Conservation laws with point constraints on the flow and their applications

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ANCONET
This talk is concerned with the recent results on conservation laws with point constraints presented in

B. Andreianov, C. Donadello, M. D. Rosini
“A second order model for vehicular traffics with local point constraints on the flow”, to appear on Mathematical Models and Methods in Applied Sciences

B. Andreianov, C. Donadello, U. Razafison, J. Y. Rolland, M. D. Rosini
“Solutions of the Aw-Rascle-Zhang system with point constraints”, to appear on Networks and Heterogeneous Media

B. Andreianov, C. Donadello, U. Razafison, M. D. Rosini
“Qualitative behaviour and numerical approximation of solutions to conservation laws with non-local point constraints on the flux and modeling of crowd dynamics at the bottlenecks”, to appear on ESAIM: Mathematical Modelling and Numerical Analysis

B. Andreianov, C. Donadello, U. Razafison, M. D. Rosini

B. Andreianov, C. Donadello, M. D. Rosini
1. The motivating problem
2. Modelling traffic flows
3. Constrained LWR
4. Non-local constraint
5. ARZ
6. Constrained ARZ
7. Two phase model (M.Benyahia, GSSI L’Aquila)
Section 1

The motivating problem
The motivating problem

Study of the traffic flow through bottlenecks, i.e. locations with reduced capacity.

Capacity of the bottleneck: the maximum number of pedestrians or vehicles that can flow through the bottleneck in a given time interval.

Typical phenomena at the bottlenecks are:

- Capacity drop
- Braess’ paradox
- Faster Is Slower effect

Possible outcomes of this study are:

- Optimization of the traffic (green waves, stop-and-go, mean travel time, etc.).
- Reducing traffic accidents.
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<td>15</td>
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1985, Heysel Stadium in Brussels (Belgium): 39 fans died as they tried to escape Liverpool supporters.

2010, Phnom Penh (Cambodia): 347 people died in a bridge stampede at the Khmer Water Festival.
Crowd Accidents

2010, Duisburg (Germany): 21 people died near an overcrowded tunnel leading into the festival.

2014, Shanghai (China): 35 people died rushing to catch (fake) money thrown from a nightclub.
### Crowd Accidents

#### Jamarat Bridge (Saudi Arabia)

<table>
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<th>Year</th>
<th>Dead</th>
<th>Injured</th>
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<tr>
<td>1994</td>
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</table>
Is it possible to avoid or to mitigate crowd accidents?
Panic management

Crowd behaviour is not always the primary cause of accidents.

Architects and city planners try to provide a safe environment for places of public assembly by using mathematical models to determine and prevent congestions.

Example

Nowadays, all public entertainment venues are equipped with doors that open outwards using crash bar latches that open when pushed.
Crowd behaviour is not always the primary cause of accidents.

Architects and city planners try to provide a safe environment for places of public assembly by using mathematical models to determine and prevent congestions.

Example

A tall column placed in the upstream of the door exit may reduce the inter-pedestrian pressure, decreasing the magnitude of clogging and making the overall outflow higher and more regular (Braess’s paradox).
Section 2

Modelling traffic flows
The prototype of first order macroscopic models for traffic flows is the Lighthill, Whitham, Richards model

\[ \begin{cases} \rho_t + [\rho \ v]_x = 0, \\ v = V(\rho), \end{cases} \quad \text{(LWR)} \]

where \( V : [0, \rho_{\text{max}}] \rightarrow [0, v_{\text{max}}] \) is non-increasing, \( V(0) = v_{\text{max}} \) and \( V(\rho_{\text{max}}) = 0 \).

The most celebrate second order model for traffic flows is the Aw, Rascle, Zhang model

\[
\begin{align*}
\rho_t + (\rho \, v)_x &= 0, \\
[\rho \, (v + p(\rho))]_t + [\rho \, (v + p(\rho)) \, v]_x &= 0,
\end{align*}
\] (ARZ)

e.g. \( p(\rho) = \rho^\gamma, \gamma > 0. \)


Zhang, “A non-equilibrium traffic model devoid of gas-like behavior”, Transportation Research Part B: Methodological 2002
Both LWR and ARZ assume that the road is \textit{homogeneous}.

However in real life this assumption is \textbf{NOT} always justified.
Both LWR and ARZ assume that the road is homogeneous.
Motivating problem

Both LWR and ARZ assume that the road is homogeneous.
Both LWR and ARZ assume that the road is **homogeneous**.

**Definition (roughly speaking)**

By point constraint we mean a pointwise bottleneck that hinders the flow.

**Applicability**

We apply the theory for **point** constraints when we are interested in the effects on the **upstream** and **downstream** and not in the dynamics **inside** the constraint location.
The concept of point constraints on the flow was introduced

- in the framework of vehicular traffic to model the presence of toll gates, construction sites, traffic lights, speed bumps, etc.

- in the framework of crowd dynamics to model the presence of doors, revolving doors, turnstiles, escalators, traffic lights, stairs, etc.
Section 3

Constrained LWR
If in \( x = 0 \) there is a constraint, then the corresponding problem is

\[
\rho_t + f(\rho)_x = 0, \quad \rho(0, x) = \rho_0(x), \quad f(\rho(t, 0^{\pm})) \leq F. \quad \text{(cLWR)}
\]

The expression of \( F \) depends on the situation at hand:

- \( F \) is constant if we have a construction site or stairs.
- \( F \) depends on time if we have a traffic light.
- \( F \) depends non-locally on \( \rho \) if we have a toll gate or door.
Consider the Riemann problem for (cLWR)

\[
\rho_t + f(\rho)_x = 0, \quad \rho(0,x) = \begin{cases} 
\rho_\ell & \text{if } x < 0, \\
\rho_r & \text{if } x > 0,
\end{cases} \quad f(\rho(t,0^\pm)) \leq F.
\]

If \( f(\mathcal{R}_{LWR}[\rho_\ell, \rho_r](0)) \leq F \), then \( \mathcal{C}\mathcal{R}_{LWR}[\rho_\ell, \rho_r] \equiv \mathcal{R}_{LWR}[\rho_\ell, \rho_r] \).

If \( f(\mathcal{R}_{LWR}[\rho_\ell, \rho_r](0)) > F \), then

\[
\mathcal{C}\mathcal{R}_{LWR}[\rho_\ell, \rho_r](x) = \begin{cases} 
\mathcal{R}_{LWR}[\rho_\ell, \hat{\rho}](x) & \text{if } x < 0, \\
\mathcal{R}_{LWR}[\hat{\rho}, \rho_r](x) & \text{if } x > 0.
\end{cases}
\]
Consider the Riemann problem for constrained LWR

\[ \rho_t + f(\rho)_x = 0, \quad \rho(0, x) \equiv \rho_l = \rho_r, \quad f(\rho(t, 0^{\pm})) \leq F. \]

If \( f(\rho_l) > F \), then the solution is
Consider the Riemann problem for constrained LWR

\[ \rho_t + f(\rho)_x = 0, \quad \rho(0, x) \equiv \rho_\ell = \rho_r, \quad f(\rho(t, 0^{\pm})) \leq F. \]

If \( f(\rho_\ell) > F \), then the solution is

- No maximum principle!
- TV bound explosion!
Well-posedness in $\textbf{BV}$

constraint $\Rightarrow$ non-classical shocks $\Rightarrow$ explosion of TV

Hence $\textbf{BV}$ is not the proper space $\Rightarrow$ introduction of the space

$$\mathcal{D} = \{ \rho \in L^1 : \psi(\rho) \in \textbf{BV} \}$$

where

$$\psi(\rho) = \int_0^\rho |f'(r)| \, dr.$$
Cauchy problem

\[
\rho_t + f(\rho)x = 0, \quad \rho(0,x) = \rho_0(x), \quad f(\rho(t,0^{\pm})) \leq F. \quad \text{(cLWR)}
\]

Definition (Colombo, Goatin '07)

\( \rho \in \mathcal{D} \) is a solution to (cLWR) if:

1) \( \forall \phi \in C^1_c, \phi \geq 0, \) and \( \forall k \in [0, \rho_{\text{max}}] \)

\[
\int_0^t \int_{\mathbb{R}} \left[ |\rho - k| \phi_t + \text{sgn}(\rho - k) \left[ f(\rho) - f(k) \right] \phi_x \right] dx \, dt
\]

\[
+ \int_{\mathbb{R}} |\rho_0(x) - k| \phi(0,x) dx
\]

\[
+ 2 \int_{\mathbb{R}^+} \left[ 1 - \frac{F(t)}{f_{\text{max}}} \right] f(k) \phi(t,0) dt \geq 0.
\]

2) \( f(\rho(t,0^-)) = f(\rho(t,0^+)) \leq F(t) \) for a.e. \( t > 0. \)
Cauchy problem

\[ \rho_t + f(\rho)_x = 0, \quad \rho(0,x) = \rho_0(x), \quad f(\rho(t,0^\pm)) \leq F. \quad \text{(cLWR)} \]

**Definition (Andreianov, Goatin, Seguin, ’10)**

\( \rho \in L^\infty \) is a solution to (cLWR) if \( \exists M > 0 \) s.t. \( \forall c_\ell, c_r \in [0, \rho_{\text{max}}] \) and \( \forall \phi \in C^1_c, \phi \geq 0 \)

\[
\begin{align*}
&\int \int \left[ |\rho - c| \phi_t + \text{sgn}(\rho - c) \left[ f(\rho) - f(c) \right] \phi_x \right] dx \ dt \\
&+ \int |\rho_0(x) - c(x)| \phi(0,x) dx \\
&+ M \int \text{dist} \left( (c_\ell, c_r), \mathcal{G}(F(t)) \right) \phi(t,0) dt \geq 0,
\end{align*}
\]

where \( \mathcal{G} \) is the set of admissible traces along \( x = 0 \) and

\[
c(x) = \begin{cases} 
  c_\ell & \text{if } x < 0, \\
  c_r & \text{if } x > 0.
\end{cases}
\]
Cauchy problem

\[ \rho_t + f(\rho)_x = 0, \quad \rho(0, x) = \rho_0(x), \quad f(\rho(t, 0^{\pm})) \leq F. \]  (cLWR)

Definition (Chalons, Goatin, Seguin ’13)

For more general fluxes, \( \rho \in \mathbb{L}^\infty \) is a solution to (cLWR) if:

1) \( \forall \phi \in C^1_c, \phi \geq 0, \) and \( \forall k \in [0, \rho_{\text{max}}] \)

\[
\iint_{\mathbb{R}_+ \times \mathbb{R}} \left[ |\rho - k| \phi_t + \text{sgn}(\rho - k) \left[ f(\rho) - f(k) \right] \phi_x \right] \, dx \, dt \\
+ \int_{\mathbb{R}} |\rho_0(x) - k| \phi(0, x) \, dx \\
+ 2 \int_{\mathbb{R}_+} \left[ f(k) - \min \left\{ f(k), F(t) \right\} \right] f(k) \phi(t, 0) \, dt \geq 0.
\]

2) \( f(\rho(t, 0^-)) = f(\rho(t, 0^+)) \leq F(t) \) for a.e. \( t > 0. \)
\[ \rho_t + f(\rho)_x = 0, \quad \rho(0, x) = \rho_0(x), \quad f(\rho(t, 0^\pm)) \leq F. \quad \text{(cLWR)} \]

**Theorem (Colombo-Goatin '07, Andreianov-Goatin-Seguin '10, Chalons, Goatin, Seguin '13)**

*If* \( F \in L^\infty \), *then there exists a semigroup* \( S^F : \mathbb{R}_+ \times L^\infty \to L^\infty \)
*for* (cLWR).

*If* \( F^1, F^2, \rho^1_0, \rho^2_0 \in L^\infty \) *and* \( F^1 - F^2, \rho^1_0 - \rho^2_0 \in L^1 \):

\[
\left\| S^F_t \rho^1_0 - S^F_t \rho^2_0 \right\|_{L^1} \leq \left\| \rho^1_0 - \rho^2_0 \right\|_{L^1} + 2 \left\| F^1 - F^2 \right\|_{L^1}.
\]

\( \mathcal{D} \) *is an invariant domain and* \( \rho_0 \in \mathcal{D} \Rightarrow \)

\[
\begin{cases}
\text{TV}(\Psi(S^F_t \rho_0)) \leq C, \\
\left\| \Psi(S^F_t \rho_0) - \Psi(S^F_s \rho_0) \right\|_{L^1} \leq C|t - s|,
\end{cases}
\]

*with* \( C > 0 \) *constant uniform w.r.t. t.*
\[ \begin{align*}
\rho_t + f(\rho)_x &= 0, & \rho(0, x) &= \rho_0(x), \\
f(\rho(t, 0)) &= q(t), & f(\rho(t, x_c^\pm)) &\leq F(t).
\end{align*} \]

**Definition (Colombo, Goatin, Rosini ’11)**

\( \rho \in L^\infty \) is a solution to the above (cIBVP) if:

1) \( \forall \phi \in C^1_c, \phi \geq 0, \) and \( \forall k \in [0, \rho_{\text{max}}] \)

\[
\begin{align*}
\int_{\mathbb{R}_+^2} \left[ |\rho - k| \phi_t + \text{sgn}(\rho - k) \ [f(\rho) - f(k)] \phi_x \right] \, dx \, dt \\
+ \int_{\mathbb{R}_+} |\rho_0(x) - k| \, \phi(0, x) \, dx \\
+ \int_{\mathbb{R}_+} \text{sgn} \left[ f^{-1}_\ast(q(t)) - k \right] \ [f(\rho(t, 0^+)) - f(k)] \, \phi(t, 0) \, dt \\
+ 2 \int_{\mathbb{R}_+} \left[ 1 - \frac{F(t)}{f_{\text{max}}} \right] f(k) \, \phi(t, x_c) \, dt &\geq 0.
\end{align*}
\]

2) \( f(\rho(t, x_c^-)) = f(\rho(t, x_c^+)) \leq F(t) \) for a.e. \( t > 0. \)
ρ_t + f(ρ)_x = 0, \quad ρ(0, x) = ρ_0(x),

f(ρ(t, 0)) = q(t), \quad f(ρ(t, x_±)) \leq F(t).

**Theorem (Colombo, Goatin, Rosini '11)**

If $q, F \in BV$, then there exists a semigroup $S^{q, F} : \mathbb{R}_+ \times \mathcal{D} \to \mathcal{D}$ for the above (clBVP).

If $q^1, q^2, F^1, F^2, ρ_0^1, ρ_0^2 \in L^\infty$ and $q^1 - q^2, F^1 - F^2, ρ_0^1 - ρ_0^2 \in L^1$:

$$\left\| S^F_t ρ_0^1 - S^F_t ρ_0^2 \right\|_{L^1} \leq \left\| ρ_0^1 - ρ_0^2 \right\|_{L^1} + \left\| q^1 - q^2 \right\|_{L^1} + 2 \left\| F^1 - F^2 \right\|_{L^1}.$$  

Moreover

$$\int_0^T \left\| f(S^q_t^1, F^1 ρ_0^1)(x^-) - f(S^q_t^2, F^2 ρ_0^1)(x^-) \right\| dt \leq$$

$$\left\| q^1(t) - q^2(t) \right\|_{L^1([0, \min\{T, τ_0\})} + 2 \left\| F^1(t) - F^2(t) \right\|_{L^1([0, \min\{T, τ_c\})}.$$
Synchronizing gateways

Colombo-Goatin-Rosini ’11:

$q_o$  

$q_b$  

$q_c$  

$X_b$  

$X_c$  

The lower graphs corresponding to the lower inflows.
Section 4

Non-local constraint
Pedestrian inflow and outflow patterns at the London Underground station

E.M. Cepolina.
Phased evacuation: An optimisation model which takes into account the capacity drop phenomenon in pedestrian flows.

Non-local constraint: explanation

High density upstream $\implies$ Capacity drop

Non-local constraint

The corresponding model is

$$\rho_t + f(\rho)_x = 0, \ f(\rho(t, 0^\pm)) \leq p \left( \int_{\mathbb{R}} w(x) \rho(t, x) \, dx \right), \ \rho(0, x) = \rho_0(x)$$

Theorem (Andreianov, Donadello, Rosini ’14)

Assume that:

\( \textbf{(F)} \) \( f \in \text{Lip}([0, \rho_{\text{max}}]; [0, f_{\text{max}}]), f(0) = 0 = f(\rho_{\text{max}}) \) and there exists \( \bar{\rho} \in ]0, \rho_{\text{max}}[ \) s.t. \( f'(\rho) (\bar{\rho} - \rho) > 0 \) a.e. in \( [0, \rho_{\text{max}}] \).

\( \textbf{(W)} \) \( w \in L^\infty(\mathbb{R}_-; \mathbb{R}_+) \) is an increasing map, \( \|w\|_{L^1(\mathbb{R}_-; \mathbb{R}_+)} = 1 \) and there exists \( i_w > 0 \) such that \( w(x) = 0 \) for any \( x \leq -i_w \).

\( \textbf{(P)} \) \( p \) is non-increasing and belongs to \( \text{Lip}([0, \rho_{\text{max}}]; ]0, f(\bar{\rho})[) \).
Theorem (Andreianov, Donadello, Rosini ’14)

Then:

(i) For any initial datum $\rho_0 \in L^\infty(\mathbb{R}; [0, \rho_{\text{max}}])$, the Cauchy problem (cLWR) admits a unique solution $\rho$. Moreover, if $\tilde{\rho} = \tilde{\rho}(t, x)$ is the solution corresponding to the initial datum $\tilde{\rho}_0 \in L^\infty(\mathbb{R}; [0, \rho_{\text{max}}])$, then for all $T > 0$ and $L > i_w$

$$\|\rho(T) - \tilde{\rho}(T)\|_{L^1([-L,L];\mathbb{R})} \leq e^{CT} \|\rho_0 - \tilde{\rho}_0\|_{L^1(|x| \leq L+MT};\mathbb{R}),$$

where $M = \text{Lip}(f)$ and $C = 2\text{Lip}(p)\|w\|_{L^\infty(\mathbb{R}_-;\mathbb{R})}$.
Theorem (Andreianov, Donadello, Rosini ’14)

Then:

(ii) If \( \rho_0 \) belongs to \( \mathcal{D} \), then the unique solution of (cLWR) verifies \( \rho(t, \cdot) \in \mathcal{D} \) for a.e. \( t > 0 \), and it satisfies

\[
TV (\Psi (\rho(t))) \leq C_t = TV (\Psi (\rho_0)) + 4 f_{\text{max}} + C \ t ,
\]

moreover, for a.e. \( t, s \) in \( \]0, T[ \) we have

\[
\| \Psi (\rho(t, \cdot)) - \Psi (\rho(s, \cdot)) \|_{L^1(\mathbb{R};\mathbb{R})} \leq |t - s| \ \text{Lip}(\Psi) \ C_T .
\]
Non-local constraint

Proof (Wave-front tracking and operator splitting methods).

Introduce $f^n \in \textbf{PLC}$ approximation of $f$, $\rho^n_0 \in \textbf{PC}$ approximation of $\rho_0$ and $p^h \in \textbf{PC}$ approximation of $p$. Define recursively

$$F^{nh}[\rho^n_0](t) = S[\rho^n_0, \Theta[\rho^n_0]](t)$$

if $t \in [0, \Delta t]$, and, if $t \in [(\ell + 1)\Delta t, (\ell + 2)\Delta t]$, $\ell \in \mathbb{N}$, then

$$F^{nh}[\rho^n_0](t) = S\left[F^{nh}[\rho^n_0]((\ell + 1)\Delta t), \Theta\left[F^{nh}[\rho^n_0]((\ell + 1)\Delta t)\right]\right](t).$$

The existence of a limit for $F^{nh}[\rho^n_0]$ as $n$ and $h$ go to infinity is ensured by the choice

$$\Delta t_h = \frac{1}{2^{h+1} \omega(0-) \text{Lip}(p)}.$$
Non-local constraint

The proof is quite technical, also because solutions of a Riemann problem in the case of a non-decreasing piecewise constant $p$ may fail to be unique, $L^1_{\text{loc}}$-continuous and consistent, see


More in general, in the above paper we investigate how the regularity of $p$ impacts the well–posedness of the problem. We show that the regularity of $p$ plays a central role in the well-posedness result. While existence still holds for the Cauchy problem when $p$ is merely continuous, it is difficult to justify uniqueness in this case.
Comparison: local VS non-local

R.M. Colombo, Rosini.
Pedestrian Flows and non-classical shocks.
Mathematical Methods in the Applied Sciences 2005

Andreianov, Donadello, Rosini.
Crowd dynamics and conservation laws with non-local constraints.
Mathematical Models and Methods in Applied Sciences 2014
Andreianov, Donadello, Razafison, Rosini, “Qualitative behaviour and numerical approximation of solutions to conservation laws with non-local point constraints on the flux and modeling of crowd dynamics at the bottlenecks”, to appear on ESAIM: Mathematical Modelling and Numerical Analysis

In this paper we investigate numerically the model (NL). We prove the convergence of a scheme based on a constraint finite volume method. We then perform ad hoc simulations to qualitatively validate the model by proving its ability to reproduce typical phenomena at the bottlenecks, such as the Braess’ paradox and the Faster Is Slower effect.
Evacuation time as a function of the position of the obstacle.
Evacuation time as a function of $v_{\text{max}}$ for different initial densities

\[ \bar{\rho} = \chi [-5.75, -2], \quad \bar{\rho}_1 = \frac{4}{5} \chi [-5.75, -2], \quad \bar{\rho}_2 = \frac{3}{5} \chi [-5.75, -2] \]
Faster is slower effect

Evacuation time as a function of $v_{\text{max}}$ for different exit efficiencies

$$p_{\beta} (\xi) = p (\beta \xi), \quad \beta = 1, \quad \beta = 0.9, \quad \beta = 0.8$$
Evacuation time as a function of $v_{\text{max}}$ for different initial locations

$$\bar{\rho} = \chi[-5.75, -2], \quad \bar{\rho}^3 = \chi[-11.75, -8], \quad \bar{\rho}^4 = \chi[-19.75, -16]$$
We are considering now a more general class of point constraints

\[ f(\rho(t,0^{\pm})) \leq q(t) = Q[\rho](t) \text{ for a.e. } t \in [0, T], \]

where

\[ Q: C^0([0, T]; L^1(\mathbb{R}; \mathbb{R})) \rightarrow L^1([0, T]; [0, f(\bar{\rho})]) \]

so that the operator \( Q \) may be non-local both in time and space.
Consider vehicular traffic through a toll booth, where the number of open gates is decided according to online data.

If the data are collected by a video camera, then we can take

\[
Q[\rho](t) = p \left( \int_{\mathbb{R}^-} \int_0^t w(x) \kappa(t - s) \rho(s, x) \, ds \, dx \right),
\]

being \( \text{supp}(w) \) the area registered by the video camera and \( \text{supp}(\kappa) \) the period of time the data are taken into account.

A time delay in adapting the number of open gates to the available data can be easily modelled.
Consider vehicular traffic through a toll booth, where the number of open gates is decided according to online data.

If the data come from a photo camera that shoots photos at times $t_i$ of the area given by $\text{supp}(w)$, then we can take

$$\tilde{Q}[\rho](t) = p\left(\sum_{t_i < t} \int_{\mathbb{R}^-} w(x) \kappa(t - t_i) \rho(t_i, x) dx\right),$$

being $\text{supp}(\kappa)$ the period of time the data are taken into account.

A time delay in adapting the number of open gates to the available data can be easily modelled.
Consider vehicular traffic through a toll booth, where the number of open gates is decided according to online data.

If the data come from local sensors located in $y_i$, then we can take

\[
\bar{Q}[\rho](t) = p \left( \sum_{i=1}^{M} \left[ w(y_i) \bar{\kappa}(t) \rho_0(y_i) + \int_0^t w'(y_i) \bar{\kappa}(t-s) f(\rho)(s, y_i) \, ds \right] \right),
\]

being $\text{supp}(\kappa)$ the period of time the data are taken into account.

A time delay in adapting the number of open gates to the available data can be easily modelled.
We introduced a concept of *entropy solutions*.

By using fixed point arguments we are able to address *solvability* of non-locally constrained problems.

Under general conditions on $Q$ we obtained the *uniqueness* of entropy solutions.

B. Andreianov, C. Donadello, U. Razafison, M. D. Rosini, “Analysis and approximation of first order traffic models with general point constraints on the flow”, in preparation
Section 5

ARZ
ARZ and the vacuum

ARZ is the $2 \times 2$ system of conservation laws

\[
\dot{\vec{Y}} + F(\vec{Y})_x = \vec{0}, \quad \vec{Y} = \begin{pmatrix} \rho \\ y \end{pmatrix}, \quad F(\vec{Y}) = v \vec{Y}, \quad v = \frac{y}{\rho} - p(\rho).
\]

away from the vacuum:

- It is strictly hyperbolic.
- It is of Temple class.
- It has the Panov renormalization property.
- Existence results are available (wave-front tracking).

Ferreira, Kondo
Glimm method and wave-front tracking for the Aw-Rascle traffic flow model.

Far East Journal of Mathematical Sciences 2010
ARZ and the vacuum

ARZ is the $2 \times 2$ system of conservation laws

$$\vec{Y}_t + F(\vec{Y})_x = 0, \quad \vec{Y} = \begin{pmatrix} \rho \\ y \end{pmatrix}, \quad F(\vec{Y}) = v \vec{Y}, \quad v = \frac{y}{\rho} - p(\rho).$$

at the vacuum: $F$ is not defined!

Vacuum (Godvik, Hanche-Olsen)

Even for initial data away from the vacuum, the solutions may attain values in the vacuum in a (possibly strictly positive) finite time.

Existence results are available (compensated compactness).

Godvik, Hanche-Olsen

Existence of solutions for the Aw-Rascle traffic flow model with vacuum.
Journal of Hyperbolic Differential Equations 2008

Lu

Existence of global bounded weak solutions to nonsymmetric systems of Keyfitz-Kranzer type.
Journal of Functional Analysis 2011
Introduce the Riemann invariants coordinates

\[ \vec{W} = (v, w) \in \mathcal{W} = \left\{(v, w) \in \mathbb{R}^2_+: v \leq w\right\} = \mathcal{W}_0 \cup \mathcal{W}_0^c \]

given by

\[
Y(\vec{W}) = p^{-1}(w - v) \begin{pmatrix} 1 \\ w \end{pmatrix}, \quad W(\vec{Y}) = \begin{pmatrix} \frac{y}{\rho} - p(\rho) \\ \frac{y}{\rho} \end{pmatrix},
\]

where

- \(v\) is the velocity,
- \(w\) is the Lagrangian marker.

Crucial for the existence results the \textbf{BV}-estimate

\[ TV(\vec{W}(t)) \leq TV(\vec{W}_0) \]
Introduce in PC the concept of \textit{physically admissible} discontinuities involving a \textit{vacuum} state:

\[
\left( \vec{W}(t, x^-), \vec{W}(t, x^+) \right) \in \mathcal{G}
\]

where

\[
\mathcal{G} \doteq \left\{ (\vec{W}_\ell, \vec{W}_r) \in \mathcal{W}^2 : \begin{array}{l}
\vec{W}_\ell \in \mathcal{W}_0 \\
\vec{W}_r \in \mathcal{W}_0 \\
\vec{W}_\ell \in \mathcal{W}_0^c \\
\vec{W}_r \in \mathcal{W}_0
\end{array} \Rightarrow \vec{W}_\ell = \vec{W}_r, \quad \Rightarrow \vec{W}_r = (w_\ell, w_\ell) \right\}.
\]
Then define the Riemann solver $\mathcal{R}_{ARZ}$

$$\mathcal{R}_{ARZ} : \mathcal{G} \rightarrow \mathcal{C}^0 ([0, +\infty[ ; \mathcal{L}^\infty (\mathbb{R} ; \mathcal{W})) ,$$

$$(\tilde{W}_l, \tilde{W}_r) \mapsto \mathcal{R}_{ARZ}[\tilde{W}_l, \tilde{W}_r],$$

by introducing first the elementary waves: shocks, rarefactions and contact discontinuities.
The Riemann solver $R_{ARZ}$

**Definition (roughly speaking)**

The solution performs either a rarefaction or a shock, followed by a contact discontinuity.

<table>
<thead>
<tr>
<th>Step</th>
<th>Conditions</th>
<th>Equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\vec{W}_l \in \mathcal{W}_c^0$ $\vec{W}_r \in \mathcal{W}_0$</td>
<td>$R_{ARZ}[\vec{W}_l, \vec{W}_r] \equiv R[\vec{W}_l, \vec{W}<em>m]$ $\vec{W}<em>m = (w</em>\ell, w</em>\ell) \in \mathcal{W}_0$</td>
</tr>
<tr>
<td>2</td>
<td>$\vec{W}_l \in \mathcal{W}_0$ $\vec{W}_r \in \mathcal{W}_c^0$</td>
<td>$R_{ARZ}[\vec{W}_l, \vec{W}_r] \left( \frac{x}{t} \right) = \begin{cases} \vec{W}_l &amp; \text{if } x &lt; v_r t \ \vec{W}_r &amp; \text{if } x &gt; v_r t \end{cases}$</td>
</tr>
<tr>
<td>3</td>
<td>$\vec{W}_l, \vec{W}<em>r \in \mathcal{W}<em>c^0$ $v_r &lt; v</em>\ell &lt; w</em>\ell$</td>
<td>$R_{ARZ}[\vec{W}_l, \vec{W}_r] \equiv S[\vec{W}_l, \vec{W}_m] + C[\vec{W}_m, \vec{W}_r]$ $\vec{W}<em>m = (v_r, w</em>\ell) \in \mathcal{W}_0^c$</td>
</tr>
<tr>
<td>4</td>
<td>$\vec{W}_l, \vec{W}<em>r \in \mathcal{W}<em>c^0$ $v</em>\ell &lt; v_r &lt; w</em>\ell$</td>
<td>$R_{ARZ}[\vec{W}_l, \vec{W}_r] \equiv R[\vec{W}_l, \vec{W}_m] + C[\vec{W}_m, \vec{W}_r]$ $\vec{W}<em>m = (v_r, w</em>\ell) \in \mathcal{W}_0^c$</td>
</tr>
<tr>
<td>5</td>
<td>$\vec{W}_l, \vec{W}<em>r \in \mathcal{W}<em>0^c$ $v</em>\ell &lt; w</em>\ell \leq v_r &lt; w_r$</td>
<td>$R_{ARZ}[\vec{W}_l, \vec{W}_r] \left( \frac{x}{t} \right) = \begin{cases} R[\vec{W}_l, \vec{W}_m] \left( \frac{x}{t} \right) &amp; \text{if } \frac{x}{t} &lt; v_r \ \vec{W}<em>r &amp; \text{if } \frac{x}{t} &gt; v_r \end{cases}$ $\vec{W}<em>m = (w</em>\ell, w</em>\ell) \in \mathcal{W}_0$</td>
</tr>
</tbody>
</table>
Section 6

Constrained ARZ
ARZ can be interpreted as a generalization of LWR possessing a family of fundamental diagram curves, rather than a single one, corresponding to different Lagrangian markers $w$.

ARZ can be interpreted as a multi population version of LWR:

The population corresponding to the Lagrangian marker $w$ is characterized by maximal speed $w$ and length $1/p^{-1}(w)$. 
Can we extend also the concept of point constraints?

Technically **YES** because ARZ has the Panov renormalization property

\[ F(\vec{Y}) = \nu \vec{Y} \]

This ensures the existence of the traces of the solutions at \( x = 0 \).
Assume to have at $x = 0$ a constraint and consider the problem

$$\vec{Y}_t + F(\vec{Y})_x = \vec{0}, \quad \vec{Y}(0, x) = \vec{Y}_0, \quad f(\vec{Y}(t, 0^\pm)) \leq F. \quad \text{(cARZ)}$$

The Riemann problem

If $f(R_{ARZ}[\vec{W}_\ell, \vec{W}_r]) \leq F$, then $C R_{ARZ}[\vec{W}_\ell, \vec{W}_r] \equiv R_{ARZ}[\vec{W}_\ell, \vec{W}_r]$.

If $f(R_{ARZ}[\vec{W}_\ell, \vec{W}_r]) > F$, then

$$C R_{ARZ}[\vec{W}_\ell, \vec{W}_r](x/t) = \begin{cases} R_{ARZ}[\vec{W}_\ell, \hat{W}(w_\ell)](x/t) & \text{if } x < 0, \\ R_{ARZ}[\hat{W}(w_\ell), \vec{W}_r](x/t) & \text{if } x > 0. \end{cases}$$

ARZ with constraint

Assume to have at $x = 0$ a constraint and consider the problem

$$
\vec{Y}_t + F(\vec{Y})_x = 0, \quad \vec{Y}(0, x) = \vec{Y}_0, \quad f(\vec{Y}(t, 0^\pm)) \leq F. \quad \text{(cARZ)}
$$

The Cauchy problem

The first results in this direction are in

Andreianov, Donadello, Razafison, Rolland, Rosini, Solutions of the Aw-Rascle-Zhang system with point constraints, to appear on Networks and Heterogeneous Media
ARZ with constraint

Assume to have at $x = 0$ a constraint and consider the problem

$$\vec{Y}_t + F(\vec{Y})_x = 0, \quad \vec{Y}(0, x) = \vec{Y}_0, \quad f(\vec{Y}(t, 0^\pm)) \leq F. \quad \text{(cARZ)}$$

The Cauchy problem

We revisit the wave-front tracking construction of physically admissible solutions possibly involving vacuum states.

$$\mathcal{G} \doteq \left\{ (\vec{W}_\ell, \vec{W}_r) \in \mathcal{W}^2 : \begin{array}{l}
\vec{W}_\ell \in \mathcal{W}_0 \\
\vec{W}_r \in \mathcal{W}_0 \\
\vec{W}_\ell \in \mathcal{W}_c^0 \\
\vec{W}_r \in \mathcal{W}_0
\end{array} \implies \vec{W}_\ell = \vec{W}_r, \quad \begin{array}{l}
\vec{W}_\ell \in \mathcal{W}_0 \\
\vec{W}_r \in \mathcal{W}_0 \\
\vec{W}_\ell \in \mathcal{W}_c^0 \\
\vec{W}_r \in \mathcal{W}_0
\end{array} \implies \vec{W}_r = (w_\ell, w_\ell) \right\}$$
Assume to have at $x = 0$ a constraint and consider the problem

$$\vec{Y}_t + F(\vec{Y})_x = 0, \quad \vec{Y}(0, x) = \vec{Y}_0, \quad f(\vec{Y}(t, 0^\pm)) \leq F. \quad \text{(cARZ)}$$

The Cauchy problem

We introduce a Kruzhkov-like family of entropies to select the admissible shocks.

$$\mathcal{E}_k(\vec{W}) = \begin{cases} 0 & \text{if } v \leq k, \\ 1 - \frac{p^{-1}(w - v)}{p^{-1}(w - k)} & \text{if } v > k, \end{cases}$$

$$Q_k(\vec{W}) = \begin{cases} 0 & \text{if } v \leq k, \\ k - \frac{q(\vec{W})}{p^{-1}(w - k)} & \text{if } v > k. \end{cases}$$

This tool allows to define rigorously the appropriate notion of admissible solution and to approximate the solutions of (cARZ).
Assume to have at $x = 0$ a constraint and consider the problem

$$\ddot{Y}_t + F(\dot{Y})_x = 0, \quad \dot{Y}(0, x) = \dot{Y}_0, \quad f(\dot{Y}(t, 0^\pm)) \leq F. \quad \text{(cARZ)}$$

The Cauchy problem

**Definition (Constrained entropy solution)**

$\tilde{W} \in L^\infty ([0, +\infty[; BV(\mathbb{R}; \mathcal{W})) \cap C^0(\mathbb{R}_+; L^1_{\text{loc}}(\mathbb{R}; \mathcal{W}))$ is a constrained entropy solution if it is a constrained weak solution and

$$\iint_{\mathbb{R}_+ \times \mathbb{R}} \left[ \mathcal{E}_k(\tilde{W}) \phi_t + Q_k(\tilde{W}) \phi_x \right] \, dx \, dt + \int_{\mathbb{R}_+} \mathcal{N}_k(\tilde{W}, q_0) \phi(t, 0) \, dt \geq 0$$

for any $\phi \in C^\infty_c ([0, +\infty[ \times \mathbb{R}; \mathbb{R})$, for any $k \in ]0, V_0]$, where

$$\mathcal{N}_k(\tilde{W}, q_0) = \begin{cases} q(\tilde{W}(t, 0)) \left[ \frac{k}{q_0(t)} - \frac{1}{p-1([w(t,0)-k]^+)} \right]^+ & \text{if } q_0(t) \neq 0 \\ k & \text{otherwise.} \end{cases}$$
Assume to have at $x = 0$ a constraint and consider the problem

$$\vec{Y}_t + F(\vec{Y})_x = 0, \quad \vec{Y}(0, x) = \vec{Y}_0, \quad f(\vec{Y}(t, 0^\pm)) \leq F. \quad \text{(cARZ)}$$

The Cauchy problem

We propose a finite volumes numerical scheme for (cARZ) and we show that it can correctly locate contact discontinuities and take the constraint into account.

Chalons, Goatin, “Transport-equilibrium schemes for computing contact discontinuities in traffic flow modeling”, Communications in Mathematical Sciences, 2007
ARZ with constraint

Assume to have at $x = 0$ a constraint and consider the problem

$$
\ddot{Y}_t + F(\dot{Y})_x = \vec{0}, \quad \dot{Y}(0, x) = \vec{Y}_0, \quad f(\dot{Y}(t, 0^{\pm})) \leq F. \quad \text{(cARZ)}
$$

The Cauchy problem

The convergence of the wave-front tracking is proved in

B. Andreianov, C. Donadello, M.D. Rosini, A second order model for vehicular traffics with local point constraints on the flow, to appear on Mathematical Models and Methods in Applied Sciences
Assume to have at $x = 0$ a constraint and consider the problem

$$\vec{Y}_t + F(\vec{Y})_x = 0, \quad \vec{Y}(0, x) = \vec{Y}_0, \quad f(\vec{Y}(t, 0^\pm)) \leq F. \quad \text{(cARZ)}$$

The Cauchy problem

The constraint $F$ is essentially constant.

$$\rho \mapsto [w - p(\rho)] \rho$$

$$\nu \mapsto [\nu + p(F/\nu)]$$
Section 7

Two phase model (M.Benyahia, GSSI L’Aquila)
Free flow \((LWR)\)
\[
\begin{align*}
\{ & u = (\rho, v) \in \Omega_f, \\
& \rho_t + [\rho v]_x = 0, \\
& v = v_f(\rho),
\}
\]

Congested flow \((ARZ)\)
\[
\begin{align*}
\{ & u = (\rho, v) \in \Omega_c, \\
& \rho_t + [\rho v]_x = 0, \\
& [\rho w]_t + [\rho w v]_x = 0.
\}
\]

Definition (Benyahia, Rosini work in progress)

Fix an initial distribution \( \bar{u} \) in \( \mathbf{BV}(\mathbb{R}; \Omega) \). Let \( u \) be a function in \( L^\infty([0, +\infty[; \mathbf{BV}(\mathbb{R}; \Omega)) \cap C^0(\mathbb{R}^+; L^1_{\text{loc}}(\mathbb{R}; \Omega)) \) satisfying the initial condition \( u(0, x) = \bar{u}(x) \) for a.e. \( x \in \mathbb{R} \). We say that \( u \) is an entropy solution to the two phase model (PT) if:

1. It is an entropy solution of LWR in \( u^{-1}(\Omega_f) \).
2. It is an entropy solution of ARZ in \( u^{-1}(\Omega_c) \).
3. If \( u \) performs a phase transition from \( u_- \) to \( u_+ \), then the speed of propagation of the phase transition satisfies the Rankine-Hugoniot conditions and \( (u_-, u_+) \in G_e \).
Theorem (Benyahia, Rosini work in progress)

For any initial distribution $\bar{u}$ in $\mathbf{BV}(\mathbb{R}; \Omega)$, the approximate solution for the Cauchy problem of (PT) with initial datum $\bar{u}$ converges (up to a subsequence) in $L^1_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R}; \Omega)$ to a function $u$ and for all $t, s \geq 0$ we have that

$$TV(u(t)) \leq TV(\bar{u}), \quad \|u(t)\|_{L^\infty(\mathbb{R}; \Omega)} \leq C,$$

$$\|u(t) - u(s)\|_{L^1(\mathbb{R}; \Omega)} \leq L |t - s|,$$

where $L = TV(\bar{u}) \max\{V_{\text{max}}, p^{-1}(W_{\text{max}}) p'(p^{-1}(W_{\text{max}}))\}$ and $C = \max\{|W_{\text{max}}|, |W_{\text{min}}|\} + V_f$. Moreover, if $V_{\text{max}} = V_f$ and $u$ performs a finite number of phase transitions, then $u$ is an entropy solution.
Let $\mathcal{R}: \Omega \times \Omega \rightarrow L^\infty(\mathbb{R}; \Omega)$ be the Riemann solver associated to the constrained (PT).

**Properties of $\mathcal{R}$**

$\mathcal{R}$ is neither $L^1_{\text{loc}}$-continuous, nor consistent.

$\mathcal{R}$ introduces non-classical shocks that do not “maximize” the flux through the constraint.

The study of the constrained Cauchy problem for (PT) will be very interesting...
THANK YOU FOR YOUR ATTENTION
If $\vec{W}_\ell \in \mathcal{W}$, $\vec{W}_r \in \mathcal{W}_0^c$ with $w_\ell = w_r = w$ and $v_r < v_\ell$, then we let

$$S[\vec{W}_\ell, \vec{W}_r](x/t) = \begin{cases} 
\vec{W}_\ell & x < \sigma t, \\
\vec{W}_r & x > \sigma t,
\end{cases} \quad \sigma = \frac{f_r - f_\ell}{\rho_r - \rho_\ell}.$$ 

The solution is then
If $\vec{W}_\ell \in \mathcal{W}, \vec{W}_r \in \mathcal{W}_0^c$ with $w_\ell = w_r = w$ and $v_r < v_\ell$, then we let

$$S[\vec{W}_\ell, \vec{W}_r](x/t) = \begin{cases} 
\vec{W}_\ell & x < \sigma t, \\
\vec{W}_r & x > \sigma t,
\end{cases} \quad \sigma = \frac{f_r - f_\ell}{\rho_r - \rho_\ell}.$$ 

The solution is then

$$\dot{x} = \sigma$$

$$\vec{W}_\ell \quad \vec{W}_r$$

$$v(t)$$

$$\rho(t)$$

$v_\ell$ $v_r$ $\rho_\ell$ $\rho_r$
If \( \tilde{W}_\ell \in \mathcal{W}_0^c \), \( \tilde{W}_r \in \mathcal{W} \) with \( w_\ell = w_r = w \) and \( v_\ell < v_r \), then we let

\[
\mathcal{R}[\tilde{W}_\ell, \tilde{W}_r](\frac{x}{t}) = \begin{cases} 
\tilde{W}_\ell & x < \lambda_1(\tilde{W}_\ell) t, \\
\left( w - p \left( \mathcal{R} \left( w - \frac{x}{t} \right) \right) \right) & \lambda_1(\tilde{W}_\ell) < \frac{x}{t} < \lambda_1(\tilde{W}_r), \\
\tilde{W}_r & x > \lambda_1(\tilde{W}_r) t.
\end{cases}
\]

Above

\[
\lambda_1(\tilde{W}) = \begin{cases} 
v - \rho(\tilde{W}) p' \left( \rho(\tilde{W}) \right) & \text{if } \tilde{W} \in \mathcal{W}_0^c, \\
w & \text{if } \tilde{W} \in \mathcal{W}_0,
\end{cases}
\]

and \( \mathcal{R} \) is the inverse function of \( \rho \mapsto p(\rho) + \rho p'(\rho) \).
Rarefactions

If $\vec{W}_\ell \in \mathcal{W}_0$, $\vec{W}_r \in \mathcal{W}$ with $w_\ell = w_r = w$ and $v_\ell < v_r$, then we let

$$
\mathcal{R}[\vec{W}_\ell, \vec{W}_r](\frac{x}{t}) = \begin{cases} 
\vec{W}_\ell & x < \lambda_1(\vec{W}_\ell) t, \\
(w - p \left( \mathcal{R} \left( w - \frac{x}{t} \right) \right)) & \lambda_1(\vec{W}_\ell) < \frac{x}{t} < \lambda_1(\vec{W}_r), \\
\vec{W}_r & x > \lambda_1(\vec{W}_r) t.
\end{cases}
$$

The solution is then

\[ f \]

\[ f \_\ell \quad f \_r \]

\[ v \_\ell \quad v \_r \]

\[ \rho \_\ell \quad \rho \_r \quad \rho \]

\[ w \]

\[ w \]

\[ v \_\ell \quad v \_r \quad v \]

\[ \text{Back} \]
If $\vec{W}_l \in \mathcal{W}_0^c$, $\vec{W}_r \in \mathcal{W}$ with $w_\ell = w_r = w$ and $v_\ell < v_r$, then we let

$$
\mathcal{R}[\vec{W}_l, \vec{W}_r](\frac{x}{t}) = \begin{cases} 
\vec{W}_l & x < \lambda_1(\vec{W}_l) t, \\
(w - p \left( \mathcal{R} \left( w - \frac{x}{t} \right) \right)) & \lambda_1(\vec{W}_l) < \frac{x}{t} < \lambda_1(\vec{W}_r), \\
\vec{W}_r & x > \lambda_1(\vec{W}_r) t.
\end{cases}
$$

The solution is then

$$
\dot{x} = \lambda_1(\vec{W}_l) t, \quad \dot{x} = \lambda_1(\vec{W}_r) v(t).
$$
Contact discontinuities

If $\vec{W}_\ell, \vec{W}_r \in \mathcal{W}_0^c$ with $v_\ell = v_r = v$ and $w_\ell \neq w_r$, then we let

$$C[\vec{W}_\ell, \vec{W}_r](x/t) = \begin{cases} \vec{W}_\ell & x < vt, \\ \vec{W}_r & x > vt. \end{cases}$$

The solution is then
Contact discontinuities

If $\vec{W}_\ell, \vec{W}_r \in \mathcal{W}_0^c$ with $v_\ell = v_r = v$ and $w_\ell \neq w_r$, then we let

$$C[\vec{W}_\ell, \vec{W}_r](x/t) = \begin{cases} 
\vec{W}_\ell & x < v t, \\
\vec{W}_r & x > v t.
\end{cases}$$

The solution is then

$$\dot{x} = v \quad \frac{\dot{x}}{t} = v(t) \quad \rho(t)$$