "graph above-graph below" condition (cf. [Diehl JHDE’09] in the $n = m = 1$ case) + conservativity condition.

The maximality can be refined: consider

$$G_{VV}^o = \{ \text{strict "graph above-graph below" condition} \}.$$  

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Well-posedness in the frame of admissible solutions

Theorem (Main result)

- **There exists an admissible solution for all** \( L^\infty \) **datum.**
- **Moreover, such solutions are** \( \mathcal{V}\mathcal{V} \) **limits.**
- If \( \vec{\rho} \) and \( \hat{\vec{\rho}} \) are admissible solutions corresponding to \( \vec{\rho}_0 \) and \( \hat{\vec{\rho}}_0 \),

\[
\sum_{h=1}^{m+n} \| \rho_h(t) - \hat{\rho}_h(t) \|_{L^1(\Omega_h)} \leq \sum_{h=1}^{m+n} \| \rho^0_h - \hat{\rho}^0_h \|_{L^1(\Omega_h)}.
\]

\( (L^1\text{-contractivity, monotonicity}) \). **Also Kato inequality holds.**

\[\implies \text{uniqueness of an admissible solution to Cauchy problem.}\]

Proof of uniqueness: Kruzhkov-per-road (doubling of variables) + existence of junction traces \( \gamma\vec{\rho}, \gamma\hat{\vec{\rho}} \implies \text{up-to-junction Kato inequality:} \)

\[
- \int_{\mathbb{R}^+} \int_{\Omega_h} \left( |\rho_h - \hat{\rho}_h| \xi_t + q_h(\rho_h, \hat{\rho}_h) \xi_x \right) \leq \text{RHS}[\gamma\vec{\rho}, \gamma\hat{\vec{\rho}}] \quad \forall \xi \geq 0
\]

By Def.II, \( \gamma\vec{\rho}, \gamma\hat{\vec{\rho}} \in \mathcal{G}_{\mathcal{V}\mathcal{V}} \). Then dissipativity \((\#)\) \[\implies \text{RHS}[\gamma\vec{\rho}, \gamma\hat{\vec{\rho}}] \leq 0.\]
Well-posedness in the frame of admissible solutions

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- **There exists an admissible solution for all** \( L^\infty \) **datum.**
- Moreover, such solutions are VV limits.
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  \( \implies \) **uniqueness of an admissible solution to Cauchy problem.**

**Proof of uniqueness**: Kruzhkov-per-road (doubling of variables) + existence of junction traces \( \gamma\tilde{\rho}, \gamma\hat{\rho} \implies \) up-to-junction Kato inequality:

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- There exists an admissible solution for all $L^\infty$ datum. Moreover, such solutions are VV limits.
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By Def.II, $\gamma\vec{\rho}, \gamma\hat{\vec{\rho}} \in \mathcal{G}_{VV}$. Then dissipativity ($\#)$ $\implies$ \text{RHS}[\gamma\vec{\rho}, \gamma\hat{\vec{\rho}}] \leq 0.$
Existence of admissible solutions

Proof of existence:

- Recall some of [Coclite, Garavello SIAM’10] results:
  - (subseq. of) VV approximations $\bar{\rho}^\varepsilon$ converges a.e. to a limit $\bar{\rho}$;
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- Standard theory: each component $\rho_h$ of $\bar{\rho}$, s.t. $\rho^\varepsilon_h \rightarrow \rho_h$, is a Kruzhkov entropy solution in $\Omega_h$.

- All $\bar{k} \in \mathcal{G}_{\bar{\varepsilon}}$ can be obtained as VV limits.
  - Tool: explicit construction of viscosity profiles $\bar{k}^\varepsilon$
    (based upon the Oleinik “graph above-graph below” condition).

- Pass to the limit $\varepsilon \rightarrow 0$ in Kato ineq. written for $\bar{\rho}^\varepsilon$ and $\bar{k}^\varepsilon$.
  - We get adapted entropy inequalities $\Rightarrow \rho$ fulfills Def. III.

Alternative proof: (for monotone junction Riemann solvers)

- use Godunov numerical scheme to construct solutions
  - Godunov scheme is well-balanced:
    $\bar{k}$ in the germ (stationary solutions) are exact discrete solutions
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- follow the same steps of passage to the limit.
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Grazie!
Thank you for your attention!