Topology 2 (Topologia 2)

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Mathematics Second Level Course

It is open to students of the **Master's degree in Mathematics (Laurea Specialistica), and bachelor's degree in Mathematics (Laurea triennale)** and to students of the **Master Mundus ALGANT program**.

When: first trimester

Where: Department of Pure and Applied Maths, Padova.

Total number of hours: about 48 (6 credits).

Examination: oral.

Description of the course

This course is an introduction to Algebraic Topology from the point of view of Sheaf Theory. It is based on lecture notes written by Pierre Schapira. Algebraic Topology is usually approached via the study of the fundamental group and of homology, defined using chain complexes, whereas, here, the accent is put on the language of categories and sheaves, with particular attention to locally constant sheaves.

Sheaves on topological spaces were invented by Jean Leray as a tool to deduce global properties from local ones. This tool turned out to be extremely powerful, and applies to many areas of Mathematics, from Algebraic Geometry to Quantum Field Theory. The functor associating to a sheaf F on a topological space X the space $F(X)$ of its global sections is left exact, but not right exact in general. The derived functors $H^j(X;F)$ encode the "obstructions" to pass from local to global. Given a ring k, the cohomology groups $H^j(X;k_X)$ of the sheaf k_X of k-valued locally constant functions is therefore a topological invariant of the space X. Indeed, it is a homotopy invariant, and we shall explain how to calculate $H^j(X;k_X)$ in various situations.

Program

Lectures will be organized as follows.

1- Linear algebra over a ring

A brief survey of linear algebra over a ring which will serve as a guide for the theory of abelian categories.

2- Categories and functors

We will expose the basic language of categories and functors. A key point is the Yoneda lemma, which asserts that a category C may be embedded in the category $C^{\wedge}$ of contravariant functors from C to the category of sets. This naturally leads to the concept of representable functor. Next, we study inductive and projective limits in some detail and with many examples.

3- Additive categories

4- Abelian categories

The aim is to construct and study the derived functors of a left (or right) exact functor F of abelian categories. Hence, we start by studying complexes (and double complexes) in additive and abelian categories. Then we briefly explain the construction of the right derived functor by using injective resolutions and later, by using F-injective resolutions. We apply these results to the case of the functors Ext and Tor.

5- Abelian sheaves on topological spaces

We study abelian sheaves on topological spaces (with a brief look at Grothendieck topologies). We construct the sheaf associated with a presheaf and the usual internal (Hom and

$\otimes$) and external operations (direct and inverse images). We also explain how to obtain locally constant or locally free sheaves when gluing sheaves.

6- Cohomology of sheaves

We will prove that the category of abelian sheaves has enough injectives and we define the cohomology of sheaves. Using the fact that the cohomology of locally constant sheaves is a homotopy invariant, we show how to calculate the cohomology of spaces by using cellular decomposition and we deduce the cohomology of some classical manifolds.

References

lecture notes written by Pierre Schapira available in
`http://www.institut.math.jussieu.fr/~schapira/`