Optimization based control of networks of discretized PDEs: application to road traffic management

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ORESTE project goals

- Optimize traffic flow in corridors
  - ramp metering
  - re-routing strategies

- Modeling approach:
  - macroscopic traffic flow models
  - discrete adjoint method for gradient computation
SUMMARY

1. Macroscopic models for road traffic
2. Discretized system
3. Quick review of the adjoint method
4. Numerical results
5. Application to optimal re-routing
6. Conclusion and perspectives
Outline of the talk

1. Macroscopic models for road traffic

2. Discretized system

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6. Conclusion and perspectives
1. Macroscopic models for road traffic

**LWR model** [Lighthill-Whitham '55, Richards '56]

Non-linear transport equation: PDE for mass conservation

\[
\partial_t \rho + \partial_x f(\rho) = 0 \quad x \in \mathbb{R}, \ t > 0
\]

- \( \rho \in [0, \rho_{\text{max}}] \) mean traffic density
- \( f(\rho) = \rho v(\rho) \) flux function

Empirical flux-density relation: fundamental diagram

![Diagram showing Greenshield '35 and Newell-Daganzo fundamental diagrams](image)

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Extension to networks

- LWR on networks:
  [Holden-Risebro, 1995; Coclite-Garavello-Piccoli, 2005; Garavello-Piccoli, 2006]
  - LWR on each road
  - Optimization problem at the junction

- Modeling of junctions with a buffer:
  [Herty-Lebacque-Moutari, 2009; Garavello-Goatin, 2012; Bressan-Nguyen, 2015]
  - Junction described with one or more buffers
  - Suitable for optimization and Nash equilibrium problems
1. Macroscopic models for road traffic

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A model for ramp metering

- Two incoming links:
  - Upstream mainline $I_1 = ]-\infty, 0[$
  - Onramp $R_1$
- Two outgoing links:
  - Downstream mainline $I_2 = ]0, +\infty[$
  - Offramp $R_2$
A model for ramp metering

Coupled PDE-ODE model:

- Classical LWR on each mainline $l_1, l_2$

\[ \partial_t \rho + \partial_x f(\rho) = 0, \quad (t, x) \in \mathbb{R}^+ \times l_i, \]

- Dynamics of the onramp described by an ODE (buffer)

\[ \frac{dl(t)}{dt} = F_{in}(t) - \gamma_{r1}(t), \quad t \in \mathbb{R}^+, \]

- $l(t) \in [0, +\infty[$ length of the onramp queue
- $F_{in}(t)$ flux entering the onramp
- $\gamma_{r1}(t)$ flux leaving the onramp (through the junction)

- Coupled at junction with an LP-optimization problem
1. Macroscopic models for road traffic

**A model for ramp metering**

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A model for ramp metering

Coupled PDE-ODE model:

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$$\partial_t \rho + \partial_x f(\rho) = 0, \quad (t, x) \in \mathbb{R}^+ \times I_i,$$

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- **Coupled at junction** with an LP-optimization problem
1. Macroscopic models for road traffic

**Junction solver: Demand & Supply**

\[
\delta(\rho_1) = \begin{cases} 
 f(\rho_1) & \text{if } 0 \leq \rho_1 < \rho^{cr}, \\
 f^{\max} & \text{if } \rho^{cr} \leq \rho_1 \leq 1,
\end{cases}
\]

\[
d(F_{in}, l) = \begin{cases} 
 \theta \gamma_{r1}^{\max} & \text{if } l(t) > 0, \\
 \min\left(F_{in}(t), \theta \gamma_{r1}^{\max}\right) & \text{if } l(t) = 0,
\end{cases}
\]

\[
\sigma(\rho_2) = \begin{cases} 
 f^{\max} & \text{if } 0 \leq \rho_2 \leq \rho^{cr}, \\
 f(\rho_2) & \text{if } \rho^{cr} < \rho_2 \leq 1,
\end{cases}
\]

\(\theta \in [0, 1]\) control parameter
Junction solver: flux maximization

1. Mass conservation: \( f(\rho_1(t, 0^-)) + \gamma_{r1}(t) = f(\rho_2(t, 0^+)) + \gamma_{r2}(t) \)

2. \( f(\rho_2(t, 0^+)) \) maximum subject to 1. and

\[
f(\rho_2(t, 0^+)) = \min \left( (1 - \beta) \delta(\rho_1(t, 0^-)) + d(F_{\text{in}}(t), l(t)), \sigma(\rho_2(t, 0^+)) \right)
\]

3. **Right of way parameter** \( P \in ]0, 1[ \) to ensure uniqueness:

\[
f(\rho_2(t, 0^+)) = Pf_1(\rho(t, 0^-)) + (1 - P) \gamma_{r1}
\]

4. Offramp treated as a sink (infinite capacity)

5. No flux from the onramp to the offramp is allowed
Riemann solver: feasible set

To find a solution of the problem we solve an *LP*-optimization problem

1. Define the spaces of the incoming fluxes
2. Consider the demands
3. Trace the supply line
4. The feasible set is given by $\Omega$

Different situations can occur:

- Demand limited case
- Supply limited case
Riemann solver: feasible set

To find a solution of the problem we solve an $LP$-optimization problem

\[ \Gamma_{r1} = \Gamma_1 \left(1 - \beta \right) + \Gamma_{r1} = \Gamma_2 \]

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\[ \Gamma_{r1} \delta(\rho_1) \]

\[ d(F_{in}, \bar{l}) \Gamma_{r1} \]

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\[ \Gamma_1 \]

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Different situations can occur:
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Riemann solver: optimal point

Demand limited case

The optimal point $Q$ is the point of maximal demands

\[ \Gamma_1(1 - \beta) + \Gamma_{r1} = \Gamma_2 \]
Riemann solver: optimal point

Supply limited case

- We introduce the **right of way parameter**
- We set optimal point $Q$ to be the point of intersection
- Different situations can occur depending on the value of the intersection point
  - $Q \in \Omega$
  - $Q \notin \Omega$

![Diagram showing the Riemann solver concept with points $\Gamma_1$, $\Gamma_2$, $\delta(\rho_1)$, and $d(F_{in}, \bar{I})$]
1. Macroscopic models for road traffic

Riemann solver: optimal point
Supply limited case

▶ We introduce the **right of way parameter**
▶ We set optimal point \( Q \) to be the point of intersection
▶ Different situations can occur depending on the value of the intersection point

\[
\begin{align*}
Q & \in \Omega \\
Q & \notin \Omega
\end{align*}
\]

\[
\Gamma_1 = \frac{\rho}{1-P} \Gamma_{r1}
\]

\[
\delta(\rho_1)
\]

\[
d(F_{in}, \bar{l})
\]
Riemann solver: optimal point

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![Diagram showing the relationship between different parameters and the optimal point $Q$.]
Riemann solver: optimal point
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Riemann solver: optimal point
Supply limited case

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- We set optimal point $Q$ to be the point of intersection
- Different situations can occur depending on the value of the intersection point
  - $Q \in \Omega$
  - $Q \notin \Omega \rightarrow \text{Optimal point: } S$
Difference with [Coclite-Garavello-Piccoli, 2005] model

The traffic distribution across the junction is given by the distribution matrix $A$:

$$A = \begin{pmatrix} 1 - \beta & 1 \\ \beta & 0 \end{pmatrix}$$
Riemann solver: theorem

**Theorem** Delle Monache & al., SIAP, 2014

Fix $P \in ]0, 1[$. For every $\rho_{1,0}, \rho_{2,0} \in [0, 1]$ and $l_0 \in [0, +\infty[$, there exists a unique admissible solution $(\rho_1(t,x), \rho_2(t,x), l(t))$ satisfying the priority (possibly in an approximate way). Moreover, for a.e. $t > 0$, it holds

$$(\rho_1(t,0^-), \rho_2(t,0^+)) = R_{l(t)}(\rho_1(t,0^-), \rho_2(t,0^+))$$
Modified Godunov scheme

- At some $\Delta t^n$, we might have multiple shocks exiting the junction at $\bar{t}$
- We divide the time step $\Delta t^n = (t^n, t^{n+1})$ into two sub-intervals $\Delta t_a = (t^n, \bar{t})$ and $\Delta t_b = (\bar{t}, t^{n+1})$
- We solve in one time step two different Riemann Problems at the junction
  - For $\Delta t_a$: Classical Godunov flux update
    \[
    \begin{align*}
    \rho_{J}^{n+1} &= \rho_J^n - \frac{\Delta t_a}{\Delta x} \left( \hat{\Gamma}_1 - F(\rho_{J-1}^n, \rho_J^n) \right) \\
    \rho_0^{n+1} &= \rho_0^n - \frac{\Delta t_a}{\Delta x} \left( F(\rho_0^n, \rho_1^n) - \hat{\Gamma}_2 \right)
    \end{align*}
    \]
  - For $\Delta t_b$: Modified flux update
    \[
    \begin{align*}
    \rho_{J}^{n+1} &= \rho_J^{\bar{t}} - \frac{\Delta t_b}{\Delta x} \left( \hat{\Gamma}_1^{\bar{t}} - F(\rho_{J-1}^n, \rho_J^{\bar{t}}) \right) \\
    \rho_0^{n+1} &= \rho_0^{\bar{t}} - \frac{\Delta t_b}{\Delta x} \left( F(\rho_0^{\bar{t}}, \rho_1^n) - \hat{\Gamma}_2^{\bar{t}} \right)
    \end{align*}
    \]
Numerical simulations
Outline of the talk

1. Macroscopic models for road traffic

2. Discretized system

3. Quick review of the adjoint method

4. Numerical results

5. Application to optimal re-routing

6. Conclusion and perspectives
Ramp metering discretized system $H(\vec{\rho}, \vec{u}) = 0$

- Dynamics and junctions solutions based on the model described earlier
- Piecewise affine system
- Control parameter $u_i$ is a constraint on the ramp inflow $r_i(k)$
Ramp metering discretized system $H(\tilde{\rho}, \tilde{u}) = 0$

Lower triangular forward system $H(\tilde{\rho}, \tilde{u}) = 0$:

- $h_{i,k} = \rho_i(k) - \rho_i(k - 1) - \{\text{flux update eq. for cell } i \text{ and time step } k\}$
- $H$ system of concatenated $h_{i,k}$
- $H$ lower triangular
- Very efficient to solve $H^T x = b$
Lower triangular forward system $H(\vec{\rho}, \vec{u}) = 0$

Figure: $\frac{\partial H}{\partial \rho}$ matrix
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Optimization of a PDE-constrained system

Optimization problem

\[
\begin{align*}
\text{minimize}_{\vec{u} \in U} & \quad J(\vec{\rho}, \vec{u}) \\
\text{subject to} & \quad H(\vec{\rho}, \vec{u}) = 0
\end{align*}
\]

- $\vec{\rho} \in \mathcal{X} \subseteq \mathbb{R}^{NT}$: state variables
- $\vec{u} \in \mathcal{U} \subseteq \mathbb{R}^{MT}$: control variables

Gradient descent: How to compute the gradient $\frac{\partial J}{\partial \vec{\rho}} \frac{\partial \vec{\rho}}{\partial \vec{u}} + \frac{\partial J}{\partial \vec{u}}$?

On trajectories, $H(\vec{\rho}, \vec{u}) = 0$ constant, thus

\[
\frac{\partial H}{\partial \vec{\rho}} \frac{\partial \vec{\rho}}{\partial \vec{u}} + \frac{\partial H}{\partial \vec{u}} = 0 \quad \mathcal{O}(T^2N^2M)
\]
Discrete adjoint method

Consider the Lagrangian

\[ L(\tilde{\rho}, \tilde{u}, \lambda) = J(\tilde{\rho}, \tilde{u}) + \lambda^T H(\tilde{\rho}, \tilde{u}) \]

thus \( \nabla_{\tilde{u}} L = \nabla_{\tilde{u}} J \):

\[ \nabla_u L(\tilde{\rho}, \tilde{u}, \lambda) = \frac{\partial J}{\partial \tilde{u}} + \frac{\partial J}{\partial \tilde{\rho}} \frac{\partial \tilde{\rho}}{\partial \tilde{u}} + \lambda^T \left( \frac{\partial H}{\partial \tilde{u}} + \frac{\partial H}{\partial \tilde{\rho}} \frac{\partial \tilde{\rho}}{\partial \tilde{u}} \right) \]

\[ = \frac{\partial J}{\partial \tilde{u}} + \lambda^T \frac{\partial H}{\partial \tilde{u}} + \left( \frac{\partial J}{\partial \tilde{\rho}} + \lambda^T \frac{\partial H}{\partial \tilde{u}} \right) \frac{\partial \tilde{\rho}}{\partial \tilde{u}} \]

Compute \( \lambda \) s.t.

\[ \frac{\partial H^T}{\partial \tilde{\rho}} \lambda = -\frac{\partial J^T}{\partial \tilde{\rho}} \quad \mathcal{O}(T^2N^2) \]

Then

\[ \nabla_{\tilde{u}} J(\tilde{\rho}, \tilde{u}) = \nabla_{\tilde{u}} L(\tilde{\rho}, \tilde{u}, \lambda) = \frac{\partial J}{\partial \tilde{u}} + \lambda^T \frac{\partial H}{\partial \tilde{u}} \quad \mathcal{O}(TN + TM) \]
Exploiting system structure

The structure of the forward system influences the efficiency adjoint system solving

**Solving for \( \lambda \)**

\[
\frac{\partial H^T}{\partial \rho} \lambda = -\frac{\partial J^T}{\partial \rho}
\]

Since \( \frac{\partial H}{\partial \rho} \) is lower triangular, \( \frac{\partial H^T}{\partial \rho} \) is an upper triangular matrix

\( \implies \) The adjoint system can be solved efficiently using backward substitution

**Complexity reduction from \( O(T^2NM) \) to \( O(NT + MT) \)**
Exploiting system structure

The structure of the forward system influences the efficiency of solving the adjoint system.

Solving for $\lambda$

$$\frac{\partial H^T}{\partial \rho} \lambda = -\frac{\partial J^T}{\partial \rho}$$

Since $\frac{\partial H}{\partial \rho}$ is lower triangular, $\frac{\partial H^T}{\partial \rho}$ is an upper triangular matrix.

$\implies$ The adjoint system can be solved efficiently using backward substitution.

Complexity reduction from $O(T^2NM)$ to $O(NT + MT)$.
Optimization algorithm

**Algorithm 1** Gradient descent loop

Pick initial control $\bar{u}_{\text{init}} \in U_{\text{ad}}$

while not converged do

$\bar{\rho} = \text{forwardSim}(\bar{u}, IC, BC)$ solve for state trajectory (forward system)

$\lambda = \text{adjointSln}(\bar{\rho}, \bar{u})$ solve for adjoint parameters (adjoint system)

$\nabla_u J = \lambda^T \frac{\partial H}{\partial u} + \frac{\partial J}{\partial u}$ compute the gradient (search direction)

$\bar{u} \leftarrow \bar{u} - \alpha \nabla_u J, \alpha \in (0, 1)$ s.t. $\bar{u} \in U_{\text{ad}}$ update the control $\bar{u}$ (update step)

end while
Remarks on discrete adjoint

Strengths:

▶ Requires only one solution of adjoint system, independent of $\dim(\mathbf{u})$ (unlike finite differences or sensitivity analysis).
▶ Extends existing simulation code
▶ General technique, can be applied in very different settings

Weaknesses:

▶ Optimization of the discretized problem, rather than approximation of the optimal solution of the continuous problem
▶ No proof of convergence
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Numerical Results: case study

Figure: I15 South, San Diego: 31 km

$N = 125$ links and $M = 9$ onramps
$T = 1800$ time-steps
$\Delta t = 4$ seconds (120 minutes time interval)

reduced congestion $\bar{c} = 100(1 - c_c/c_{nc})$, with $c = \max\{TTT - VMT/v_{max}, 0\}$
4. Numerical results

Numerical Results

Density and queue lengths without control

Density and queue difference with control
Model Predictive Control

Performance under noisy input data: MPC loop

- initial conditions at time $t$ and boundary fluxes on $T_h$ (noisy inputs)
- optimal control policy on $T_h$
- forward simulation on $T_u \leq T_h$ using optimal controls and exact initial and boundary data
- $t \rightarrow t + T_u$

Comparison with ALINEA (local feedback control without boundary conditions)

Figure: Congestion reduction and noise robustness
4. Numerical results

Running time

Convergence

Simulation time
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Application to optimal re-routing

System Optimal Dynamic Traffic Assignment problem with Partial Control:
Multi-commodity flow:
\[ \rho_i(k) = \sum_{c \in C} \rho_{i,c}(k) \]

accounting for compliant \( c_c \in C \) and non-compliant \( c_n \) users

- full Lagrangian paths known for the controllable agents
- knowledge of the aggregate split ratios for the non-controllable (selfish) agents.

Goal: Control compliant users to optimize traffic flow
Application to optimal re-routing

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Goal: Control compliant users to optimize traffic flow
Numerical study: synthetic network [Ziliaskopoulos, 2001]

- Normal conditions: link capacity between cell 3 and cell 4 = 6
  Optimal total travel time: 178

- Incident between cell 3 and cell 4: capacity goes to 0
  Optimal total travel time: 211 (otherwise 244)
5. Application to optimal re-routing

**Synthetic network: partial control**

Total travel time reduction as a function of the percentage of vehicles that are rerouted:

almost optimal allocation can be achieved by controlling $\sim60\%$ of the demand
5. Application to optimal re-routing

Numerical study: real case

Figure: I210 with parallel arterial route, Arcadia (13 km)

\[ N = 24 \text{ links} \]
\[ 1 \text{ hour time-horizon} \]
\[ \Delta t = 30 \text{ seconds} \]
Numerical Results

50% capacity drop between min 10 and min 30
Numerical Results

Arterial capacity used:

(a) 40% of arterial capacity
(b) 50% of arterial capacity
(c) full arterial utilization
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In summary

Results:
- Definition of appropriate junction solvers for ramp metering and multi-commodity
- Formulation of the corresponding finite horizon optimal control problems
- Gradient computation with $O(NT + MT)$ complexity
- Numerical tests showing the benefits on real field applications

Possible extensions:
- Variable speed limit, maximal queue length, ...
- Selfish agents response (Stackelberg games)
References


Thank you for your attention!