Minimax Approximation and Remez Algorithm

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Minimax approximation seeks the polynomial of degree n that approximates the given function in the given interval such that the absolute maximum error is minimized. The error is defined here as the difference between the function and the polynomial.

Chebyshev Proved that such polynomial exists and that it is unique. He also gave the criteria for a polynomial to be a minimax polynomial[1]. Assuming that the given interval is $[a, b]$ Chebyshev's criteria states that if $P_n(X)$ is the minimax polynomial of degree n then there must be at least $(n+2)$ points in this interval at which the error function attains the absolute maximum value with alternating sign as shown in figure 1 for $n = 3$ and by the following equations:

$$a \leq x_0 < x_1 < \cdots < x_{n+1} \leq b$$

$$F(x_i) - P_n(x_i) = (-1)^i E \quad (1)$$

$$i = 0, 1, \ldots, n+1$$

$$E = \pm \max_{a \leq x \leq b} |F(x) - P_n(x)| \quad (2)$$

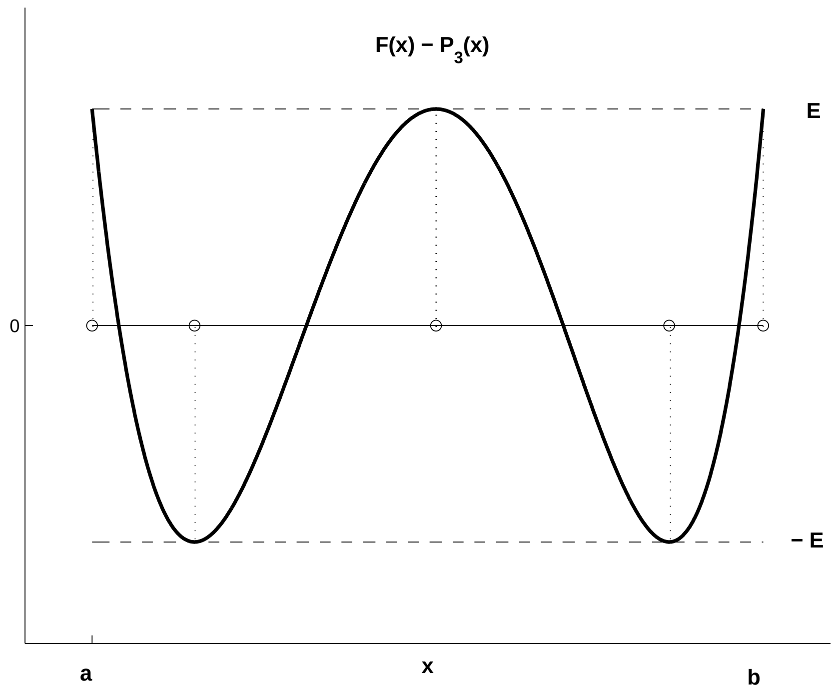

Figure 1: Example of a minimax third order polynomial that conforms to the Chebychev criteria

The minimax polynomial can be computed analytically up to $n = 1$. For higher order a numerical method due to Remez [2] has to be employed.

Remez algorithm is an iterative algorithm. We start the first iteration by an arbitrary set of $(n + 2)$ points in the given interval. Each iteration is composed of two steps. In the first step we compute the coefficients such that the error function takes equal magnitude with alternating sign at $(n+2)$ given points.

$$F(x_i) - P_n(x_i) = (-1)^i E \quad (3)$$

$$F(x_i) - [c_0 + c_1(x_i - a) + c_2(x_i - a)^2 + \cdots + c_n(x_i - a)^n] = (-1)^i E \quad (4)$$

$$c_0 + c_1 h_i + \cdots + c_n h_i^n + (-1)^i E = F(x_i) \quad (5)$$

$$i = 0, 1, 2, \ldots, n+1$$

Equation 5 is a system of $(n+2)$ linear equations in the $(n+2)$ unknowns $\{c_0, c_1, \ldots, c_n, E\}$. These equations are proved to be independent [2] hence we can solve them using any method from linear algebra to get the values of the coefficients as well as the error at the given $(n+2)$ points.

After the first step we compute the coefficients such that the error function at the given $(n+2)$ points is equal in magnitude and alternating in sign. However the magnitude of this error is the not the absolute maximum magnitude in the given interval $[a, b]$. Therefore the minimax condition is still not met. We need to move to a new set of points.

The second step of Remez algorithm seeks a new set of $(n+2)$ points that approach the $(n+2)$ points of the minimax condition.

The second step is called the exchange step. There are two exchange techniques. In the first exchange technique we exchange a single point in the current set of $(n+2)$ points to get a new set of points while in the second exchange technique we exchange all the points of the current set of $(n+2)$ points to get a new set points.

We start the second step by noting that the error alternates in sign at the $(n+2)$ points of the first step therefore the error function has $(n+1)$ roots, one root in each of the the intervals: $[x_0, x_1], [x_1, x_2], \ldots, [x_n, x_{n+1}]$. We compute these roots using any numerical method such as the method of chords or bisection. We denote these roots by $z_0, z_1, \ldots, z_n$. We divide the interval $[a, b]$ into the $(n+2)$ intervals: $[a, z_0], [z_0, z_1], [z_1, z_2], \ldots, [z_{n-1}, z_n], [z_n, b]$. In each of these intervals we compute the point at which the error attains its maximum or minimum value and denote these points by $x_0^*, x_1^*, \ldots, x_{n+1}^*$.

We can carry out the last step numerically by computing the root of the derivative of the error function if such root exists otherwise we compute the error function at the endpoints of the interval and pick the one that gives larger absolute value.

We define k such that

$$k = \max_i |F(x_i^*) - P_n(x_i^*)| \tag{6}$$

In the single point exchange technique we exchange x_k by x_k^* while in the multiple exchange technique we exchange all the $(n+2)$ points $\{x_i\}$ by $\{x_i^*\}$.

We use this new set of $(n+2)$ points in the first step of the following iteration. We repeat the two steps a number of times until the difference between the old $(n+2)$ points and the new $(n+2)$ points lies below a given threshold. Figure 2 illustrates the second step graphically for a third order polynomial.

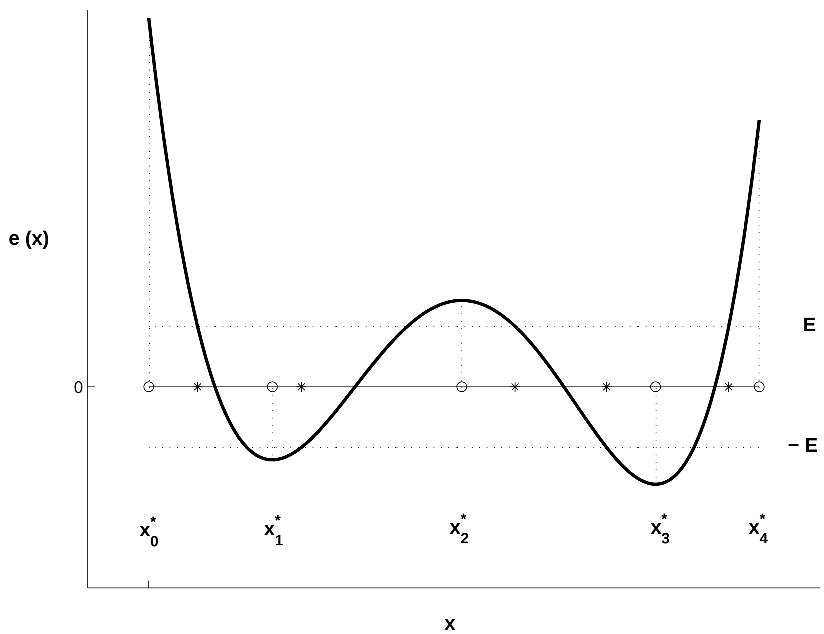

Figure 2: Illustration of second step of Remez algorithm

At the end of Remez algorithm we approach the Chebychev minimax condition hence the magnitude of the error function at the final set of the $(n+2)$ points (E) represents the maximum absolute value of the approximation error.

References

[1] N. L. Carothers, *A Short Course on Approximation Theory*. http://personal.bgsu.edu/∼carother/Approx.html, 1998.

[2] L. Veidinger, "On the numerical determination of the best approximations in the Chebychev sense," *Numerische Mathematik*, vol. 2, pp. 99–105, 1960.