Domain and Event Structure Semantics for Petri Nets with Read and Inhibitor Arcs*

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Abstract

We propose a functorial concurrent semantics for Petri nets extended with *read* and *inhibitor* arcs, that we call *inhibitor nets*. Along the lines of the seminal work by Winskel on safe (ordinary) nets, the truly concurrent semantics is given at a categorical level via a chain of coreflections leading from the category **SW-IN** of semi-weighted inhibitor nets to the category **Dom** of finitary prime algebraic domains (equivalent to the category **PES** of prime event structures). As an intermediate semantic model, we introduce *inhibitor event structures*, an event based model able to faithfully capture the dependencies among events which arise in the presence of read and inhibitor arcs. Inhibitor event structures generalise several event structure models in the literature, like prime, asymmetric and bundle event structures.

Keywords: Petri nets, read and inhibitor arcs, true concurrency, unfolding, categorical semantics, event structures, domains.

Introduction

Several generalisations of Petri nets [33, 36] have been proposed in the literature to overcome the expressiveness limitations arising from the simplicity of the classical model. At a very basic level Petri nets have been extended with two new kinds of arcs, namely *read arcs* (also called test, activator or positive contextual arcs) [13, 30, 21, 39] and *inhibitor arcs* (also called negative contextual arcs) [2, 30, 21] which allow a transition to check for the presence, resp. absence of resources (tokens), which are not affected by the firing of the transition. Read arcs are able

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to faithfully represent the situations where a resource is read but not consumed (read-only accesses). They have been used to model concurrent accesses to shared data (e.g., read operations in a database) [37, 14], to study temporal efficiency in asynchronous systems [39] and to give a truly concurrent semantics to concurrent constraint programs [29, 8]. Inhibitor arcs have been introduced in [2] to solve a synchronisation problem not expressible in classical Petri nets. A study of the expressiveness of inhibitor arcs, along with a comparison with other extensions proposed in the literature, namely priorities, exclusive-or transitions and switches, is carried out in [19, 32]. In particular it is worth stressing that inhibitor arcs make the model Turing complete, essentially because they allow to simulate the zero-testing operation of RAM machines which cannot be expressed neither by flow nor by read arcs. Inhibitor arcs have been employed, for example, for performance evaluation of distributed systems [3], to provide π-calculus with a net-based semantics [10] and to show the existence of an expressiveness gap between two different semantics of a process algebra based on Linda coordination primitives [11].

The purpose of this paper is to provide a truly concurrent semantics for *inhibitor nets*, i.e., Petri nets extended with read and inhibitor arcs.

Generally speaking, a truly concurrent semantics provides a description of the behaviour of a system, where the events in computations and their mutual relationships, notably causality, conflict and concurrency, are made explicit. This information can be useful for several purposes, e.g., to distribute independent branches of a computation over distinct processors, or, when causality is interpreted as "information flow", to verify the functional dependencies or non-interference properties between components ([14, 16]). Moreover, a concurrent semantics can represent a good basis for the development of effective verification techniques. In fact, an explicit representation of concurrency, which does not consider all the possible interleavings of concurrent events, may help to attack the state explosion problem [26, 15].

As discussed in detail below, the greater expressiveness arising from the introduction of inhibitor arcs is paid in terms of an increase of the complexity of the causal structure of computations, where the dependencies among events cannot be reduced simply to causality and conflict. To capture these dependencies the theory must be extended in a quite non-trivial way. The resulting semantic model turns out to have an applicability which goes beyond inhibitor nets, being suited to model, in general, formalisms where events can be disabled/enabled several times by other events. In particular it has been used profitably to model the concurrent semantics for graph transformation systems (see [4]).

We remark that, whenever one is interested only in reachability properties, read arcs can be safely replaced by self-loops, and, restricting to safe nets, also inhibitor arcs can be encoded by means of flow arcs, using a complementation technique. However, these encodings do not preserve the concurrency properties of a system. For instance, consider the safe inhibitor net N in Fig. 1, where place s inhibits transitions t_1 and t_2 (an inhibitor arc from a place s to a transition t is depicted as a dotted line from s to t, ending with an empty circle). This net can be transformed into the safe net N' in Fig. 1 with only read arcs by introducing a complement place $\bar{s}$ for s (a read arc is represented by an undirected, horizontal line). Place $\bar{s}$ is marked if and only if s was not marked and each transition having s in its pre-set has $\bar{s}$ in its post-set, and vice versa. Then read arcs can be replaced by self-loops, obtaining the net N'' in Fig. 1.

The marking graph of the nets N' and N'', when restricted to the places originally in N, is the same as that of N. However it is easy to see that the operations of complementation and introduction of self-loops radically change the dependency relations between transitions and thus the concurrency of the system. For instance, the complementation operation introduces a cycle

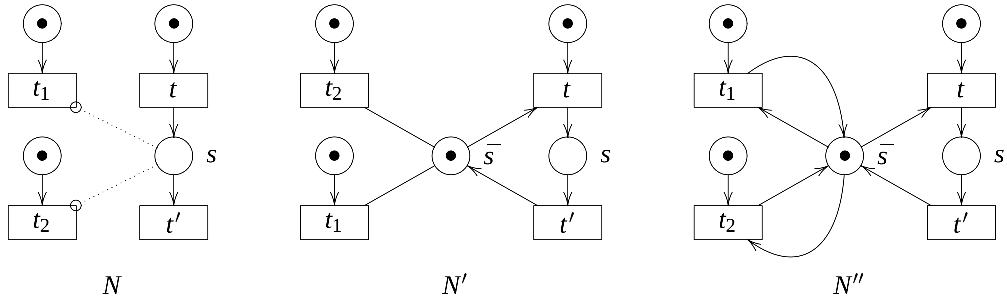


Figure 1: Encoding read and inhibitor arcs via flow arcs do not preserve concurrency.

of flow arcs involving t and t'. Observe also that while in the original net N transitions t_1 and t_2 could fire in parallel in the initial marking, in the transformed net N'', after the introduction of self-loops, they are forced to fire sequentially.

In the development of the concurrent semantics for inhibitor nets we follow the seminal work on ordinary safe nets of [31, 41], where the semantics is given at a categorical level via a chain of coreflections (special kinds of adjunctions), leading from the category **S-N** of safe (marked) P/T nets to the category **Dom** of finitary prime algebraic domains, through the categories **O-N** of occurrence nets and **PES** of prime event structures (PES's), the last step being an equivalence of categories. The diagram below represents the mentioned chain of coreflections. Given functors F and G, we write $F \vdash G$ when F is right adjoint to G. The same symbol is used, possibly rotated, in diagrams. The symbol $\hookrightarrow$ indicates inclusion functors.

$$\mathbf{S\text{-}N} \underset{\mathscr{U}}{\overset{\hookleftarrow}{\underset{\perp}{\longrightarrow}}} \mathbf{O\text{-}N} \underset{\mathscr{E}}{\overset{\mathscr{N}}{\underset{\perp}{\rightleftarrows}}} \mathbf{PES} \underset{\mathscr{L}}{\overset{\mathscr{P}}{\underset{\sim}{\rightleftarrows}}} \mathbf{Dom}$$

As shown in [27, 28] essentially the same construction applies to the wider category of *semi-weighted* nets, i.e., (possibly non-safe) P/T nets where the initial marking is a set and transitions can generate at most one token in each post-condition. A generalisation to the whole category of P/T nets is also possible, as shown in [28], but it requires some additional technical machinery and it allows one to obtain a proper adjunction rather than a coreflection.

A categorical semantics defined via an adjunction can be considered satisfactory under many respects. First, the semantic mapping is a functor, i.e., it "respects" the notion of morphism between systems, which formalises the idea of "simulation". Moreover, given a functor, its adjoint (if it exists) is unique up to natural isomorphism. Hence, when there is an obvious functor mapping semantic models back into the category of systems (e.g., occurrence nets are special nets, and thus the functor is simply the inclusion) the semantics can be defined canonically as the functor in the opposite direction, forming an adjunction. Finally, several operations on nets (systems) may be expressed at categorical level as limit/colimit constructions (see [41, 27]). Since left/right adjoint functors preserve colimits/limits, a semantics defined via an adjunction turns out to be compositional with respect to such operations.

The categorical unfolding approach has been extended in [6] to nets with read arcs, referred to as *contextual nets* (see also [40]). There, the key observation is that prime event structures

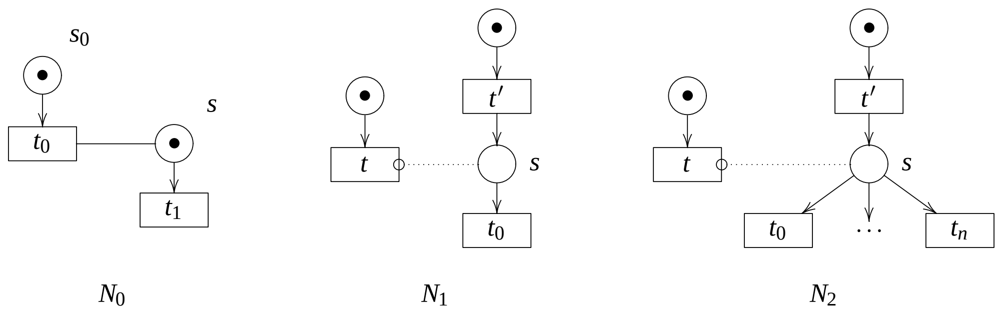
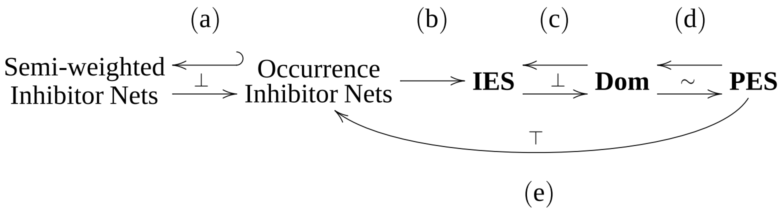


Figure 2: Some basic contextual and inhibitor nets.

are not adequate to model in a direct way the dependencies between transition occurrences in a contextual net. The problem is illustrated by the net N_0 of Fig. 2 where the same place s is "read" by transition t_0 and "consumed" by transition t_1. The firing of t_1 prevents t_0 to be executed, so that t_0 can never follow t_1 in a computation, while the converse is not true, since t_1 *can* fire after t_0. This situation can be interpreted naturally as an *asymmetric conflict* between the two transitions and cannot be represented faithfully in a PES. To model the behaviour of contextual nets, the paper [6] introduces *asymmetric event structures* (AES's), an extension of prime event structures where the symmetric conflict is replaced by an asymmetric conflict relation. Such a feature is obviously still necessary to be able to model the dependencies arising between events in inhibitor nets, but the nonmonotonic features related to the presence of inhibitor arcs (negative conditions) make the situation far more complicated.

Consider the safe net N_1 in Fig. 2 where the place s, which inhibits transition t, is in the post-set of transition t' and in the pre-set of t_0. The execution of t' inhibits the firing of t, which can be enabled again by the firing of t_0. Thus t can fire before or after the "sequence" $t';t_0$, but not in between the two transitions. Roughly speaking there is a sort of atomicity of the sequence $t';t_0$ with respect to t. The situation can be more involved since many transitions $t_0, \ldots, t_n$ may have the place s in their pre-set (see the net N_2 in Fig. 2). Therefore, after the firing of t', the transition t can be re-enabled by any of the conflicting transitions $t_0, \ldots, t_n$. This leads to a sort of *or*-causality, but only when t fires after t'. With a logical terminology we can say that t causally depends on the implication $t' \Rightarrow t_0 \vee t_1 \vee \ldots \vee t_n$.

To face these additional complications in this paper we introduce *inhibitor event structures* (IES's), a generalisation of PES's and AES's equipped with a ternary relation, called *DE-relation (disabling-enabling relation)* and denoted by $\vdash\!\circ\,(\cdot,\cdot,\cdot)$, which allows one to model the dependencies between transitions in N_2 simply as $\vdash\!\circ\,(\{t'\},t,\{t_0,\ldots,t_n\})$. As we will see, the DE-relation is sufficient to represent both causality and asymmetric conflict and thus concretely it is the only relation of an IES. Using inhibitor event structures and the DE-relation as basic tools we will extend Winskel's approach to (semi-weighted) inhibitor nets, providing this class of nets with a coreflective concurrent semantics. The proposed constructions are informally summarised by the diagram below.



As in the case of ordinary and contextual nets, the connection between nets and event structures is established via an unfolding construction which maps each net into an occurrence net (step (a) in the diagram). The complex structure of inhibitor net computations makes it hard to find an appropriate notion of *occurrence inhibitor net*. We identify two distinct, in our opinion both reasonable, notions of occurrence inhibitor net, and correspondingly we provide two different unfolding constructions which associate to each semi-weighted inhibitor net an occurrence inhibitor net. In both cases the unfolding construction gives rise to a functor which is right adjoint to the inclusion. The unfolding can be naturally abstracted to an IES, having the transitions of the net as events (step (b) in the diagram).

Finally, we establish a close relationship between IES's and prime algebraic domains (step (c) in the diagram), generalising the equivalence between **PES** and **Dom**. As already pointed out in [12], when dealing with inhibitor nets a deterministic computation is not uniquely determined by the events which occur in it. More concretely, in a deterministic process the absence of a token in an inhibitor place which enables a transition, may arise in two different situations: because the transition producing the token has not fired yet, or because the transition removing the token has already fired. For instance, the net N_1 of Fig. 2 admits two possible executions involving all its transitions, namely $t; t'; t_0$ and $t'; t_0; t$, which should not be identified from the point of view of causality. To deal with this problem a deterministic process, as defined in [12], includes also a partition of the inhibitor arcs into *before* and *after* arcs. Intuitively, the fact that an inhibitor arc from s to t is classified as "before" means that t must be executed before the place s is filled, while if it is an "after" arc then t must be executed after the token has been removed from s.

In a similar way, a *configuration* of an IES is not uniquely identified as a set of events, but some additional information has to be added which plays a basic role also in the definition of the order on configurations. More concretely, a configuration of an IES is a set of events endowed with a *choice relation* which chooses one among the possible different orders of execution of events constrained by the DE-relation. The configurations of an IES, endowed with a suitable computational order, form a prime algebraic domain, and Winskel's equivalence between **PES** and **Dom** generalises to a coreflection between the category **IES** of inhibitor event structures and **Dom**. By exploiting such coreflection one can recover a domain (or, equivalently, prime event structure) semantics for inhibitor nets.

Answering a question which was left open in the conference version of the paper [5], also the construction leading from occurrence i-nets to PES's and domains is given a universal characterisation as a coreflection (step (e) in the diagram). By analogy with contextual nets one could expect that the coreflection between occurrence i-nets and prime algebraic domains factorizes through **IES**, namely, that the functor from **Dom** to the category of occurrence i-nets could be "decomposed" in two functors, from **Dom** to IES's and from IES's to occurrence i-nets, respectively, establishing coreflections between the corresponding categories. We show that this is not possible, discussing how this fact is related to the complex kinds of dependencies among events expressible in IES's.

The rest of the paper is organised as follows. Section 1 presents the category of inhibitor nets and focuses on the subcategory of semi-weighted inhibitor nets which we shall work with. Section 2 introduces the categories of occurrence inhibitor nets and the corresponding unfolding constructions. Section 3 presents some background material regarding prime and asymmetric event structures, and their relationship with prime algebraic domains. Then Section 4 introduces inhibitor event structures, and presents the coreflection between the corresponding category and the category of domains. Section 5 shows how the unfoldings can be abstracted to an IES and a PES semantics. The construction which maps the unfoldings into PES's is characterised as a coreflection. Finally Section 6 draws some conclusions and directions of future research. An Appendix collects the full proofs of the results in the paper.

Some of the results in this paper appeared in CONCUR 2000 proceedings [5]. See also the PhD theses [4, 9] for a wider treatment of the semantics of Petri nets with read and inhibitor arcs, with applications to process calculi.

1 The category of inhibitor nets

Inhibitor nets are an extension of ordinary Petri nets where, by means of read and inhibitor arcs, transitions can check both for the presence and for the absence of tokens in places of the net. This section, after giving the basics of *(marked) inhibitor P/T nets*, turns the class of inhibitor nets into a category **IN** by introducing a suitable notion of morphism.

To give the formal definition we need some notation for sets and multisets. Let A be a set. The powerset of A is denoted by $\mathbf{2}^A$. A *multiset* of A is a function $M : A \to \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers. The set of multisets of A is denoted by μA. The usual operations and relations on multisets, like multiset union $+$ or multiset difference $-$, are used. We write $M \leq M'$ if $M(a) \leq M'(a)$ for all $a \in A$. If $M \in \mu A$, we denote by $[\![M]\!]$ the multiset defined as $[\![M]\!](a) = 1$ if $M(a) > 0$ and $[\![M]\!](a) = 0$ otherwise, obtained by changing all non-zero coefficients of M to 1; sometimes $[\![M]\!]$ will be confused with the corresponding subset $\{a \in A \mid [\![M]\!](a) = 1\}$ of A. A *multirelation* $f : A \to B$ is a multiset of $A \times B$. We will limit our attention to finitary multirelations, namely multirelations f such that the set $\{b \in B \mid f(a,b) > 0\}$ is finite. Multirelation f induces in an obvious way a (possibly partial) function $\mu f : \mu A \to \mu B$, defined as $\mu f(\sum_{a\in A} n_a \cdot a) = \sum_{b\in B} \sum_{a\in A} (n_a \cdot f(a,b)) \cdot b$.[1] If f satisfies $f(a,b) \leq 1$ for all $a \in A$ and $b \in B$, i.e. $f = [\![f]\!]$, then we sometimes confuse it with the corresponding set-relation and write $f(a,b)$ for $f(a,b) = 1$.

DEFINITION 1 (INHIBITOR NET) *A* (marked) inhibitor Petri net (i-net) *is a tuple* $N = \langle S, T, F, C, I, m \rangle$, *where*

- S *is a set of* places*;*
- T *is a set of* transitions*;*
- $F = \langle F_{pre}, F_{post} \rangle$ *is a pair of multirelations from* T *to* S*;*

[1] The function μf can be partial since infinite coefficients are disallowed in multisets. For instance, given the multirelation $f : \mathbb{N} \to \{0\}$ with $f(n,0) = 1$ for all $n \in \mathbb{N}$, then μf is undefined on the multiset $\sum_{n\in\mathbb{N}} 1 \cdot n$.

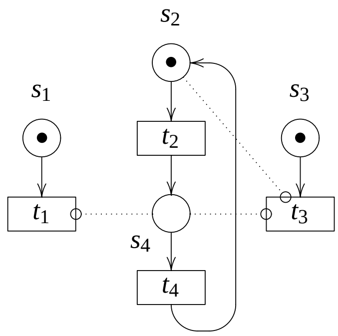


Figure 3: A safe inhibitor net N_3.

- *C and I are relations between T and S, called the* context *and* inhibitor relation, *respectively;*
- *m is a multiset of S, called the* initial marking.

If the inhibitor relation I is empty then N is called a contextual net (c-net).

We assume, as usual, that $S \cap T = \emptyset$. Moreover, we require that for each transition $t \in T$, there exists a place $s \in S$ such that $F_{pre}(t,s) > 0$. In the following when considering an i-net N, we will assume that $N = \langle S,T,F,C,I,m \rangle$. Moreover superscripts and subscripts on the net names carry over the names of the net components. For instance $N_i = \langle S_i,T_i,F_i,C_i,I_i,m_i \rangle$.

Let N be an i-net. As usual, the functions from μT to μS induced by the multirelations F_{pre} and F_{post} are denoted by ${}^\bullet(\ )$ and $(\ )^\bullet$, respectively. If $A \in \mu T$ is a multiset of transitions, ${}^\bullet A$ is called its *pre-set*, while $A^\bullet$ is called its *post-set*. Moreover, by $\underline{A}$ we denote the *context* of A, defined as $\underline{A} = C([\![A]\!])$, and by ${}^{\circledcirc}A = I([\![A]\!])$ the *inhibitor set* of A. The same notation is used to denote the functions from S to $\mathbf{2}^T$ defined as, for $s \in S$, ${}^\bullet s = \{t \in T \mid F_{post}(t,s) > 0\}$, $s^\bullet = \{t \in T \mid F_{pre}(t,s) > 0\}$, $\underline{s} = \{t \in T \mid C(t,s)\}$ and ${}^{\circledcirc}s = \{t \in T \mid I(t,s)\}$. For instance, for transition t_3 in the i-net N_3 of Fig. 3, we have ${}^\bullet t_3 = s_3$, $t_3{}^\bullet = 0$ and ${}^{\circledcirc}t_3 = \{s_2,s_4\}$. Considering place s_4 we obtain ${}^\bullet s_4 = \{t_2\}$, $s_4{}^\bullet = \{t_4\}$ and ${}^{\circledcirc}s_4 = \{t_1,t_3\}$.

A finite multiset of transitions A is enabled at a marking M, if M contains the pre-set of A and an additional multiset of tokens which covers the context of A. Furthermore the places of the inhibitor set of A must be empty both before and after the firing of the transitions in A.

DEFINITION 2 (TOKEN GAME) *Let N be an i-net and let M be a* marking *of N, i.e., a multiset $M \in \mu S$. A finite multiset $A \in \mu T$ is* enabled *at M if (i) ${}^\bullet A + \underline{A} \leq M$ and (ii) $[\![M + A^\bullet]\!] \cap {}^{\circledcirc}A = \emptyset$. The* transition relation *between markings is defined as*

$$M\,[A\rangle\,M' \qquad \textit{iff} \qquad A \textit{ is enabled at } M \textit{ and } M' = M - {}^\bullet A + A^\bullet.$$

Step and firing sequences, as well as reachable markings, are defined in the usual way. For instance, in the net N_3 of Fig. 3 a possible firing sequence starting from the initial marking is $s_1 + s_2 + s_3\,[t_2\rangle\,s_1 + s_4 + s_3\,[t_4\rangle\,s_1 + s_2 + s_3\,[t_1\rangle\,s_2 + s_3$.

DEFINITION 3 (I-NET MORPHISM) *Let* N_0 *and* N_1 *be i-nets. An* i-net morphism $h : N_0 \to N_1$ *is a pair* $h = \langle h_T, h_S \rangle$*, where* $h_T : T_0 \to T_1$ *is a partial function and* $h_S : S_0 \to S_1$ *is a multirelation such that (1)* $\mu h_S(m_0) = m_1$ *and (2) for each* $t_0 \in T_0$,

(a) $\mu h_S({}^\bullet t_0) = {}^\bullet h_T(t_0)$

(b) $\mu h_S(t_0{}^\bullet) = h_T(t_0)^\bullet$

(c) $\mu h_S(\underline{t_0}) = \underline{h_T(t_0)}$

(d) $[\![h_S]\!]^{-1}({}^{\circledcirc}h_T(t_0)) \subseteq {}^{\circledcirc}t_0$.

where we recall that $[\![h_S]\!]$ *is the set relation underlying the multirelation* h_S*. We denote by* **IN** *the category having i-nets as objects and i-net morphisms as arrows, and by* **CN** *its full subcategory having contextual nets as objects.*

Conditions (1), (2.a) and (2.b) are the defining conditions of Winskel's morphisms on ordinary nets. Condition (2.c) takes into account read arcs.[2] Note that the left-hand side of the equality is a multiset, while the right-hand side is a set. Hence this condition imposes $\mu h_S(\underline{t_0})$ to be a set (each element must occur with multiplicity 1) and to coincide with $\underline{h_T(t_0)}$. Condition (2.d) regarding the inhibitor arcs can be better explained by recalling that morphisms are intended to represent simulations: in order to map computations of N_0 into computations of N_1 morphisms are required to preserve preconditions and contexts, while, dually, inhibitor conditions must be reflected, since they are negative conditions. In fact observe that condition (2.d) on inhibiting places can be rewritten as

$$s_1 \in [\![\mu h_S(s_0)]\!] \wedge I_1(h_T(t_0), s_1) \quad \Rightarrow \quad I_0(t_0, s_0),$$

which shows more explicitly that inhibitor arcs are reflected. In particular, if h_S is a *total function* then

$$I_1(h_T(t_0), h_S(s_0)) \quad \Rightarrow \quad I_0(t_0, s_0).$$

It is easy to show that i-net morphisms are closed under composition.

PROPOSITION 4 (COMPOSITION OF I-NET MORPHISMS) *The class of i-net morphisms is closed under composition.*

Proof . See the Appendix.

Observe that i-net morphisms can be seen as a generalisation of the process mappings of [9, 12]. More precisely, processes of inhibitor nets in the style of Goltz-Reisig for a net N can be defined as special morphisms from a (deterministic) occurrence i-net to the net N (see [4]).

By the next proposition i-net morphisms preserve the token game, and thus marking reachability.

PROPOSITION 5 (MORPHISMS PRESERVE THE TOKEN GAME) *Let* N_0 *and* N_1 *be i-nets, and let* $h = \langle h_T, h_S \rangle : N_0 \to N_1$ *be an i-net morphism. Then for each* $M, M' \in \mu S_0$ *and* $A \in \mu T_0$

$$M\,[A\rangle\,M' \quad \Rightarrow \quad \mu h_S(M)\,[\mu h_T(A)\rangle\,\mu h_S(M').$$

Therefore i-net morphisms preserve reachable markings, i.e., if M_0 *is a reachable marking in* N_0 *then* $\mu h_S(M_0)$ *is reachable in* N_1.

[2] The category of contextual nets considered in [6] is isomorphic to **CN**, although there the inhibitor relation is absent rather than empty.

Proof. Suppose that $M\,[A\rangle\,M'$. Thus $^\bullet A + \underline{A} \leq M$ and $[\![M + A^\bullet]\!] \cap {}^{\circledcirc}A = \emptyset$.

First notice that $\mu h_T(A)$ is enabled at $\mu h_S(M)$. The proof of Condition (i) in the definition of enabling (see Definition 2), i.e., $^\bullet\mu h_T(A) + \underline{\mu h_T(A)} \leq \mu h_S(M)$, is essentially the same as for ordinary nets, adapted to take into account also the read arcs (see [6] for details). As for Condition (ii), which involves the inhibiting places, notice that

$$
\begin{aligned}
&[\![\mu h_S(M) + \mu h_T(A)^\bullet]\!] \cap {}^{\circledcirc}\mu h_T(A) = \qquad \text{[by (2.b) in the definition of morphism]}\\
&\quad = [\![\mu h_S(M) + \mu h_S(A^\bullet)]\!] \cap {}^{\circledcirc}\mu h_T(A)\\
&\quad = [\![\mu h_S(M + A^\bullet)]\!] \cap {}^{\circledcirc}\mu h_T(A)\\
&\quad = \emptyset
\end{aligned}
$$

The last passage is justified by observing that if $s_1 \in [\![\mu h_S(M + A^\bullet)]\!] \cap {}^{\circledcirc}\mu h_T(A)$, then there is $s_0 \in [\![M + A^\bullet]\!]$ such that $s_1 \in [\![\mu h_S(s_0)]\!]$ and $s_1 \in {}^{\circledcirc}h_T(A)$. By condition (2.d) in the definition of i-net morphism, this implies $s_0 \in {}^{\circledcirc}A$ and therefore $s_0 \in [\![M + A^\bullet]\!] \cap {}^{\circledcirc}A$, which instead is empty by hypothesis.

It is now immediate to conclude that $\mu h_S(M)\,[\mu h_T(A)\rangle\,\mu h_S(M')$. □

As in [41, 28, 6] we will restrict our attention to a subclass of nets where each token produced in a computation has a uniquely determined history. The next definition introduces the corresponding subcategory of **IN**.

DEFINITION 6 (SEMI-WEIGHTED AND SAFE I-NETS) *A* semi-weighted *i-net is an i-net N such that the initial marking m is a set and F_{post} is a relation (i.e., $t^\bullet$ is a set for all $t \in T$). We denote by* **SW-IN** *the full subcategory of* **IN** *having semi-weighted i-nets as objects; the corresponding subcategory of c-nets is denoted by* **SW-CN**.

A semi-weighted i-net is called safe *if also F_{pre} is a relation and each reachable marking is a set.*

An example of semi-weighted net which is not safe is given in Fig. 4.(a). As mentioned above, the basic property of semi-weighted nets, which will be essential in the unfolding construction, is that any token produced in a computation of the net has a uniquely determined history. More precisely, the tokens in the initial marking are uniquely identified by the place where they are and, inductively, any other token produced along the computation can be identified with the set of tokens consumed to produce it, the transition fired and the name of the place where the token is. For instance, referring to net N_4 in Fig. 4.(a), the token in s' in the initial marking is identified as s'. The token produced in s after the firing of t' corresponds to $\langle\langle\{s'\},t\rangle,s\rangle$. The property of uniqueness of causal history ceases to hold for general i-nets, as one can immediately verify by considering the simple net N_5 in Fig. 4.(b), where even the two tokens in the initial marking are indistinguishable. For a detailed discussion about the role of semi-weightedness see, e.g., [27].

2 Occurrence i-nets and the unfolding constructions

Generally speaking, an occurrence net provides a static representation of some computations of a net, in which the events (firing of transitions) and the relationships between events are made explicit. In [40, 6] the notion of (nondeterministic) occurrence net has been generalised to the

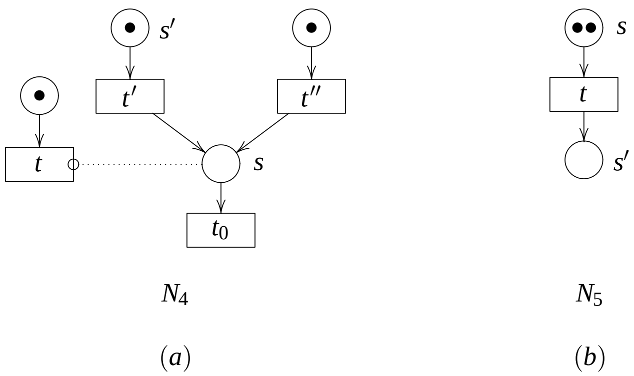


Figure 4: (a) A semi-weighted i-net which is not safe and (b) a non semi-weighted i-net.

case of nets with read arcs. Here, the presence of the inhibitor arcs and the complex kind of dependencies they induce on transitions, makes it hard to fix a unique notion of occurrence i-net.

In this section we present two different, in our opinion both reasonable, notions of occurrence i-net and, correspondingly, we develop two unfolding constructions.

In the first construction, given an i-net N, we consider the underlying contextual net N_c, obtained from N by forgetting the inhibitor arcs. Then, disregarding the inhibitor arcs, we apply to N_c the unfolding construction for contextual nets defined in [6], which produces an occurrence contextual net $\mathscr{U}_a(N_c)$. Finally, if a place s and a transition t were originally connected by an inhibitor arc in the net N, then we insert an inhibitor arc between each occurrence of s and each occurrence of t in $\mathscr{U}_a(N_c)$, thus obtaining the unfolding $\mathscr{U}_i(N)$ of the net N. Then the characterisation of the unfolding as a universal construction can be lifted from contextual to inhibitor nets.

It is worth observing that in this way the unfolding of an inhibitor net is decidable, in the sense that the problem of establishing if a possible transition occurrence actually appears in the unfolding is decidable. This fact may be helpful if one wants to use the unfolding in practice to prove properties of the modelled system. The price to pay is that, differently from what happens for ordinary and contextual nets, some of the transitions in the unfolding may not be firable, since they are generated without taking care of inhibitor places. Therefore not all the transitions of the unfolding correspond to a concrete firing of a transition of the original net, but only those which are executable.

In the second approach, the dependency relations (of causality and asymmetric conflict) for a net are defined only with respect to a fixed assignment for the net which specifies, for any inhibitor arc (t,s), if the inhibited transition t is executed before or after the place s is filled, and in the second case which one of the transitions in the post-set $s^{\bullet}$ of the inhibitor place consumes the token. Then the firability of a transition t amounts to the existence of an assignment which is acyclic on the transitions which must be executed before t. Relying on this idea we can define a notion of occurrence net where each transition is really executable. The corresponding unfolding construction produces a net where the mentioned problem of the existence of non-firable transitions disappears. However, in this way, as a consequence of the

Turing completeness of inhibitor nets (see, e.g., [1]) the produced unfolding is not decidable.

2.1 Lifting the unfolding from contextual to inhibitor nets

In the first approach, the unfolding construction disregards the inhibitor arcs. Consequently the notion of occurrence i-net is defined considering only the dependencies induced by flow and read arcs. As mentioned in the introduction, these dependencies can be fully captured by using two relations that we call read causality and read asymmetric conflict (the qualification "read" is due to the fact that they consider read arcs only, disregarding inhibitor arcs).

DEFINITION 7 (READ CAUSALITY) *Let N be a safe i-net. The* read causality relation *is defined as the least transitive relation $<_r$ on $S \cup T$ such that, for all $s \in S$ and $t, t' \in T$,*

1. $s <_r t$ *if* $s \in {}^\bullet t$,
2. $t <_r s$ *if* $s \in t^\bullet$,
3. $t <_r t'$ *if* $t^\bullet \cap \underline{t'} \neq \emptyset$.

Clauses (1) and (2) above are standard (see Fig. 5.(a)). The only novelty with respect to ordinary nets is the last clause stating that a transition causally depends on transitions generating tokens in its context (see Fig. 5.(b)).

DEFINITION 8 (READ ASYMMETRIC CONFLICT) *Let N be a safe i-net. The* read asymmetric conflict $\nearrow_r$ *is defined by taking, for all $t, t' \in T$, $t \nearrow_r t'$ if one of the following conditions holds:*

1. $\underline{t} \cap {}^\bullet t' \neq \emptyset$
2. $t \neq t' \ \wedge \ {}^\bullet t \cap {}^\bullet t' \neq \emptyset$
3. $t <_r t'$.

To understand the above definition consider an i-net N where each transition is intended to represent a single event and thus can fire at most once. Clause (1) considers the basic case of asymmetric conflict: if a transition t' consumes a token in the context of t (see Fig. 5.(d)), then, as already discussed, the firing of t' prevents the firing of t. Notice that asymmetric conflict determines an order of execution locally to each computation: if $t \nearrow_r t'$ and t, t' fire in the same computation then t must precede t'. Therefore a set of transitions in a cycle of asymmetric conflict cannot occur in the same computation, a fact that can be naturally interpreted as a kind of conflict. This explains clause (2) which capture the usual symmetric conflict as an asymmetric conflict in both directions (see Fig. 5.(c)). Asymmetric conflict can be also seen as a *weak* form of *causal dependency*, in the sense that if $t \nearrow t'$ then t precedes t' in all computations containing both transitions. Hence in clause (3) we also let $t \nearrow_r t'$ whenever $t <_r t'$.

DEFINITION 9 (READ CONCURRENCY) *Let N be a safe i-net. A set of places $X \subseteq S$ is called* read concurrent, *written $conc_r(X)$, if for all $x, y \in X$, $\neg(x <_r y)$, the set of read causes of X, i.e., $\{y : \exists x \in X .\ y <_r x\}$ is finite and $\nearrow_r$ is acyclic on such a set.*

Intuitively, the last requirement in the definition above corresponds to the absence of conflicts in the causes of X.

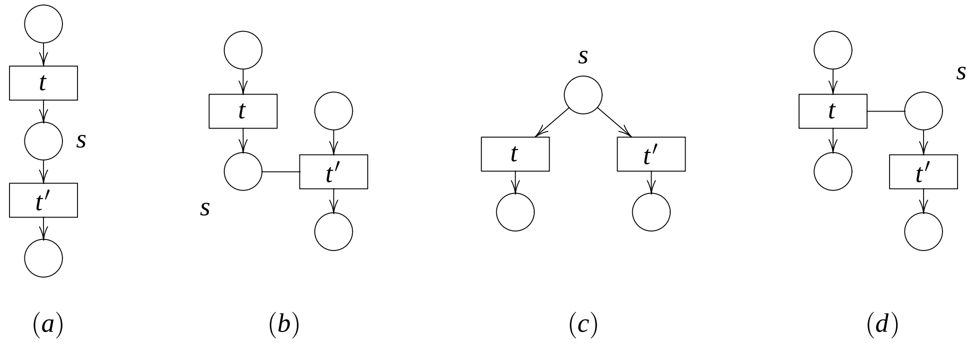


Figure 5: Read causality and asymmetric conflict: (a), (b) $t <_r t'$ and (c), (d) $t \nearrow_r t'$.

Figure 6: Not all events of an occurrence i-net are executable.

DEFINITION 10 (OCCURRENCE I-NETS) *An* occurrence i-net *N is a safe i-net N where read causality $<_r$ is a finitary partial order, read asymmetric conflict $\nearrow_r$ is acyclic on the causes of each transition, there are no backward conflicts (for all $s \in S$, $|{}^\bullet s| \leq 1$) and the initial marking is $m = \{s \in S \mid {}^\bullet s = \emptyset\}$.*

The full subcategory of **SW-IN** *having occurrence i-nets as objects is denoted by* **O-IN**, *while* **O-CN** *denotes the category of occurrence c-nets, namely the full subcategory of* **O-IN** *having only c-nets as objects.*

We remark that, since the above definition does not take into account the inhibitor arcs of the net, we are not guaranteed that each transition in an occurrence i-net is firable. For instance, N_6 in Fig. 6 is an occurrence i-net, but the only transition t can never fire.

It is worth introducing now some functors relating the categories of nets defined so far (see Fig. 7).

DEFINITION 11 *We denote by $\mathscr{R}_{ic}$:* **SW-IN** $\rightarrow$ **SW-CN** *the functor which maps each i-net into the underlying c-net with an empty inhibitor relation, defined as $\mathscr{R}_{ic}(\langle S,T,F,C,I,m\rangle) = \langle S,T,F,C,\emptyset,m\rangle$, and by $\mathscr{I}_{ci}$:* **SW-CN** $\rightarrow$ **SW-IN** *the obvious inclusion.*

The relations $\leq_r$ and $\nearrow_r$ associated to an i-net N are exactly the relations of causality and asymmetric conflict of the underlying c-net. Therefore the category of occurrence c-nets **O-CN** is the same as in [6] or [40], and occurrence i-nets are semi-weighted i-nets N such that $\mathscr{R}_{ic}(N)$ is an occurrence c-net.

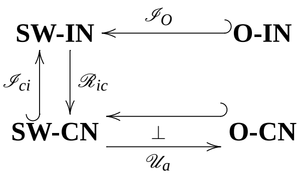


Figure 7: Functors relating semi-weighted (occurrence) c-nets and i-nets.

The paper [6] defines an unfolding functor $\mathscr{U}_a$: **SW-CN** → **O-CN**, mapping each semi-weighted c-net to an occurrence c-net.

DEFINITION 12 (UNFOLDING OF CONTEXTUAL NETS) *Let N be a semi-weighted contextual net. The* unfolding $\mathscr{U}_a(N) = \langle S', T', F', C', \emptyset, m' \rangle$ *of the net N and the* folding morphism $f_N = \langle f_T, f_S \rangle : \mathscr{U}_a(N) \to N$ *are the unique occurrence contextual net and morphism satisfying the following equations:*

$$
\begin{array}{rl}
m' = & \{\langle \emptyset, s\rangle \mid s \in m\} \\
S' = & m' \cup \{\langle t', s\rangle \mid t' \in T' \ \wedge\ s \in f_T(t')^{\bullet}\} \\
T' = & \{t' \mid t' = \langle M_p, M_c, t\rangle \ \wedge\ t \in T \ \wedge\ M_p \cup M_c \subseteq S' \ \wedge\ M_p \cap M_c = \emptyset \\
 & \quad \wedge\ conc_r(M_p \cup M_c) \wedge \mu f_S(M_p) = {}^{\bullet}t \ \wedge\ \mu f_S(M_c) = \underline{t}\}
\end{array}
$$

$$
\begin{array}{lll}
F'_{pre}(t', s') & \text{iff} & t' = \langle M_p, M_c, t\rangle \ \wedge\ s' \in M_p \quad (t \in T) \\
F'_{post}(t', s') & \text{iff} & s' = \langle t', s\rangle \quad (s \in S) \\
C'(t', s') & \text{iff} & t' = \langle M_p, M_c, t\rangle \ \wedge\ s' \in M_c \quad (t \in T) \\
f_T(t') = t & \text{iff} & t' = \langle M_p, M_c, t\rangle \\
f_S(s', s) & \text{iff} & s' = \langle x, s\rangle \quad (x \in T' \cup \{\emptyset\})
\end{array}
$$

As usual, places and transitions in the unfolding represent respectively tokens and firing of transitions in the original net. Each item of the unfolding is a copy of an item in the original net, enriched with the corresponding "history". The folding morphism f maps each item of the unfolding to the corresponding item in the original net. In the mentioned paper, the functor $\mathscr{U}_a$ is shown to be right adjoint to the inclusion functor of **O-CN** into **SW-CN**.

THEOREM 13 *The unfolding construction over contextual nets extends to a functor $\mathscr{U}_a$:* **SW-CN** → **O-CN** *which is right adjoint to the inclusion functor.*

By suitably using the functors $\mathscr{R}_{ic}$ and $\mathscr{I}_{ci}$ we can lift both the construction and the result from contextual nets to inhibitor nets.

DEFINITION 14 (UNFOLDING) *Let N be a semi-weighted i-net. Consider the occurrence c-net $\mathscr{U}_a(\mathscr{R}_{ic}(N)) = \langle S', T', F', C', \emptyset, m' \rangle$ and the folding morphism $f_N : \mathscr{U}_a(\mathscr{R}_{ic}(N)) \to \mathscr{R}_{ic}(N)$. Define an inhibiting relation on the net $\mathscr{U}_a(\mathscr{R}_{ic}(N))$ by taking for $s' \in S'$ and $t' \in T'$*

$$I'(s', t') \quad \textit{iff} \quad I(f_N(s'), f_N(t')).$$

Then the unfolding $\mathscr{U}_i(N)$ of the net N is the occurrence i-net $\langle S', T', F', C', I', m' \rangle$ and the folding morphism *is given by f_N seen as a function from $\mathscr{U}_i(N)$ into N.*

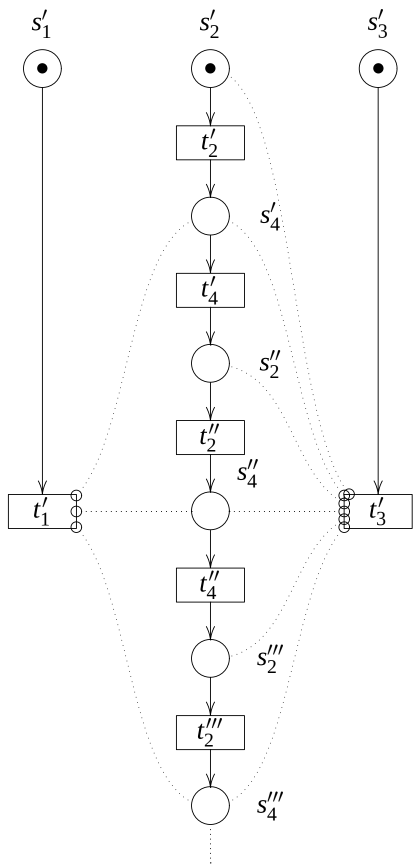


Figure 8: Part of the unfolding $\mathscr{U}_i(N_3)$ of i-net N_3 of Fig. 3.

The fact that $\mathscr{U}_i(N)$ is an occurrence i-net immediately follows from its construction. Furthermore, since the place component of f_N is a total function, according to condition (2.d) in the definition of i-net morphism, the unfolding $\mathscr{U}_i(N)$ can be characterised as the *least* i-net which extends $\mathscr{U}_a(\mathscr{R}_{ic}(N))$ with the addition of inhibitor arcs in a way that $f_N : \mathscr{U}_i(N) \to N$ is a well defined i-net morphism.

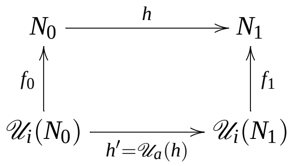
Fig. 8 presents (part of) the unfolding $\mathscr{U}_i(N_3)$ of the i-net N_3 of Fig. 3. Occurrences of an item x are denoted by $x', x'', \ldots$. Observe the unfolding includes an instance of transition t_3, although it is not executable.

The unfolding construction is functorial, namely we can define a functor $\mathscr{U}_i$: **SW-IN** $\to$ **O-IN**, which acts on arrows as $\mathscr{U}_a \circ \mathscr{R}_{ic}$. In other words, given $h : N_0 \to N_1$, the arrow $\mathscr{U}_i(h) : \mathscr{U}_i(N_0) \to \mathscr{U}_i(N_1)$ is obtained by interpreting h as a morphism between the c-nets underlying N_0 and N_1, taking its image via $\mathscr{U}_a$, and then considering the map $\mathscr{U}_a(h)$ as an arrow from $\mathscr{U}_i(N_0)$ to $\mathscr{U}_i(N_1)$.

PROPOSITION 15 *The unfolding construction extends to a functor* $\mathscr{U}_i$: **SW-IN** $\rightarrow$ **O-IN**, *which acts on arrows as* $\mathscr{U}_a \circ \mathscr{R}_{ic}$.

Proof. The only thing to verify is that given an i-net morphism $h : N_0 \rightarrow N_1$, the morphism $h' = \mathscr{U}_a \circ \mathscr{R}_{ic}(h) : \mathscr{U}_a(\mathscr{R}_{ic}(N_0)) \rightarrow \mathscr{U}_a(\mathscr{R}_{ic}(N_1))$, seen as a mapping $h' : \mathscr{U}_i(N_0) \rightarrow \mathscr{U}_i(N_1)$ is still an i-net morphism.

First notice that the following diagram, where f_0 and f_1 are the folding morphisms, commutes by construction (although h', in principle, may not be an i-net morphism).



Conditions (1) and (2.a)-(2.c), not involving inhibitor arcs, are automatically verified since h' is a morphism between the underlying c-nets. Let us prove the validity of condition (2.d), as expressed by the remark which follows Definition 3, namely

$$s'_1 \in [\![\mu h'_S(s'_0)]\!] \;\wedge\; I'_1(h'_T(t'_0), s'_1) \quad \Rightarrow \quad I'_0(t'_0, s'_0).$$

Assume $s'_1 \in [\![\mu h'_S(s'_0)]\!] \;\wedge\; I'_1(h'_T(t'_0), s'_1)$. Hence, $f_{1S}(s'_1) \in [\![\mu(f_{1S} \circ h'_S)(s'_0)]\!]$ and, by definition of the unfolding, $I_1(f_{1T}(h'_T(t'_0)), f_{1S}(s'_1))$. Therefore, by commutativity of the diagram

$$f_{1S}(s'_1) \in [\![\mu h_S(f_{0S}(s'_0))]\!] \quad \text{and} \quad I_1(h_T(f_{0T}(t'_0)), f_{1S}(s'_1))$$

Being h an i-net morphism, by condition (2.d) in Definition 3, we have that

$$I_0(f_{0T}(t'_0), f_{0S}(s'_0))$$

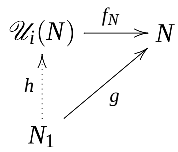
and therefore, by definition of the unfolding, $I'_0(t'_0, s'_0)$, which is the desired conclusion. □

We can now state the main result of this section, establishing a coreflection between semi-weighted i-nets and occurrence i-nets. It essentially relies on Theorem 13 which characterises the unfolding for c-nets as an universal construction.

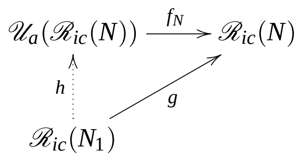
THEOREM 16 (COREFLECTION BETWEEN **SW-IN** AND **O-IN**) *The unfolding functor* $\mathscr{U}_i$: **SW-IN** $\rightarrow$ **O-IN** *is right adjoint to the obvious inclusion functor* $\mathscr{I}_O$: **O-IN** $\rightarrow$ **SW-IN** *and thus establishes a coreflection between* **SW-IN** *and* **O-IN**.

The component at an object N in **SW-IN** *of the counit of the adjunction,* $f : \mathscr{I}_O \circ \mathscr{U}_i \stackrel{.}{\rightarrow} 1$, *is the folding morphism* $f_N : \mathscr{U}_i(N) \rightarrow N$.

Proof. Let N be a semi-weighted i-net, let $\mathscr{U}_i(N) = \langle S', T', F', C', I', m' \rangle$ be its unfolding and let $f_N : \mathscr{U}_i(N) \rightarrow N$ be the folding morphism as in Definition 14. We have to show that for any occurrence i-net N_1 and for any morphism $g : N_1 \rightarrow N$ there exists a unique morphism $h : N_1 \rightarrow \mathscr{U}_i(N)$ such that the following diagram commutes:



The *existence* is readily proved by observing that an appropriate choice is $h = \mathcal{U}_i(g)$. The commutativity of the diagram simply follows by the commutativity of the diagram involving the underlying c-nets and morphisms, namely



With a little abuse of notation, we have denoted with the same symbol the morphism between the underlying c-nets and the same mapping seen as a morphism between the i-nets.

Also *uniqueness* follows easily by the universal property of the construction for c-nets given by Theorem 13. In fact let $h' : N_1 \to \mathcal{U}_i(N)$ be another i-net morphism such that $f_N \circ h' = g$. This means that h' is another c-net morphism which makes commute the diagram involving the underlying c-nets. This implies that, as desired, h and h' coincide. $\square$

2.2 Executable occurrence i-nets

The second approach is inspired by the notion of deterministic process of an i-net in [9]. As mentioned in the introduction, the inhibitor arcs of the net underlying a process are partitioned into two subsets: the *before* inhibitor arcs and *after* inhibitor arcs. Then the dependencies induced by such a partition are required to be acyclic in order to guarantee the firability of all the transitions of the net in a single computation. Following this idea, to ensure that each transition of a nondeterministic occurrence net is firable in some computation, we require, for each transition t, the existence of a so-called *assignment* which partitions the inhibitor arcs into before and after arcs, without introducing cyclic dependencies on the transitions which must be executed before t.

DEFINITION 17 (ASSIGNMENT) *Let N be a safe i-net. An* assignment *for N is a function $\rho : I \to T$ such that, for all $(t,s) \in I$, $\rho(t,s) \in {}^\bullet s \cup s^\bullet$.*

Intuitively, an assignment ρ specifies for each inhibitor arc (t,s), if the transition t fires *before* or *after* the place s receives a token. If $\rho(t,s) \in {}^\bullet s$ then (t,s) is a before arc, while if $\rho(t,s) \in s^\bullet$ then (t,s) is an after arc. In the last case, since the place s may be in the pre-set of several transitions, the assignment specifies also which of the transitions in $s^\bullet$ consumes the token.

Given a safe net N, once an assignment ρ for N is fixed, new dependencies arise between the transitions of the net, formalised by means of the relations $\prec_i^\rho$ and $\nearrow_i^\rho$. We define $t \prec_i^\rho t'$ iff $\exists s \in {}^{\circledcirc}t' \cap {}^\bullet t.\ \rho(t',s) = t$ and $t \nearrow_i^\rho t'$ iff $\exists s \in {}^{\circledcirc}t \cap t'^\bullet.\ \rho(t,s) = t'$. Observe that, as suggested by the adopted symbols, the additional dependencies can be seen as a kind of causality and asymmetric conflict, respectively. In fact if $t \prec_i^\rho t'$, then t' can happen only after t has removed the token from s, and thus t acts as a kind of cause for t'. If $t \nearrow_i^\rho t'$ then if both t and t' happen in the same computation then necessarily t occurs before t', since t' generates a token in a place s which inhibits t, while according to the interpretation of ρ, t must occur before the place s is filled.

Under a fixed assignment ρ, we can introduce a kind of generalised causality and asymmetric conflict by joining the "read" relations $\leq_r$ and $\nearrow_r$ defined in the previous subsection with

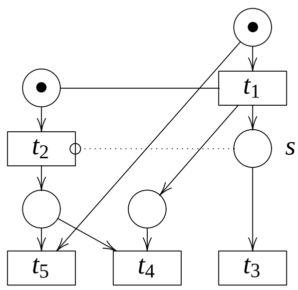


Figure 9: An executable occurrence i-net for which there exists no assignment ρ making relation $\nearrow\!\!\!\!\times_{\rho}$ acyclic on the causes of each transition in the net.

the additional dependencies induced by the inhibitor arcs. We define $<_{\rho}= (<_r \cup \prec_i^{\rho})^+$ and $\nearrow\!\!\!\!\times_{\rho} =<_{\rho} \cup \nearrow_r \cup \nearrow_i^{\rho}$, i.e., $\nearrow\!\!\!\!\times_{\rho}$ records both kinds of dependency. Furthermore, for $x \in S \cup T$ we denote by $\lfloor x \rfloor_{\rho}$ the set $\{t \in T \mid t \leq_{\rho} x\}$, and similarly, for $X \subseteq S \cup T$, we define $\lfloor X \rfloor_{\rho} = \bigcup \{\lfloor x \rfloor_{\rho} \mid x \in X\}$.

Now we are ready to introduce executable occurrence i-nets, which refine occurrence i-nets by constraining all the transitions of the net to be firable.

DEFINITION 18 (EXECUTABLE OCCURRENCE I-NET) *An* executable occurrence i-net *is a safe i-net N such that*

- *for all $t \in T$ there exists an assignment ρ such that $(\nearrow\!\!\!\!\times_{\rho})_{\lfloor t \rfloor_{\rho}}$ is acyclic and $\lfloor t \rfloor_{\rho}$ is finite,*
- *for all $s \in S$ $|{}^{\bullet}s| \leq 1$, and*
- *$m = \{s \in S \mid {}^{\bullet}s = \emptyset\}$.*

It is not difficult to see that each executable occurrence i-net is an occurrence i-net. We denote by $\mathbf{O\text{-}IN^e}$ the full subcategory of **O-IN** having executable occurrence i-nets as objects.

We remark that it is not possible to require the existence of a single assignment ρ such that $\nearrow\!\!\!\!\times_{\rho}$ is acyclic on $\lfloor t \rfloor_{\rho}$ for each transition t of the net. For instance, such an assignment does not exist for the net in Fig. 9, although each of its transitions can fire in some computation (thus for each transition t there exists an assignment ρ for which $\nearrow\!\!\!\!\times_{\rho}$ is acyclic on its causes $\lfloor t \rfloor_{\rho}$). In fact, for the assignment $\rho(t_2, s) = t_1$ the relation $\nearrow\!\!\!\!\times_{\rho}$ is cyclic on $\lfloor t_4 \rfloor_{\rho}$, while for $\rho(t_2, s) = t_3$ the relation $\nearrow\!\!\!\!\times_{\rho}$ is cyclic on $\lfloor t_5 \rfloor_{\rho}$.

Now, a notion of concurrent set of places of an executable occurrence i-net can be naturally defined.

DEFINITION 19 (CONCURRENCY) *A set of places $M \subseteq S$ is called* concurrent, *written conc(M), if there is an assignment ρ such that*

i. for all $s, s' \in M$ $\neg(s <_{\rho} s')$,

ii. $\lfloor M \rfloor_{\rho}$ is finite and *iii.* $\nearrow\!\!\!\!\times_\rho$ *is acyclic on* $\lfloor M \rfloor_\rho$.

It is possible to show that, as for ordinary and contextual nets, a set of places M is concurrent if and only if there is a reachable marking in which all the places of M contain a token.

PROPOSITION 20 *Let N be an executable occurrence i-net and let $M \subseteq S$. Then $conc(M)$ iff there exists a reachable marking M' such that $M \subseteq M'$.*

Proof. See the Appendix.

The above immediately implies a basic property of executable occurrence i-nets, namely the fact that each transition of such a net can fire in some computation (and thus each place contains a token at some reachable marking).

PROPOSITION 21 *Let N be an executable occurrence i-net. Then for each transition $t \in T$ there exists a reachable marking such that t is enabled at M.*

Proof. Immediate from the previous proposition and the definition of executable occurrence i-net. □

We introduce now an unfolding construction, that, when applied to a semi-weighted i-net N, produces an executable occurrence i-net.

DEFINITION 22 ((EXECUTABLE) UNFOLDING) *Let N be a semi-weighted i-net. The* (executable) unfolding $\mathcal{U}_i^{\mathbf{e}}(N) = \langle S', T', F', C', I', m' \rangle$ *of the net N and the* folding morphism $f_N = \langle f_T, f_S \rangle : \mathcal{U}_i^{\mathbf{e}}(N) \to N$ *are the unique executable occurrence i-net and i-net morphism satisfying the equations given in Definition 12, with the following changes:*

$$\begin{aligned} T' = \quad & \{ t' \mid t' = \langle M_p, M_c, t \rangle \ \wedge\ t \in T \ \wedge\ M_p \cup M_c \subseteq S' \ \wedge\ M_p \cap M_c = \emptyset \\ & \quad \wedge\ conc(M_p \cup M_c) \wedge \mu f_S(M_p) = {}^{\bullet}t \ \wedge\ \mu f_S(M_c) = \underline{t} \\ & \quad \wedge\ \exists \rho .\ (\lfloor t' \rfloor_\rho \ \textit{finite} \ \wedge \nearrow\!\!\!\!\times_\rho \ \textit{acyclic on} \ \lfloor t' \rfloor_\rho) \} \end{aligned}$$

$$I'(t', s') \qquad \text{iff} \qquad f_S(s', s) \ \wedge\ I(f_T(t'), s)$$

The main difference with respect to the unfolding of contextual nets is the fact that we refer here to a notion of concurrency which takes into account also the effect of inhibitor arcs.

Figure 10 presents (part of) the executable unfolding of the i-net N_3 of Fig. 3. Occurrences of an item x are denoted by $x', x'', \ldots$. Observe that the non-executable occurrence of transition t_3 is not included in this unfolding.

As one would expect, the two proposed unfolding constructions are tightly related, in the sense that $\mathcal{U}_i^{\mathbf{e}}(N)$ can be obtained from $\mathcal{U}_i(N)$ simply by removing the non-executable transitions (e.g., compare Fig. 8 and Fig. 10). This fact can be exploited elegantly to prove the universality of the executable unfolding as follows. First of all, let $\Pi : \mathbf{O\text{-}IN} \to \mathbf{O\text{-}IN^e}$ be the *pruning functor* which maps each occurrence i-net $N = \langle S, T, F, C, I, m \rangle$ to the net $N' = \langle S', T', F', C', I', m \rangle$, where T' is the subset of executable transitions, S' is the subset of reachable places and the relations F', C' and I' are the obvious restrictions of the original relations. The construction extends in an obvious way to a functor, mapping each morphism $f : N_1 \to N_2$ into the restriction $f_{\Pi(N_1)} : \Pi(N_1) \to \Pi(N_2)$ which is well-defined since morphisms preserve the token game and thus the executability of transitions.

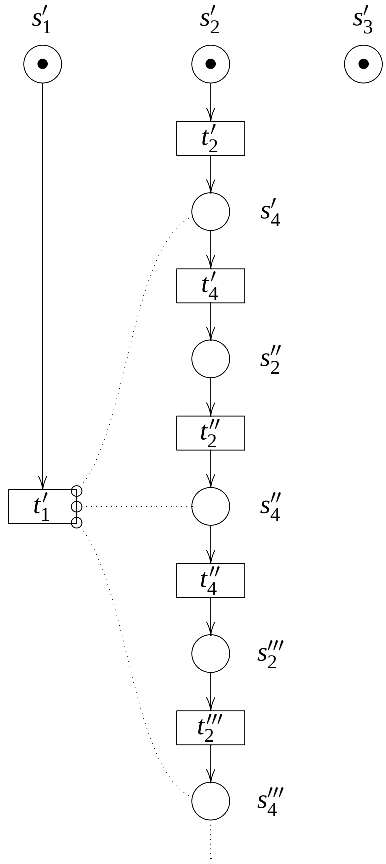


Figure 10: Part of the unfolding $\mathscr{U}_i^{\mathbf{e}}(N_3)$ of i-net N_3 of Fig. 3.

Next one can show that, given an executable occurrence i-net N and any occurrence i-net N', a morphism $f : N \to N'$ is also a morphism from N to $\Pi(N')$, and thus that the pruning functor $\Pi : \mathbf{O\text{-}IN} \to \mathbf{O\text{-}IN^e}$ is right adjoint to the inclusion functor $\mathscr{J}^{\mathbf{e}} : \mathbf{O\text{-}IN^e} \hookrightarrow \mathbf{O\text{-}IN}$, and $\mathbf{O\text{-}IN^e}$ is a coreflective subcategory of **O-IN**. At this point, one can formally state the relationship between $\mathscr{U}_i^{\mathbf{e}}(N)$ and $\mathscr{U}_i(N)$, which provides also an indirect proof of the universality of the new unfolding construction.

PROPOSITION 23 *For any semi-weighted i-net N, $\mathscr{U}_i^{\mathbf{e}}(N) = \Pi(\mathscr{U}_i(N))$. Therefore $\mathscr{U}_i^{\mathbf{e}}$ is right adjoint to the inclusion functor $\mathscr{I}_O^{\mathbf{e}} : \mathbf{O\text{-}IN^e} \to \mathbf{SW\text{-}IN}$ and they establish a coreflection between* **SW-IN** *and* $\mathbf{O\text{-}IN^e}$.

3 Prime and asymmetric event structures, and their relation with domains

In this background section we recall some basic notions and results on prime event structures and domains, as developed in [31, 41]. Furthermore we give some intuition on how such results have been extended in [6] to structures with asymmetric conflict. These notions and results will be useful later in the treatment of inhibitor event structures.

3.1 Prime event structures and domains.

Prime event structures. *Prime event structures* (PES) [31] are a simple event-based model of concurrent computations in which events are considered as atomic and instantaneous steps, which can appear only once in a computation. The relationships between events are expressed by two binary relations: *causality* and *conflict*.

DEFINITION 24 (PRIME EVENT STRUCTURES) *A* prime event structure (PES) *is a tuple $P = \langle E, \leq, \# \rangle$, where E is a set of* events *and $\leq$, # are binary relations on E called* causality relation *and* conflict relation *respectively, such that:*

1. *the relation $\leq$ is a partial order and $\lfloor e \rfloor = \{e' \in E : e' \leq e\}$ is finite for all $e \in E$;*
2. *the relation # is irreflexive, symmetric and hereditary with respect to $\leq$, i.e., $e \# e'$ and $e' \leq e''$ imply $e \# e''$ for all $e, e', e'' \in E$;*

Let $P_0 = \langle E_0, \leq_0, \#_0 \rangle$ and $P_1 = \langle E_1, \leq_1, \#_1 \rangle$ be two PES*'s. A* PES-morphism *$f : P_0 \to P_1$ is a partial function $f : E_0 \to E_1$ such that for all $e_0, e_0' \in E_0$, assuming that $f(e_0)$ and $f(e_0')$ are defined:*

1. $\lfloor f(e_0) \rfloor \subseteq f(\lfloor e_0 \rfloor)$;
2. (a) $f(e_0) = f(e_0') \wedge e_0 \neq e_0' \quad \Rightarrow \quad e_0 \#_0 e_0'$;
   (b) $f(e_0) \#_1 f(e_0') \quad \Rightarrow \quad e_0 \#_0 e_0'$;

The category of prime event structures and PES*-morphisms is denoted by* **PES**.

An event can occur only after some other events (its causes) have taken place, and the execution of an event can prevent the execution of other events. This is formalised via the notion of *configuration* of a PES $P = \langle E, \leq, \# \rangle$, which is a subset of events $C \subseteq E$ such that for all $e, e' \in C$ $\neg(e\#e')$ (*conflict-freeness*) and $\lfloor e \rfloor \subseteq C$ (*left-closedness*). Given two configurations $C_1 \subseteq C_2$ if $e_0, \ldots, e_n$ is any linearisation of the events in $C_2 - C_1$, compatible with causality, then

$$C_1 \subseteq C_1 \cup \{e_0\} \subseteq C_1 \cup \{e_0, e_1\} \subseteq \ldots \subseteq C_2$$

is a sequence of well-defined configurations. Therefore subset inclusion can be safely thought of as a computational ordering on configurations. The set of configurations of a prime event structure P, ordered by subset inclusion, is denoted by $Conf(P)$.

Prime algebraic domains. A preordered or partially ordered set $\langle D, \sqsubseteq \rangle$ will be often denoted simply as D, by omitting the (pre)order relation. Given an element $x \in D$, we write $\downarrow x$ to denote the set $\{y \in D \mid y \sqsubseteq x\}$. Given a subset $X \subseteq D$, the *least upper bound* and *greatest lower bound* of X, when they exist, are denoted by $\bigsqcup X$ and $\bigsqcap X$, respectively. A subset $X \subseteq D$ is *compatible*, written $\uparrow X$, if there exists an upper bound $d \in D$ for X (i.e., $x \sqsubseteq d$ for all $x \in X$). It is *pairwise compatible* if $\uparrow \{x, y\}$ (often written $x \uparrow y$) for all $x, y \in X$. A subset $X \subseteq D$ is *directed* if any finite subset of X has an upper bound in X. The partial order D is *complete* (CPO) if any directed subset of X has a least upper bound in D.

Let D be a CPO. Recall that an element $e \in D$ is *compact* if for any directed set $X \subseteq D$, $e \sqsubseteq \bigsqcup X$ implies $e \sqsubseteq x$ for some $x \in X$. The set of *compact* elements of D is denoted by $\mathsf{K}(D)$.

DEFINITION 25 (PRIME ALGEBRAIC FINITARY COHERENT POSET) *A partial order D is called* coherent *(*pairwise complete*) if for all pairwise compatible $X \subseteq D$, there exists the least upper bound $\bigsqcup X$ of X in D.*

A complete prime *of D is an element $p \in D$ such that, for any compatible $X \subseteq D$, if $p \sqsubseteq \bigsqcup X$ then $p \sqsubseteq x$ for some $x \in X$. The set of complete primes of D is denoted by $Pr(D)$. The partial order D is called* prime algebraic *if for any element $d \in D$ we have $d = (\bigsqcup \downarrow d \cap Pr(D))$. The set $\downarrow d \cap Pr(D)$ of complete primes of D below d will be denoted $Pr(d)$. We say that D is* finitary *if for each compact element $e \in \mathsf{K}(D)$ the set $\downarrow e$ is finite.*

Coherent, prime algebraic, finitary partial orders will be referred to as (Winskel's) domains.

Being not expressible as the least upper bound of other elements, the complete primes of D can be seen as elementary indivisible pieces of information (events). Thus prime algebraicity expresses the fact that any element can be obtained by composing these elementary blocks of information.

The definition of morphism between domains is based on the notion of immediate precedence. Given a domain D and two distinct elements $d \neq d' \in D$ we say that d is an *immediate predecessor* of d', written $d \prec d'$ if

$$d \sqsubseteq d' \ \wedge \forall d'' \in D. \ (d \sqsubseteq d'' \sqsubseteq d' \ \Rightarrow \ d'' = d \ \vee \ d'' = d')$$

Moreover we write $d \preceq d'$ if $d \prec d'$ or $d = d'$. According to the informal interpretation of domain elements sketched above, $d \prec d'$ intuitively means that d' is obtained from d by adding a quantum of information. Domain morphisms are required to preserve such relation.

DEFINITION 26 (CATEGORY **Dom**) *Let D_0 and D_1 be domains. A* domain morphism $f : D_0 \to D_1$ *is a function, such that:*

- $\forall x,y \in D_0$, *if* $x \preceq y$ *then* $f(x) \preceq f(y)$. ($\preceq$-preserving)
- $\forall X \subseteq D_0$, *X pairwise compatible,* $f(\bigsqcup X) = \bigsqcup f(X)$; (Additive)
- $\forall X \subseteq D_0$, $X \neq \emptyset$ *and compatible,* $f(\bigsqcap X) = \bigsqcap f(X)$; (Stable)

We denote by **Dom** *the category having domains as objects and domain morphisms as arrows.*

Relating prime event structures and domains. Both event structures and domains can be seen as models of systems where computations are built out from atomic pieces. Formalising this intuition, in [41] the category **Dom** is shown to be equivalent to the category **PES**, the equivalence being established by two functors $\mathscr{L}$: **PES** $\to$ **Dom** and $\mathscr{P}$: **Dom** $\to$ **PES**

$$\mathbf{PES} \underset{\mathscr{L}}{\overset{\mathscr{P}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\sim}}}} \mathbf{Dom}$$

The functor $\mathscr{L}$ associates to each PES the poset $Conf(P)$ of its configurations which can be shown to be a domain. The image via $\mathscr{L}$ of a PES-morphism $f : P_0 \to P_1$ is the obvious extension of f to sets of events.

The definition of the functor $\mathscr{P}$, mapping domains back to PES's requires the introduction of the notion of prime interval.

DEFINITION 27 (PRIME INTERVAL) *Let $\langle D, \sqsubseteq \rangle$ be a domain. A* prime interval *is a pair $[d,d']$ of elements of D such that $d \prec d'$. Let us define*

$$[c,c'] \leq [d,d'] \quad \textit{if } (c = c' \sqcap d) \ \wedge \ (c' \sqcup d = d'),$$

and let $\sim$ be the equivalence obtained as the transitive and symmetric closure of (the preorder) $\leq$.

The intuition that a prime interval represents a pair of elements differing only for a "quantum" of information is confirmed by the fact that there exists a bijective correspondence between $\sim$-classes of prime intervals and complete primes of a domain D (see [31]). More precisely, the map

$$[d,d']_\sim \mapsto p,$$

where p is the only element in $Pr(d') - Pr(d)$, is an isomorphism between the $\sim$-classes of prime intervals of D and the complete primes $Pr(D)$ of D, whose inverse is the function:

$$p \mapsto [\bigsqcup\{c \in D \mid c \sqsubset p\}, p]_\sim.$$

The above machinery allows us to give the definition of the functor $\mathscr{P}$ "extracting" an event structure from a domain.

DEFINITION 28 (FROM DOMAINS TO PES'S) *The functor $\mathscr{P}$:* **Dom** $\to$ **PES** *is defined as follows:*

- *given a domain D, $\mathscr{P}(D) = \langle Pr(D), \leq, \# \rangle$ where*

$$p \leq p' \quad \textit{iff} \quad p \sqsubseteq p' \qquad \textit{and} \qquad p\#p' \quad \textit{iff} \quad \neg(p \uparrow p');$$

- *given a domain morphism $f : D_0 \to D_1$, the morphism $\mathscr{P}(f) : \mathscr{P}(D_0) \to \mathscr{P}(D_1)$ is the function:*

$$\mathscr{P}(f)(p_0) = \begin{cases} p_1 & \text{if } p_0 \mapsto [d_0, d_0']_\sim, \ f(d_0) \prec f(d_0') \\ & \text{and } [f(d_0), f(d_0')]_\sim \mapsto p_1; \\ \\ \bot & \text{otherwise, i.e., if } f(d_0) = f(d_0'). \end{cases}$$

3.2 Asymmetric event structures and domains

Asymmetric event structures have been introduced in [6] as a generalisation of prime event structures where the conflict relation is allowed to be non-symmetric. Formally, an *asymmetric event structure (*AES*)* is a triple $G = \langle E, \leq, \nearrow \rangle$, where E is a set of events, $\leq$ is the causality relation and $\nearrow$ is a binary relation on E called *asymmetric conflict*.

The notion of configuration extends smoothly to AES's, the main difference being the fact that the computational order between configurations is not simply set-inclusion. In fact, a configuration C can be extended with an event e' only if for any event $e \in C$, it does not hold that $e' \nearrow e$ (since, in this case, e would disable e').

The set of configurations of an AES with such a computational order is a domain. The corresponding functor from the category **AES** of asymmetric event structures to category **Dom** has a left adjoint which maps each domain to the corresponding prime event structure (each PES can be seen as a special AES). Hence Winskel's equivalence between **PES** and **Dom** generalises to a coreflection between **AES** and **Dom**.

4 Inhibitor event structures

This section introduces the class of event structures that we consider adequate for modelling the complex phenomena which arise in the dynamics of inhibitor nets. Furthermore we establish a connection between IES's and domains, by showing that the equivalence between **PES** and **Dom** generalises to the existence of a categorical coreflection between **IES** and **Dom**. We finally study the problem of removing the non-executable events from an IES, by characterising the full subcategory $\mathbf{IES^e}$, consisting of the IES's where all events are executable, as a coreflective subcategory of **IES**.

4.1 The category of inhibitor events structures

Let us fix some notational conventions. Given a set X, by $\mathbf{2}^X_{\mathit{fin}}$ we denote the set of finite subsets of X and by $\mathbf{2}^X_1$ the set of subsets of X of cardinality at most one (singletons or the empty set). In the sequel generic subsets of events will be denoted by upper case letters $A, B, \ldots$, and singletons or empty subsets by $a, b, \ldots$

DEFINITION 29 (PRE-INHIBITOR EVENT STRUCTURE) *A* pre-inhibitor event structure *(pre-IES) is a pair $I = \langle E, \vdash\!\circ \rangle$, where E is a set of* events *and $\vdash\!\circ \; \subseteq \mathbf{2}^E_1 \times E \times \mathbf{2}^E$ is a ternary relation called* disabling-enabling relation (DE-relation *for short*).

Informally, if $\mapstochar\!\!\!\multimap (\{e'\}, e, A)$ then the event e' inhibits the event e, which can be enabled again by one of the events in A. The first argument of the relation can be also the empty set $\emptyset$, $\mapstochar\!\!\!\multimap (\emptyset, e, A)$ meaning that the event e is inhibited in the initial state of the system. Moreover the third argument (the set of events A) can be empty, $\mapstochar\!\!\!\multimap (\{e'\}, e, \emptyset)$ meaning that there are no events that can re-enable e after it has been disabled by e'.

The DE-relation is sufficient to represent both causality and asymmetric conflict and thus, concretely, it is the only relation of a (pre-)IES. This is formalised in the definition below, which introduces generalised (or-) causality, asymmetric conflict and conflict (over sets of events) as relations derived from the DE-relation.

DEFINITION 30 (DEPENDENCY RELATIONS) *Let $I = \langle E, \mapstochar\!\!\!\multimap \rangle$ be a pre-*IES*. The relations of* (generalised) causality $< \subseteq \mathbf{2}^E \times E$, asymmetric conflict $\nearrow \subseteq E \times E$ *and* conflict $\# \subseteq \mathbf{2}^E_{fin}$ *are defined by the following set of rules:*

$$\frac{\mapstochar\!\!\!\multimap (\emptyset, e, A) \quad \#_p A}{A < e}\ (<1) \qquad \frac{A < e \quad \forall e' \in A.\ A_{e'} < e' \quad \#_p(\cup\{A_{e'} \mid e' \in A\})}{(\cup\{A_{e'} \mid e' \in A\}) < e}\ (<2)$$

$$\frac{\mapstochar\!\!\!\multimap (\{e'\}, e, \emptyset)}{e \nearrow e'}\ (\nearrow 1) \qquad \frac{e \in A < e'}{e \nearrow e'}\ (\nearrow 2) \qquad \frac{\#\{e, e'\}}{e \nearrow e'}\ (\nearrow 3)$$

$$\frac{e_0 \nearrow \ldots \nearrow e_n \nearrow e_0}{\#\{e_0, \ldots, e_n\}}\ (\#1) \qquad \frac{A' < e \quad \forall e' \in A'.\ \#(A \cup \{e'\})}{\#(A \cup \{e\})}\ (\#2)$$

where $\#_p A$ means that the events in A are pairwise conflicting, namely $\#\{e, e'\}$ for all $e, e' \in A$ with $e \neq e'$. We will use the infix notation for the binary conflicts, writing $e \# e'$ instead of $\#\{e, e'\}$. Moreover we will write $e < e'$ to indicate $\{e\} < e'$.

To understand the basic rule (<1) notice that if $\mapstochar\!\!\!\multimap (\emptyset, e, \{e'\})$ then the event e can be executed only after e' has fired. This is exactly what happens in a PES when e' causes e, or in symbols when $e' < e$. Here, more generally, if $\mapstochar\!\!\!\multimap (\emptyset, e, A)$ then we can imagine A as a set of disjunctive causes for e, since at least one of the events in A will appear in every history of the event e; intuitively we can think that e causally depends on $\bigvee A$. This generalisation of causality, restricted to the case in which the set A is pairwise conflicting (namely all distinct events in A are in conflict), is represented as $A < e$. Notice that under the assumption that A is pairwise conflicting, when $A < e$ *exactly one* event in A appears in each history of e. Therefore, in particular, for any event $e' \in A$, if e and e' are executed in the same computation then surely e' must precede e. Similar notions of or-causality have been studied in general event structures [41], flow event structures [7] and in bundle event structures [24, 25].

As for rule $(\nearrow 1)$, note that, if $\mapstochar\!\!\!\multimap (\{e'\}, e, \emptyset)$ then e can never follow e' in a computation since there are no events which can re-enable e after the execution of e'. Instead the converse order of execution is admitted, namely e *can* fire before e'. This situation is naturally interpreted as an *asymmetric conflict* between the two events and it is written $e \nearrow e'$. According to the "weak causality" interpretation of asymmetric conflict (if $e \nearrow e'$ then e precedes e' in all computations containing both events) rule $(\nearrow 2)$ imposes asymmetric conflict to include (also generalised) causality, by asking that $A < e$ implies $e' \nearrow e$ for all $e' \in A$.

In rule $(\#1)$ cycles of asymmetric conflict are used to define a notion of conflict on sets of events. If $e_0 \nearrow e_1 \ldots e_n \nearrow e_0$ then all such events cannot appear together in the same computation, since each one should precede the others. This fact is formalised via a conflict relation on sets of events $\#\{e_0, e_1 \ldots, e_n\}$. In particular, binary (symmetric) conflict corresponds to asymmetric conflict in both directions as expressed by rule $(\nearrow 3)$.

Rule (< 2) generalises the transitivity of the causality relation. If $A < e$ and for every event $e' \in A$ we can find a set of events $A_{e'}$ such that $A_{e'} < e'$, then the union of all such sets, namely $\cup\{A_{e'} \mid e' \in A\}$, can be seen as (generalised) cause of e, provided that it is pairwise conflicting. Observe that in particular, if $\{e'\} < e$ and $\{e''\} < e'$ then $\{e''\} < e$. Rule $(\#2)$ expresses a kind of hereditarity of the conflict with respect to causality. Suppose $A' < e$ and that any event $e' \in A'$ is in conflict with A, namely $\#(A \cup \{e'\})$ for any $e' \in A'$. Since by definition of $<$ the execution of e must be preceded by an event in A' we can conclude that also e is in conflict with A, i.e., $\#(A \cup \{e\})$. In particular by taking $A' = \{e'\}$ and $A = \{e''\}$ we obtain that if $\{e'\} < e$ and $\#\{e', e''\}$ then $\#\{e, e''\}$.

The intended meaning of the relations $<$, $\nearrow$ and $\#$ is summarised below.

$A < e$	means that in every computation where e is executed, there is exactly one event $e' \in A$ which is executed and it precedes e;
$e' \nearrow e$	means that in every computation where both e and e' are executed, e' precedes e;
$\#A$	means that there are no computations where all events in A are executed.

Notice that, due to the greater generality of IES's, the rules defining the dependency relations are more involved than in PES's and AES's, and it is not possible to give a separate definition of the various relations. In fact, according to rules (< 1) and (< 2) one can derive $A' < e$ only provided that the events in A' are pairwise conflicting. Asymmetric conflict is in turn induced both by generalised causality (rule $(\nearrow 2)$) and by conflict (rule $(\nearrow 3)$). Finally, the conflict relation is defined by using the asymmetric conflict (rule $(\#1)$) and it is inherited along causality (rule $(\#2)$). From a technical point of view, the set of rules in Definition 30 can be interpreted as a monotone operator over the lattice $\mathbf{2}^{\mathbf{2}^{E} \times E} \times \mathbf{2}^{E \times E} \times \mathbf{2}^{\mathbf{2}^{E}_{fin}}$, so that the relations defined by mutual recursion are, formally, the least fixed point of such operator.

Inhibitor event structures properly generalise prime and asymmetric event structures; moreover, when applied to (the encoding into IES's of) prime and asymmetric event structures the above rules induce the usual relations of causality and (asymmetric) conflict. For what regards the treatment of disjunctive or-causality (relation $<$) the presented rules resembles also the equivalence rules for bundle event structures in [25].

An inhibitor event structure is defined as a pre-IES where events related by the DE-relation satisfy a few further requirements suggested by the intended meaning of such relation. Furthermore the causality and asymmetric conflict relations must be induced "directly" by the DE-relation.

DEFINITION 31 (INHIBITOR EVENT STRUCTURE) *An* inhibitor event structure (IES) *is a pre-*IES $I = \langle E, \vdash\!\circ \rangle$ *satisfying, for all* $e \in E$, $a \in \mathbf{2}^{E}_{1}$ *and* $A \subseteq E$,

1. *if* $\vdash\!\circ\,(a, e, A)$ *then* $\#_p A$ *and* $\forall e' \in a.\ \forall e'' \in A.\ e' < e''$;

2. *if* $A < e$ *then* $\mapsto(\emptyset, e, A)$*;*

3. *if* $e \nearrow e'$ *then* $\mapsto(\{e'\}, e, \emptyset)$.

Note that we can have $\mapsto(\emptyset, e, \emptyset)$, meaning that event e can never be executed. In this case, by rule (< 1), we deduce $\emptyset < e$ and thus, by rule (#2), we have $\#\{e\}$, i.e., the event e is in conflict with itself. Similarly if $\mapsto(e, e', A)$, with $e \in A$, by condition (2) above, necessarily $e < e$ and thus the event e is not executable. In an analogous way, if $\mapsto(e, e, A)$ then $e \nearrow e$ and thus e is not executable.

We next define the category of IES's by introducing a notion of IES-morphism which, as discussed later, generalises both PES and AES-morphisms.

DEFINITION 32 (CATEGORY **IES**) *Let* $I_0 = \langle E_0, \mapsto_0 \rangle$ *and* $I_1 = \langle E_1, \mapsto_1 \rangle$ *be two* IES*'s. An* IES-morphism $f : I_0 \to I_1$ *is a partial function* $f : E_0 \to E_1$ *such that for all* $e_0, e_0' \in E_0$, $A_1 \subseteq E_1$, *assuming that* $f(e_0)$ *and* $f(e_0')$ *are defined:*

1. $f(e_0) = f(e_0') \wedge e_0 \neq e_0' \quad \Rightarrow \quad e_0 \#_0 e_0'$*;*

2. $A_1 < f(e_0) \quad \Rightarrow \quad \exists A_0 \subseteq f^{-1}(A_1).\ A_0 < e_0$*;*

3. $\mapsto_1(\{f(e_0')\}, f(e_0), A_1) \quad \Rightarrow \quad \exists A_0 \subseteq f^{-1}(A_1).\ \exists a_0 \subseteq \{e_0'\}.\ \mapsto_0(a_0, e_0, A_0)$.

We denote by **IES** *the category of inhibitor event structures and* IES*-morphisms.*

Condition (1) is the usual condition of event structure morphisms which allows one to confuse only conflicting branches of computations. As formally proved later in Proposition 35 condition (2) can be seen as a generalisation of the requirement of preservation of causes, namely of the property $\lfloor f(e) \rfloor \subseteq f(\lfloor e \rfloor)$, of PES (and AES) morphisms. Finally, condition (3), as it commonly happens for event structures morphisms, just imposes the preservation of computations by asking, whenever some events in the image are constrained in some way, that stronger constraints are present in the pre-image. More precisely suppose that $\mapsto_1(\{f(e_0')\}, f(e_0), A_1)$. Thus we can have a computation where $f(e_0')$ is executed first and $f(e_0)$ can be executed only after one of the events in A_1. Alternatively the computation can start with the execution of $f(e_0)$. According to condition (3), e_0 and e_0' are subject in I_0 to the same constraint of their images or, when $a_0 = \emptyset$ or $A_0 = \emptyset$, to stronger constraints selecting one of the possible orders of execution. It is worth stressing that, since $A_i < e_i$ can be equivalently expressed as $\mapsto(\emptyset, e_i, A_i)$, condition (2) is essentially a variation of (3), which is needed to cover the case in which the first argument of the DE-relation is the empty set.

The next proposition gives some useful properties of IES-morphisms, which are basically generalisations of analogous properties holding in the case of prime and asymmetric event structures.

PROPOSITION 33 *Let* I_0 *and* I_1 *be* IES*'s and let* $f : I_0 \to I_1$ *be an* IES*-morphism. For any* $e_0, e_0' \in E_0$,

1. *if* $f(e_0) < f(e_0')$ *then* $\exists A_0.\ e_0 \in A_0 < e_0'$ *or* $e_0 \# e_0'$*;*

2. *if* $f(e_0) \nearrow f(e_0')$ *then* $e_0 \nearrow e_0'$.

Proof . See the Appendix. In particular the above results are useful in showing that IES-morphisms are closed under composition and thus that category **IES** is well-defined.

PROPOSITION 34 *The* IES*-morphisms are closed under composition.*

Proof . See the Appendix.

The category **PES** of prime event structures can be viewed as a full subcategory of **IES**. This result substantiates the claim that IES's (and constructions on them) are a "conservative" extension of PES's.

PROPOSITION 35 (PRIME AND INHIBITOR EVENT STRUCTURES) *Let* $\mathscr{J}_i : \mathbf{PES} \to \mathbf{IES}$ *be the functor defined as follows. To any* PES $P = \langle E, \leq, \# \rangle$ *the functor* $\mathscr{J}_i$ *associates the* IES $\langle E, \vdash\!\circ \rangle$ *where the DE-relation is defined by* $\vdash\!\circ\, (\emptyset, e, \{e''\})$ *if* $e'' < e$ *and* $\vdash\!\circ\, (\{e'\}, e, \emptyset)$ *if* $e \# e'$*, and for any* PES *morphism* $f : P_1 \to P_2$ *its image* $\mathscr{J}_i(f)$ *is* f *itself. Then the functor* $\mathscr{J}_i$ *is a full embedding of* **PES** *into* **IES***.*

More generally, it is possible to show that the category of asymmetric event structures introduced in [6] fully embeds into **IES** (see [4]). Also (extended) bundle event structures [25] and prime event structures with possible events [35] can be seen as special classes of IES's. As we will discuss later, the categorical treatment of IES's and the results relating IES's and domains specialises to such event structure models.

4.2 Saturation of pre-IES's

Given a pre-IES I satisfying only condition (1) of Definition 31, it is always possible to "saturate" the relation $\vdash\!\circ$ in order to obtain an IES where the relations of causality and (asymmetric) conflict are exactly the same as in I. Intuitively, in a PES-like structure where only "direct" causality and conflict between events are given, the saturation would amount to taking the transitive closure of causality and to inherit conflict along causality. The DE-relation derived from the unfolding of an i-net will be not saturated, hence the saturation operation will play a central role in defining the IES semantics of an i-net (see Definition 55).

PROPOSITION 36 *Let* $I = \langle E, \vdash\!\circ \rangle$ *be a pre-*IES *satisfying condition (1) of Definition 31. Then* $\overline{I} = \langle E, \vdash\!\circ^{s} \rangle$*, where* $\vdash\!\circ^{s} = \vdash\!\circ \cup \{(\emptyset, e, A) \mid A < e\} \cup \{(\{e'\}, e, \emptyset) \mid e \nearrow e'\}$ *is a* IES*, called the* saturation *of* I*. Moreover the relations of causality, asymmetric conflict and conflict in* $\overline{I}$ *are the same as in* I*.*

The next technical lemma will be quite useful later to prove that some mappings between IES's are well-defined IES-morphisms (see Propositions 52 and 56 and Lemma 61). It singles out some sufficient conditions for a function between pre-IES's to be a well-defined IES-morphism between the IES's obtained by saturating them.

LEMMA 37 *Let* $I_i = \langle E_i, \vdash\!\circ_i \rangle$ $(i \in \{0,1\})$ *be pre-*IES*'s satisfying condition (1) of Definition 31, let* $\overline{I_i} = \langle E_i, \vdash\!\circ_i^{s} \rangle$*, and let* $<_i$*,* $\nearrow_i$ *and* $\#_i$ *be the relations of causality, asymmetric conflict and conflict in* I_i*. Let* $f : E_0 \to E_1$ *be a partial function such that for each* $e_0, e'_0 \in E_0$ *and* $A_1 \subseteq E_1$*:*

1. $f(e_0) = f(e'_0) \wedge e_0 \neq e'_0 \quad \Rightarrow \quad e_0 \#_0 e'_0$*;*

2. $\mapsto_1(\emptyset, f(e_0), A_1) \quad \Rightarrow \quad \exists A_0 \subseteq f^{-1}(A_1).\ A_0 <_0 e_0$;

3. $\mapsto_1(f(e'_0), f(e_0), \emptyset) \quad \Rightarrow \quad e_0 \nearrow_0 e'_0$;

4. $\mapsto_1(\{f(e'_0)\}, f(e_0), A_1) \wedge A_1 \neq \emptyset \ \Rightarrow\ \exists A_0 \subseteq f^{-1}(A_1). \exists a_0 \subseteq \{e'_0\}.\ \mapsto^s_0(a_0, e_0, A_0)$.

Then $f : \overline{I_0} \to \overline{I_1}$ is an IES*-morphism.*

Proof . See the Appendix.

4.3 The domain of configurations of inhibitor event structures

The domain associated to an IES is obtained by considering the family of its configurations with a suitable order. Since here computations involving the same events may be different from the point of view of causality, a configuration is not uniquely identified as a set of events, but some additional information has to be added which plays a basic role also in the definition of the order on configurations. More concretely, a configuration of an IES is a set of events endowed with a *choice relation* (playing a role similar to assignments for occurrence i-nets) which chooses among the possible different orders of execution of events constrained by the DE-relation.

Consider a set of events C of an inhibitor event structure I, and suppose $e', e, e'' \in C$ and $\mapsto(\{e'\}, e, A)$ for some A, with $e'' \in A$. We already noticed that in this case there are two possible orders of execution of the three events (either $e; e'; e''$ or $e'; e''; e$), which cannot be identified from the point of view of causality. A choice relation for C must choose one of them by specifying that e precedes e' or that e'' precedes e. To ease the definition of the notion of choice relation, we first introduce, for a given set of events C, the set *choices*(C), a relation on C which "collects" *all* the possible precedences between events induced by the DE-relation. A choice relation for C is then defined as suitable subset of *choices*(C). To ensure that all the events in the configuration are executable in the specified order, the choice relation is also required to satisfy suitable properties of acyclicity and finitariness.

DEFINITION 38 (CHOICE) *Let $I = \langle E, \mapsto \rangle$ be an* IES *and let $C \subseteq E$. We denote by choices(C) the set*

$$\{(e, e') \mid \exists A.\ \mapsto_C(\{e'\}, e, A)\} \cup \{(e'', e) \mid \exists a.\ \exists A.\ \mapsto_C(a, e, A) \wedge\ e'' \in A\} \subseteq C \times C,$$

where the restriction of $\mapsto(,,)$ to C is defined by $\mapsto_C(a, e, A)$ if and only if $\mapsto(a, e, A')$ for some A', with $e \in C$, $a \subseteq C$ and $A = A' \cap C$.
A choice *for C is a relation $\hookrightarrow_C \subseteq$ choices(C) such that*

1. *if $\mapsto_C(a, e, A)$ then $\exists e' \in a.\ e \hookrightarrow_C e'$ or $\exists e'' \in A.\ e'' \hookrightarrow_C e$;*

2. *$\hookrightarrow_C$ is* acyclic*;*

3. *$\forall e \in C.\ \{e' \in C \mid e' \hookrightarrow^*_C e\}$ is* finite.

Condition (1) intuitively requires that whenever the DE-relation permits two possible orders of execution, the relation $\hookrightarrow_C$ chooses one of them. The fact that $\hookrightarrow_C \subseteq$ *choices*(C) ensures that $\hookrightarrow_C$ imposes precedences only between events involved in the DE-relation. Conditions (2) and (3) guarantee that the precedences specified by $\hookrightarrow_C$ do not give rise to cyclic situations and that each event must be preceded only by finitely many others. Notice that the acyclicity of $\hookrightarrow_C$ ensures that exactly one of the two possible choices in condition (1), namely either $\exists e' \in a.\ e \hookrightarrow_C e'$ or $\exists e'' \in A.\ e'' \hookrightarrow_C e$ is taken. Otherwise, if $e \hookrightarrow_C e'$ and $e'' \hookrightarrow_C e$, since necessarily $e' < e''$ and thus $e' \hookrightarrow_C e''$, the relation $\hookrightarrow_C$ would be cyclic. It is worth observing that conditions (2) and (3) can be equivalently rephrased by saying that $\hookrightarrow_C^*$ is a finitary partial order.

Configurations of PES's (and AES's, see [6]) are required to be conflict free and downward closed with respect to causality. The following proposition shows that the property of admitting a choice implies a generalisation of causal closedness and conflict freeness. Furthermore any choice certainly agrees with the asymmetric conflict (since both relations impose an order of execution on events).

PROPOSITION 39 *Let $I = \langle E, \vdash\!\circ \rangle$ be an* IES *and let $C \subseteq E$ be a subset of events such that there exists a choice $\hookrightarrow_C$ for C. Then*

1. *for any $e \in C$, if $A < e$ then $A \cap C \neq \emptyset$;*
2. *$\nearrow_C \subseteq \hookrightarrow_C$;*
3. *for any $A \subseteq C$ it is not the case that $\#A$.*

Proof. 1. Observe that if $A < e$, by definition of IES, $\vdash\!\circ (\emptyset, e, A)$. Therefore, if $A \cap C = \emptyset$ then we would have $\vdash\!\circ_C(\emptyset, e, \emptyset)$. Therefore no relation over C could be a choice, since condition (1) of Definition 38 could not be satisfied.

2. Consider $C \subseteq E$ and $e, e' \in C$. If $e \nearrow e'$ then, by definition of IES, $\vdash\!\circ (\{e'\}, e, \emptyset)$ and thus $\vdash\!\circ_C(\{e'\}, e, \emptyset)$. Therefore, if $\hookrightarrow_C$ is a choice for C, by condition (1) in Definition 38, necessarily $e \hookrightarrow_C e'$.

3. Let $A \subseteq C$ and suppose that $\#A$. Then it is easy to show that C contains a cycle of asymmetric conflict, and thus by point (2), any choice for C would be cyclic as well, contradicting the definition.

The proof of the fact that if $\#A$ for some $A \subseteq C$ then C contains a cycle of asymmetric conflict proceeds by induction on the height of the derivation of $\#A$. The base case in which the last rule in the derivation is $(\#1)$, namely

$$\frac{e_0 \nearrow \ldots \nearrow e_n \nearrow e_0}{\#\{e_0, \ldots, e_n\}} \quad (\#1)$$

is trivial. Suppose instead that the last rule in the derivation is $(\#2)$, namely

$$\frac{A'' < e \quad \forall e' \in A''.\ \#(A' \cup \{e'\})}{\#(A' \cup \{e\})} \quad (\#2)$$

In this case, by point (1), there exists $e'' \in A'' \cap C$. Since $\#(A' \cup \{e''\})$ by the second premise of the rule, and $A' \cup \{e''\} \subseteq C$ we conclude by inductive hypothesis. □

A configuration of an IES is now introduced as a set of events endowed with a choice relation. Proposition 39 above shows how this definition generalises the notion of PES and AES configuration.

DEFINITION 40 (CONFIGURATION) *Let* $I = \langle E, \vdash\!\circ \rangle$ *be an* IES*. A* configuration *of* I *is a pair* $\langle C, \hookrightarrow_C \rangle$*, where* $C \subseteq E$ *is a set of events and* $\hookrightarrow_C \subseteq C \times C$ *is a* choice *for* C*.*

In the sequel, with abuse of notation, we will often denote a configuration and the underlying set of events with the same symbol C, referring to the corresponding choice relation as $\hookrightarrow_C$.

As the reader probably noticed, the notions of choice and that of assignment are strictly related. Formally, as we will see later, each occurrence i-net N can be mapped to an IES and, for any subset $X \subseteq T$, an assignment ρ for N such that $X = \lfloor X \rfloor_\rho$ and $\nearrow_\rho$ is acyclic and finitary on X, uniquely determines a choice turning X in a configuration of the IES corresponding to N.

We already know that the existence of a choice implies the causal closedness and conflict freeness of a configuration. Moreover, if C is a configuration, given any $e \in C$ and $A < e$, not only $A \cap C \neq \emptyset$, but since by definition of $<$ necessarily $\#_p A$, we have that $A \cap C$ contains exactly one event. More generally, for the same reason, if C is a configuration and $\vdash\!\circ (a, e, A)$ for some $e \in C$, then $A \cap C$ contains at most one element, and if it is non-empty then $a \subseteq C$. The last assertion is obvious if $a = \emptyset$, while if $a = \{e'\}$ it follows from Proposition 39.(1), recalling that $e' < e''$ for all $e'' \in A$.

The next technical proposition shows a kind of maximality property of the choice relation for a configuration. It states that if a choice for C relates two events, then any other choice for C must establish an order between such events. Consequently two compatible choices on the same set of events must coincide.

PROPOSITION 41 *Let* $\langle C_i, \hookrightarrow_{C_i} \rangle$ *for* $i \in \{1, 2\}$ *be configurations of an* IES I*.*

1. *If* $e, e' \in C_1 \cap C_2$ *and* $e \hookrightarrow_{C_1} e'$ *then* $e \hookrightarrow_{C_2} e'$ *or* $e' \hookrightarrow^*_{C_2} e$*.*
2. *If* $C_1 = C_2$ *and* $\hookrightarrow^*_{C_1} \subseteq \hookrightarrow^*_{C_2}$ *then* $\hookrightarrow_{C_1} = \hookrightarrow_{C_2}$*, namely the two configurations coincide.*

Proof. See the Appendix.

The next definition introduces a computational order on the set of configurations of an IES.

DEFINITION 42 (EXTENSION) *Let* $I = \langle E, \vdash\!\circ \rangle$ *be an* IES *and let* C *and* C' *be configurations of* I*. We say that* C' extends C *and we write* $C \sqsubseteq C'$*, if*

1. $C \subseteq C'$*;*
2. $\forall e \in C.\ \forall e' \in C'.\ e' \hookrightarrow_{C'} e \quad \Rightarrow \quad e' \in C$*;*
3. $\hookrightarrow_C \subseteq \hookrightarrow_{C'}$*.*

The poset of all configurations of I*, ordered by extension, is denoted by* $Conf(I)$*.*

The extension relation defined on IES's configurations is a generalisation of that introduced in [6] for AES's. The basic idea is that a configuration C can be extended only by adding events which are not supposed to happen before other events already in C, as expressed by condition (2). Moreover the extension relation takes into account the choice relations of the two configurations. Intuitively, condition (3) serves to ensure, together with (2), that the past history of events in C remains the same in C'.

The history of an event in a configuration C is formally defined as a suitable subconfiguration of C.

DEFINITION 43 (HISTORY) *Let I be an* IES *and let $C \in Conf(I)$ be a configuration. For any $e \in C$ we define the* history *of e in C as the configuration $\langle C[\![e]\!], \hookrightarrow_{C[\![e]\!]} \rangle$, where $C[\![e]\!] = \{e' \in C \mid e' \hookrightarrow_C^* e\}$ and $\hookrightarrow_{C[\![e]\!]} = \hookrightarrow_C \cap (C[\![e]\!] \times C[\![e]\!])$.*

It is not difficult to see that $\langle C[\![e]\!], \hookrightarrow_{C[\![e]\!]} \rangle$ is a well-defined configuration. The only fact that is not obvious is the validity of condition (1) in the definition of choice (Definition 38). Now, if $\vdash\!\circ_{C[\![e]\!]}(a, e', A)$ then $\vdash\!\circ_C(a, e', A')$ with $a \subseteq C[\![e]\!]$, $e' \in C[\![e]\!]$ and $A = A' \cap C[\![e]\!]$. Being C a configuration, it must be $e' \hookrightarrow_C e_0$ for $e_0 \in a$ or $e_1 \hookrightarrow_C e'$ for some $e_1 \in A'$. In the first case, $e_0 \in a \subseteq C[\![e]\!]$ and thus $e' \hookrightarrow_{C[\![e]\!]} e_0$, while in the second case, since $e' \in C[\![e]\!]$, by definition of history we must have $e_1 \in C[\![e]\!]$, thus $e_1 \hookrightarrow_{C[\![e]\!]} e'$.

Recall that, by definition, the reflexive and transitive closure of a choice is a finitary partial order, and thus each history $C[\![e]\!]$ is a *finite* configuration. Furthermore, it is easy to see that $C[\![e]\!] \sqsubseteq C$.

The next lemma shows that, given a pairwise compatible set of configurations $X \subseteq Conf(I)$ of an IES I, its greatest lower bound and least upper bound can be computed componentwise. Furthermore, for any C_1 and C_2 in X, if they contain a common event e, then the history of e in the two configurations is the same, namely $C_1[\![e]\!] = C_2[\![e]\!]$.

LEMMA 44 *Let $X \subseteq Conf(I)$ be a pairwise compatible set of configurations of an* IES *I and let $C_1, C_2 \in X$. Then*

1. *if $e \hookrightarrow_{C_1}^* e'$ and $e' \in C_2$ then $e \in C_2$ and $e \hookrightarrow_{C_2}^* e'$;*
2. *if $e \in C_1 \cap C_2$ then $C_1[\![e]\!] = C_2[\![e]\!]$;*
3. *$C_1 \sqcap C_2 = C_1 \cap C_2$, with $\hookrightarrow_{C_1 \cap C_2} = \hookrightarrow_{C_1} \cap \hookrightarrow_{C_2}$;*
4. *the least upper bound of X exists, and it is given by*

$$\bigsqcup X = \langle \bigcup_{C \in X} C, \bigcup_{C \in X} \hookrightarrow_C \rangle.$$

Proof. See the Appendix.

By exploiting such properties, we can prove that the poset of configurations of an IES has the desired algebraic structure.

THEOREM 45 (CONFIGURATIONS FORM A DOMAIN) *Let I be an* IES*. Then $\langle Conf(I), \sqsubseteq \rangle$ is a domain. The complete primes of $Conf(I)$ are the possible histories of events in I, i.e.*

$$Pr(Conf(I)) = \{C[\![e]\!] \mid C \in Conf(I), e \in C\}.$$

Proof. Let us start by showing that for each $C \in Conf(I)$ and $e \in C$, the configuration $C[\![e]\!]$ is a complete prime element. Suppose $C[\![e]\!] \sqsubseteq \bigsqcup X$ for $X \subseteq Conf(I)$ pairwise compatible. Therefore there exists $C_1 \in X$ such that $e \in C_1$. Since C_1 and $C[\![e]\!]$ are bounded by $\bigsqcup X$, by Lemma 44.(2), $C[\![e]\!] = C_1[\![e]\!]$. Observing that $C_1[\![e]\!] \sqsubseteq C_1$, it follows that, as desired, $C[\![e]\!] \sqsubseteq C_1$.

Now, by a set-theoretical calculation exploiting the definition of history (Definition 43) and the characterisation of the least upper bound in Lemma 44, we obtain

$$C = \bigsqcup_{e \in C} C[\![e]\!] = \bigsqcup Pr(C).$$

This shows that $Conf(I)$ is prime algebraic and that $Pr(Conf(I)) = \{C[\![e]\!] \mid C \in Conf(I), e \in C\}$.

The fact that $Conf(I)$ is coherent has been proved in Lemma 44.(4). Finally, the finitariness of $Conf(I)$ follows from prime algebraicity and the fact that $C[\![e]\!]$ is finite for each $C \in Conf(I)$ and $e \in C$. $\Box$

We remark that if P is a PES and $I = \mathscr{J}_i(P)$ is its encoding into IES's, then for each configuration of I the choice relation is uniquely determined as the restriction of causality to the configuration. Therefore the domain of configurations $Conf(I)$ defined in this section coincides with the domain $Conf(P)$ as defined by Winskel. A similar situation arises for the IES encoding of asymmetric event structures [6], PES with possible events [34] and (extended) bundle event structures [25].

4.4 A coreflection between IES and Dom

To prove that the construction which associates the domain of configurations to an IES lifts to a functor from **IES** to **Dom**, a basic result is the fact that IES-morphisms preserve configurations. Observe that since configurations are not simply sets of events it is not completely obvious, a priori, what should be the image of a configuration through a morphism. Let $f : I_0 \to I_1$ be an IES-morphism and let C_0 be a configuration of I_0. According to the intuition underlying IES (and general event structure) morphisms, we expect that any possible execution of the events in C_0 can be simulated in $f(C_0)$. But the converse implication is not required to hold, namely the level of concurrency in $f(C_0)$ may be higher. For instance we can map two causally related events $e_0 \leq e_1$ to a pair of concurrent events. Hence we cannot pretend that the whole image of the choice relation of C_0 is a choice for $f(C_0)$, but just that there is a choice for $f(C_0)$ included in such image. By the properties of choices, there is only one choice on $f(C_0)$ included in the image of $\hookrightarrow_{C_0}$, which is obtained as the intersection of the image of $\hookrightarrow_{C_0}$ with $choices(f(C_0))$.

Given a function $f : X \to Y$ and a relation $r \subseteq X \times X$, we will denote by $f(r)$ the relation in Y defined as $f(r) = \{(y,y') \mid \exists (x,x') \in r.\ f(x) = y \ \wedge\ f(x') = y'\}$.

LEMMA 46 *Let $f : I_0 \to I_1$ be an* IES*-morphism and let $\langle C_0, \hookrightarrow_0\rangle \in Conf(I_0)$. Then the pair $\langle C_1, \hookrightarrow_1\rangle$ with $C_1 = f(C_0)$ and $\hookrightarrow_1 = f(\hookrightarrow_0) \cap choices(f(C_0))$, namely the unique choice relation on C_1 included in $f(\hookrightarrow_{C_0})$, is a configuration in I_1. Moreover the function $f^* : Conf(I_0) \to Conf(I_1)$ which associates to each configuration C_0 the configuration C_1 defined as above, is a domain morphism.*

Proof. See the Appendix.

The previous lemma implies that the construction taking an IES into its domain of configurations can be viewed as a functor.

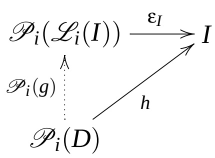
PROPOSITION 47 *There exists a functor $\mathscr{L}_i$: **IES** $\to$ **Dom** defined as $\mathscr{L}_i(I) = Conf(I)$ for each* IES *I and $\mathscr{L}_i(f) = f^*$ for each* IES*-morphism $f : I_0 \to I_1$.*

A functor going back from domains to IES's, namely $\mathscr{P}_i$: **Dom** $\to$ **IES** can be obtained simply as the composition of the functor $\mathscr{P}$: **Dom** $\to$ **PES**, defined by Winskel, with the full embedding $\mathscr{J}_i$ of **PES** into **IES** discussed in Proposition 35. The functor $\mathscr{P}_i$ is left adjoint to $\mathscr{L}_i$ and thus they establish a coreflection between **IES** and **Dom**.

THEOREM 48 (COREFLECTION BETWEEN **IES** AND **Dom**) *The functor* $\mathscr{P}_i : \mathbf{Dom} \to \mathbf{IES}$ *is left adjoint to* $\mathscr{L}_i : \mathbf{IES} \to \mathbf{Dom}$. *The counit of the adjunction at an* IES I *is the function* $\varepsilon_I : \mathscr{P}_i \circ \mathscr{L}_i(I) \to I$, *mapping each history of an event e into the event e itself, i.e.,* $\varepsilon_I(C[\![e]\!]) = e$, *for all* $C \in Conf(I)$ *and* $e \in C$.

Proof (Sketch). Let I be an IES and let $\varepsilon_I : \mathscr{P}_i(\mathscr{L}_i(I)) \to I$ be the function defined as $\varepsilon_I(C[\![e]\!]) = e$, for all $C \in Conf(I)$ and $e \in C$. It is not difficult to prove that ε_I is a well-defined IES-morphism (see the full proof in the Appendix).

We have to show that given any domain $(D, \sqsubseteq)$ and IES-morphism $h : \mathscr{P}_i(D) \to I$, there is a unique domain morphism $g : D \to \mathscr{L}_i(I)$ such that the following diagram commutes:



The morphism $g : D \to \mathscr{L}_i(I)$ can be defined as follows. Given $d \in D$, observe that $C_d = \langle Pr(d), \sqsubset_{Pr(d)} \rangle$ is a configuration of $\mathscr{P}_i(D)$, where $\sqsubset_{Pr(d)} = \sqsubset \cap (Pr(d) \times Pr(d))$. Therefore we can define

$$g(d) = h^*(C_d).$$

The fact that $h^*(C_d)$ is a configuration in I and thus an element of $\mathscr{L}_i(I)$, follows from Lemma 46. Moreover g is a domain morphism. In fact it is $\preceq$*-preserving*, *Additive* and *Stable* (see the full proof in the Appendix).

The rest of the proof essentially relies on a general result which holds of any domain morphism $f : D \to \mathscr{L}_i(I)$ having as target the domain of configurations of an IES: for all $p \in Pr(D)$, $|\, f(p) - \bigcup f(Pr(p) - \{p\}) \,| \leq 1$ and

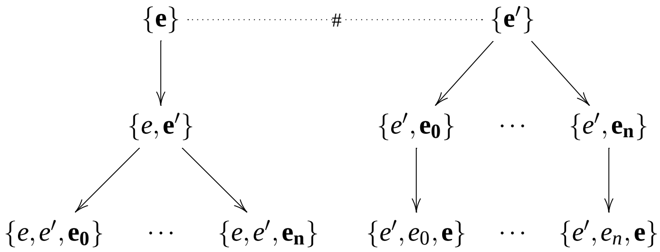
$$\mathscr{P}_i(f)(p) = \begin{cases} \bot & \text{if } f(p) - \bigcup f(Pr(p) - \{p\}) = \emptyset \\ f(p)[\![e]\!] & \text{if } f(p) - \bigcup f(Pr(p) - \{p\}) = \{e\} \end{cases}$$

Exploiting such result, the fact that morphism g defined as above makes the diagram commute and its uniqueness follow as easy consequences. □

It is worth stressing that the above result, together with Winskel's equivalence between the category **Dom** of domains and the category **PES** of prime event structures, allows one to translate an IES I into a PES $\mathscr{P}(\mathscr{L}_i(I))$.

COROLLARY 49 *The functor* $\mathscr{J}_i : \mathbf{PES} \to \mathbf{IES}$ *is left adjoint of* $\mathscr{P} \circ \mathscr{L}_i : \mathbf{IES} \to \mathbf{PES}$. *The unit will be denoted by* $\kappa : 1 \to \mathscr{P} \circ \mathscr{L}_i \circ \mathscr{J}_i$.

The universal characterisation of the construction intuitively ensures that the obtained PES is the "best approximation" of I in the category **PES**. By the characterisation of the complete prime elements in the domain of configurations (see Theorem 45) we have that the events in $\mathscr{P}(\mathscr{L}_i(I))$ are the possible histories of the events in I. The picture below depicts the PES corresponding to a basic IES containing the events $\{e, e', e_0, \ldots, e_n\}$ related by the DE-relation as $\vdash\!\circ\, (\{e'\}, e, \{e_0, \ldots, e_n\})$. We explicitly represent a history of an event e as a set of events, where e appears in boldface style.



As observed before, asymmetric event structures [6], (extended) bundle event structures [25], prime event structures with possible events [35] can be seen as subcategories of **IES**. Let **ES** be any of such subcategories. Since **ES** includes all the prime event structures, it is easy to prove that the coreflection between **IES** and **Dom** restricts to a coreflection between **ES** and **Dom** [4].

4.5 Removing non-executable events

The non-executability of events in an IES is not completely captured by the proof system of Definition 30, in the sense that we cannot derive $\#\{e\}$ for every non-executable event. Here we propose a semantic approach to rule out unused events from an IES, namely we simply remove from a given IES all events which do not appear in any configuration. Nicely, this can be done functorially and the subcategory $\mathbf{IES^e}$ of IES's where all events are executable turns out to be a coreflective subcategory of **IES**. Moreover, the coreflection between **IES** and **Dom** restricts to a coreflection between $\mathbf{IES^e}$ and **Dom**.

We start defining the subcategory of IES's where all events are executable.

DEFINITION 50 *We denote by* $\mathbf{IES^e}$ *the full subcategory of* **IES** *consisting of the* IES*'s* $I = \langle E, \vdash\!\circ \rangle$ *such that for any* $e \in E$ *there exists* $C \in Conf(I)$ *with* $e \in C$.

Any IES is turned into an $\mathbf{IES^e}$ object by forgetting the events which do not appear in any configuration. The next definition introduces the functor $\Psi : \mathbf{IES} \to \mathbf{IES^e}$ performing such construction.

DEFINITION 51 *We denote by* $\Psi : \mathbf{IES} \to \mathbf{IES^e}$ *the functor mapping each* IES *I into the* $\mathbf{IES^e}$ *object* $\Psi(I) = \overline{\langle \psi(E), \vdash\!\circ_{\psi(E)} \rangle}$*, where* $\overline{(.)}$ *denotes saturation (see Proposition 36) and* $\psi(E)$ *is the set of executable events in I, namely*

$$\psi(E) = \{e \in E \mid \exists C \in Conf(I).\ e \in C\}.$$

Moreover if $f : I_0 \to I_1$ *is an* IES*-morphism then* $\Psi(f) = f_{|\psi(E_0)}$*. With* $\mathcal{J}_{ies} : \mathbf{IES^e} \to \mathbf{IES}$ *we denote the inclusion.*

The fact that $\Psi(I)$ is an $\mathbf{IES^e}$ object follows easily from its definition. The well-definedness of $\Psi(f)$ for any IES-morphism f is basically a consequence of the fact that, by Lemma 46, an IES-morphism preserves configurations and thus also executable events.

PROPOSITION 52 *Let I_0 and I_1 be* IES*'s and let $f : I_0 \to I_1$ be an* IES*-morphism. Then $\Psi(f) : \Psi(I_0) \to \Psi(I_1)$, defined as above, is an* IES*-morphism. Hence Ψ is a well-defined functor.*

Proof . See the Appendix.

It is easy to verify that, if I is a $\mathbf{IES^e}$ object and I' an arbitrary IES, then any IES-morphism $f : I \to \Psi(I')$ is also a morphism $f : I \to I'$. This implies that the inclusion of $\mathbf{IES^e}$ into **IES** is left adjoint to Ψ, i.e., $\Psi \vdash \mathscr{J}_{ies}$, and thus that $\mathbf{IES^e}$ is a coreflective subcategory of **IES**.

PROPOSITION 53 (RELATING **IES** AND $\mathbf{IES^e}$) $\Psi \vdash \mathscr{J}_{ies}$

Finally observe that the functor $\mathscr{P}_i : \mathbf{Dom} \to \mathbf{IES}$ maps each domain into the encoding of a PES, which is clearly an object in $\mathbf{IES^e}$. Therefore it is easy to prove that the coreflection between **IES** and **Dom** restricts to a coreflection between $\mathbf{IES^e}$ and **Dom**.

COROLLARY 54 *Let $\mathscr{P}_i^{\mathbf{e}} : \mathbf{Dom} \to \mathbf{IES^e}$ and $\mathscr{L}_i^{\mathbf{e}} : \mathbf{IES^e} \to \mathbf{Dom}$ denote the restrictions of the functors $\mathscr{P}_i$ and $\mathscr{L}_i$. Then $\mathscr{P}_i^{\mathbf{e}} \dashv \mathscr{L}_i^{\mathbf{e}}$.*

5 Event structure semantics for i-nets

To provide an event structure and a domain semantics for i-nets we investigate the relationship between occurrence i-nets and inhibitor event structures. The kind of dependencies arising among transitions in an occurrence i-net can be represented naturally by the DE-relation, and therefore the IES corresponding to an occurrence i-net is obtained by forgetting the places and taking the transitions of the net as events. Furthermore morphisms between occurrence i-nets restrict to morphisms between the corresponding IES's, and therefore the semantics can be given via a functor $\mathscr{E}_i : \mathbf{O\text{-}IN} \to \mathbf{IES}$. The construction, when applied to an executable occurrence i-net, restricts to a functor $\mathscr{E}_i^{\mathbf{e}} : \mathbf{O\text{-}IN^e} \to \mathbf{IES^e}$.

When combined with the coreflection between **IES** and **Dom** and with Winskel's equivalence between **Dom** and **PES**, this result allows us to obtain a functor from **O-IN** to **PES**. Answering a question left open in [5], we show that such functor admits a left adjoint providing a coreflection between **O-IN** and **PES**.

The analogy with contextual nets breaks for the fact that, while in [6] the coreflection between **O-CN** and **PES** is expressed as the composition of two coreflections, between **O-CN** and the category **AES** of asymmetric event structures and between **AES** and **PES**, here, in the case inhibitor nets, the functor from **PES** to **O-IN** does not factorize through the category **IES**. An object level construction can be easily performed, associating to each IES a corresponding i-net. However such a construction does not give rise to a functor and, actually, we show that there is no functor from **IES** to **O-IN** forming a coreflection with $\mathscr{E}_i$. The last part of this section briefly discusses the origin of this problem, showing that it is intimately connected to or-causality.

5.1 From occurrence i-nets to IES's and PES's

Let us show first how an IES can be extracted from an occurrence i-net.

DEFINITION 55 *Let N be an occurrence i-net. The pre-*IES *associated to N is defined as $I_N^p = \langle T, \mapsto_N^p \rangle$, with $\mapsto_N^p \subseteq \mathbf{2}_1^T \times T \times \mathbf{2}^T$, given by: for $t, t' \in T$, $t \neq t'$ and $s \in S$*

1. *if* $t^\bullet \cap ({}^\bullet t' \cup \underline{t'}) \neq \emptyset$ *then* $\vdash\!\circ^p_N(\emptyset, t', \{t\})$

2. *if* $({}^\bullet t \cup \underline{t}) \cap {}^\bullet t' \neq \emptyset$ *then* $\vdash\!\circ^p_N(\{t'\}, t, \emptyset)$*;*

3. *if* $s \in {}^{\circledcirc}t$ *then* $\vdash\!\circ^p_N({}^\bullet s, t, s^\bullet)$.

The IES *associated to N, denoted by* $I_N = \langle T, \vdash\!\circ_N \rangle$*, is obtained by saturating* I^p_N*, i.e.,* $I_N = \overline{I^p_N}$.

The first two clauses of the definition encode, by using the DE-relation, the causal dependencies and the asymmetric conflicts induced by the flow and read arcs (we could have written "if $t <_r t'$ then $\vdash\!\circ^p_N(\emptyset, t', \{t\})$" and "if $t \nearrow_r t'$ then $\vdash\!\circ^p_N(\{t'\}, t, \emptyset)$"). The last clause fully exploits the expressiveness of the DE-relation to represent the dependencies induced by inhibitor places. Notice that I^p_N is a pre-IES satisfying also condition (1) of the definition of IES. Thus, by Proposition 36, it can be saturated to obtain the IES I_N.

The next proposition shows that the transition component of an i-net morphism is an IES-morphism between the corresponding IES's.

PROPOSITION 56 *Let* N_0 *and* N_1 *be occurrence i-nets and let* $h : N_0 \to N_1$ *be an i-net morphism. Then* $h_T : I_{N_0} \to I_{N_1}$ *is a* IES*-morphism.*

Proof . See the Appendix.

By the above proposition we get the existence of a functor which maps each i-net to the corresponding IES defined as in Definition 55 and each i-net morphism to its transition component.

DEFINITION 57 *We denote by* $\mathscr{E}_i : \mathbf{O\text{-}IN} \to \mathbf{IES}$ *the functor defined as* $\mathscr{E}_i(N) = I_N$ *for each occurrence i-net N and* $\mathscr{E}_i(h : N_0 \to N_1) = h_T$ *for each morphism* $h : N_0 \to N_1$.

By exploiting the relation between choices and assignments mentioned before, one can verify that if N is an executable occurrence i-net then $\mathscr{E}_i(N)$ is an $\mathbf{IES^e}$ object. Therefore the functor $\mathscr{E}_i$ restricts to a functor $\mathscr{E}^{\mathbf{e}}_i : \mathbf{O\text{-}IN^e} \to \mathbf{IES^e}$.

The coreflection between **IES** ($\mathbf{IES^e}$) and **Dom** can be finally used to obtain a domain semantics, and, by exploiting Winskel's equivalence, a prime event structure semantics for semi-weighted i-nets. As explained in Section 4.4, the PES semantics is obtained from the IES semantics by introducing an event for each possible different history of events in the IES.

Figure 11 presents part of the domain associated to the net N_3 of Fig. 3, namely of $\mathscr{L}_i(\mathscr{E}_i(\mathscr{U}_i(N_3))) = \mathscr{L}^{\mathbf{e}}_i(\mathscr{E}^{\mathbf{e}}_i(\mathscr{U}^{\mathbf{e}}_i(N_3)))$. The choice relation for each configuration is implicitly represented by the order in which events are mentioned in the corresponding set. Observe that several distinct configurations contains exactly the same events.

5.2 From prime event structures to occurrence i-nets

In [41] Winskel maps each prime event structure into a canonical occurrence net, via a free construction which generates for each set of events related in a certain way by the dependency relations a unique place that induces that kind of relation on the events. We next show how this construction can be generalised to inhibitor nets.

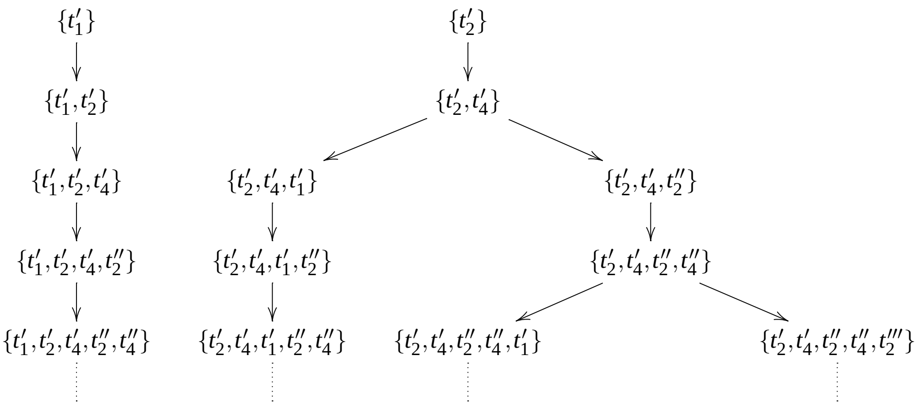


Figure 11: Part of the domain $\mathscr{L}_i(\mathscr{E}_i(\mathscr{U}_i(N_3))) = \mathscr{L}_i^{\mathbf{e}}(\mathscr{E}_i^{\mathbf{e}}(\mathscr{U}_i^{\mathbf{e}}(N_3)))$ associated to the net N_3 in Fig. 3.

DEFINITION 58 (FROM PES'S TO OCCURRENCE I-NETS) *Let $P = \langle E, \leq, \# \rangle$ be a* PES *and let $\nearrow$ denote the corresponding asymmetric conflict relation, i.e., $\nearrow = < \cup \#$. Then $\mathscr{N}_i(P)$ is the i-net $N = \langle S,T,F,C,I,m \rangle$ defined as follows, where A,B range over $\mathbf{2}^E$ and $e \in E$,*

- $m = \{\langle \emptyset, A, B\rangle \mid \forall a \in A.\ \forall b \in B.\ a \nearrow b \ \wedge\ \#_p B\}$;
- $S = m \cup \{\langle \{e\}, A, B\rangle \mid \forall e' \in A \cup B.\ e < e' \ \wedge\ \forall a \in A.\ \forall b \in B.\ a \nearrow b \ \wedge\ \#_p B\}$;
- $T = E$;
- $F = \langle F_{pre}, F_{post}\rangle$, *with*

  $$F_{pre} = \{(e,s) \mid s = \langle x,A,B\rangle \in S,\ e \in B\},$$
  $$F_{post} = \{(e,s) \mid s = \langle \{e\},A,B\rangle \in S\};$$
- $C = \{(e,s) \mid s = \langle x,A,B\rangle \in S,\ e \in A\}$.
- $I = \{(e,s) \mid s = \langle x,A,B\rangle \ \wedge\ ((\exists e' \in x.\ e \nearrow e') \vee (\exists e' \in B.\ e' < e))\}$

The definition of m, S, T and C is similar to the construction in [6], which associates a canonical contextual net to an asymmetric event structure. The transitions of net $\mathscr{N}_i(P)$ are the events of P and the places are triples of the form $\langle x,A,B\rangle$, with $x,A,B \subseteq E$, and $|x| \leq 1$, added to induce the same dependencies between events as those existing in P. A place $\langle x,A,B\rangle$ is a precondition for all the events in B and a context for all the events in A. Moreover, if $x = \{e\}$, such a place is a postcondition for e, otherwise if $x = \emptyset$ the place belongs to the initial marking. Therefore each place gives rise to a conflict between each pair of (distinct) events in B and to an asymmetric conflict between each pair of events $a \in A$ and $b \in B$.

With the same spirit, the net is saturated with all the inhibitor arcs inducing the correct dependencies among events. Consider a place $s = \langle x, A, B \rangle$ and two events e, e'. To understand the second branch of the disjunction in the definition of I above, assume that $e' \in B$ and $e' < e$. Then place s is in the pre-set of e' and thus it must be emptied by the firing of e' before the execution of e. Hence we force s to inhibit e in $\mathscr{N}_i(P)$, i.e., we insert the pair (e, s) in I. The first branch of the disjunction is motivated by analogous considerations.

Two technical lemmata follow which will play a crucial role in the proof of the main result of this section. The first one can be proved as Lemma 7.2 in [6], hence its proof is omitted. In the sequel, given an i-net N and a transition $t \in T$, we will denote by $\lceil \{t\} \rceil$ its set of consequences, namely $\lceil \{t\} \rceil = \{t' \in T \mid t <_r t'\}$. For notational convenience the consequences are defined also for the empty set by $\lceil \emptyset \rceil = T$.

LEMMA 59 *Let N_0, N_1 be occurrence i-nets and let $h : N_0 \to N_1$ be a morphism. For $s_0 \in S_0$ and $s_1 \in S_1$, if $h_S(s_0, s_1)$ then*

1. $h_T(^\bullet s_0) = {}^\bullet s_1$;
2. $s_0{}^\bullet = h_T^{-1}(s_1{}^\bullet) \cap \lceil {}^\bullet s_0 \rceil$;
3. $\underline{s_0} = h_T^{-1}(\underline{s_1}) \cap \lceil {}^\bullet s_0 \rceil$;
4. $h_T^{-1}({}^{\circledcirc}s_1) \subseteq {}^{\circledcirc}s_0$.

LEMMA 60 *Let P be a* PES, *let N_0 be an occurrence i-net and let $h_T : \mathscr{J}_i(P) \to \mathscr{E}_i(N_0)$ be an* IES*-morphisms (recall that $\mathscr{J}_i$ is the full embedding of* **PES** *into* **IES** *defined in Proposition 35). Then there exists a unique h_S such that $h = \langle h_T, h_S \rangle : \mathscr{N}_i(P) \to N_0$ is an i-net morphism.*

Proof. See the Appendix.

The next lemma shows that constructing the occurrence i-net for a given PES and then taking the corresponding IES, one recovers (an IES isomorphic to) the original PES.

LEMMA 61 *For any* PES *P, the identity over the events $\rho_P : \mathscr{J}_i(P) \to \mathscr{E}_i(\mathscr{N}_i(P))$ is an* IES*-isomorphism.*

Proof. See the Appendix.

We can thus present the main result of this section, which shows that the functor $\mathscr{N}_i$ is left adjoint to the functor $\mathscr{P}\mathscr{L}_i\mathscr{E}_i$, mapping each occurrence i-net into the corresponding PES.

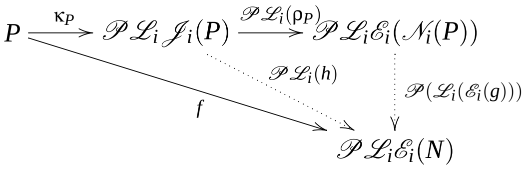
THEOREM 62 *The construction $\mathscr{N}_i$ extends to a functor $\mathscr{N}_i :$* **PES** $\to$ **O-IN** *and $\mathscr{N}_i \dashv \mathscr{P}\mathscr{L}_i\mathscr{E}_i$.*

Proof. Let us prove that $\mathscr{N}_i \dashv \mathscr{P}\mathscr{L}_i\mathscr{E}_i$ with unit $\eta_P : P \to \mathscr{P}\mathscr{L}_i\mathscr{E}_i(\mathscr{N}_i(P))$ defined as $\eta_P = \kappa_P; \mathscr{P}\mathscr{L}_i(\rho_P)$

$$P \xrightarrow{\kappa_P} \mathscr{P}\mathscr{L}_i\mathscr{J}_i(P) \xrightarrow{\mathscr{P}\mathscr{L}_i(\rho_P)} \mathscr{P}\mathscr{L}_i\mathscr{E}_i(\mathscr{N}_i(P))$$

where κ_P is the unit of the coreflection between **IES** and **PES** (see Corollary 49) and ρ_P is the identity on events (see Lemma 61).

We must show that for any PES P, occurrence i-net N and morphism $f : P \to \mathscr{P}\mathscr{L}_i\mathscr{E}_i(N)$ there is a unique arrow $g : \mathscr{N}_i(P) \to N$ such that the outer triangle commutes



Since, by Corollary 49 $\mathscr{J}_i \dashv \mathscr{P}\mathscr{L}_i$, there is a unique $h : \mathscr{J}_i(P) \to \mathscr{E}_i(N)$ such that the left triangle above commutes.

Furthermore, by Lemma 60, h uniquely extends to a morphism $g : \mathscr{N}_i(P) \to N$ such that $\mathscr{E}_i(g) = g_T = h$, thus making the right triangle in the diagram above commute (recall that ρ_P is the identity on events). This proves the existence of the morphism g we were looking for. Uniqueness follows from the observation that the existence of two distinct choices for g would violate the uniqueness of h. □

Observe that the image of the functor $\mathscr{N}_i$ is entirely included in $\mathbf{O\text{-}IN^e}$, i.e., for any PES P the net $\mathscr{N}_i(P)$ is an executable occurrence i-net. Hence $\mathscr{N}_i$ naturally restricts to a functor $\mathscr{N}_i^{\mathbf{e}} : \mathbf{PES} \to \mathbf{O\text{-}IN^e}$, which, by general arguments, is left adjoint to $\mathscr{P} \circ \mathscr{L}_i^{\mathbf{e}} \circ \mathscr{E}_i^{\mathbf{e}}$. Also note that since the functors $\mathscr{I}_O \circ \mathscr{N}_i, \mathscr{I}_O^{\mathbf{e}} \circ \mathscr{N}_i^{\mathbf{e}} : \mathbf{PES} \to \mathbf{SW\text{-}IN}$ clearly coincide, as a byproduct we immediately have that also their right-adjoints are the same, i.e., the two proposed constructions (with or without non-executable events) lead to the same PES (and domain).

5.3 From IES's to i-nets: a negative result

We finally show that, differently from what happens for contextual nets and asymmetric event structures, the coreflection between **O-IN** and **PES** described above does not factorize through the category **IES**, i.e., that there is no left adjoint functor $\mathscr{M}_i : \mathbf{IES} \to \mathbf{O\text{-}IN}$ which forms a coreflection with $\mathscr{E}_i$.

More generally we can show that there is no functor $\mathscr{M}_i : \mathbf{IES} \to \mathbf{O\text{-}IN}$ such that, $\mathscr{E}_i \circ \mathscr{M}_i$ is naturally isomorphic to the identity. Assume by contradiction that there is such a functor. Consider two IES's I_0 and I_1, obtained by saturating the pre-IES's $\langle\{e_0, e_0'\}, \{(\emptyset, e_0, \{e_0'\})\}\rangle$ (where $e_0' < e_0$) and $\langle\{e_1, e_1', e_1''\}, \{(\{e_1'\}, e_1'', \{e_1\})\}\rangle$.

Since $\mathscr{E}_i(\mathscr{M}_i(I_1)) \simeq I_1$ and the only way to generate a triple where all components are non-empty is to have an inhibiting place, in $\mathscr{M}_i(I_1)$ there must be a place $s_1 \in {}^{\circledcirc}e_1'' \cap e_1'^{\bullet} \cap {}^{\bullet}e_1$ (see Fig. 12.(b)). Since the mapping $h : I_0 \to I_1$ such that $h(e_0) = e_1$ and $h(e_0') = e_1'$ is a well-defined IES-morphism, there must exist an i-net morphism $\mathscr{M}_i(h) : \mathscr{M}_i(I_0) \to \mathscr{M}_i(I_1)$. This implies that there are places $s_0' \in e_0'^{\bullet}$ and $s_0 \in {}^{\bullet}e_0$ such that $\mathscr{M}_i(h)(s_0', s_1)$ and $\mathscr{M}_i(h)(s_0, s_1)$. Since $\mathscr{M}_i(h)$ is an occurrence i-net morphism, necessarily $s_0 = s_0'$, otherwise we would have $s_0 \# s_0'$, hence $e_0 \# e_0'$ in $\mathscr{M}_i(I_0)$ and thus in $\mathscr{E}_i(\mathscr{M}_i(I_0))$, contradicting the assumption $\mathscr{E}_i(\mathscr{M}_i(I_0)) \simeq I_0$ (see Fig. 12.(a)).

Consider now the IES I_2 which is obtained by saturation of the pre-IES $\langle\{e_2, e_2', e_2''\}, \{(\emptyset, e_2, \{e_2', e_2''\}), (\{e_2'\}, e_2'', \emptyset), (\{e_2''\}, e_2', \emptyset)\}\rangle$ (where $e_2' \# e_2''$ and $\{e_2', e_2''\} < e_2$) and the IES-morphism $f : I_2 \to I_0$, defined by $f(e_2) = e_0$ and $f(e_2') = f(e_2'') = e_0'$. Since there is an

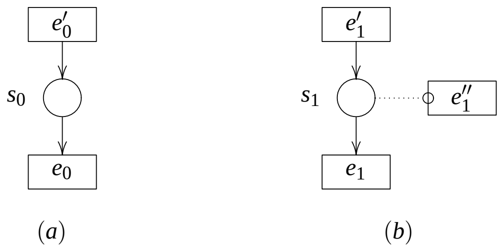


Figure 12: (a) Part of $\mathscr{M}_i(I_0)$ and (b) Part of $\mathscr{M}_i(I_1)$.

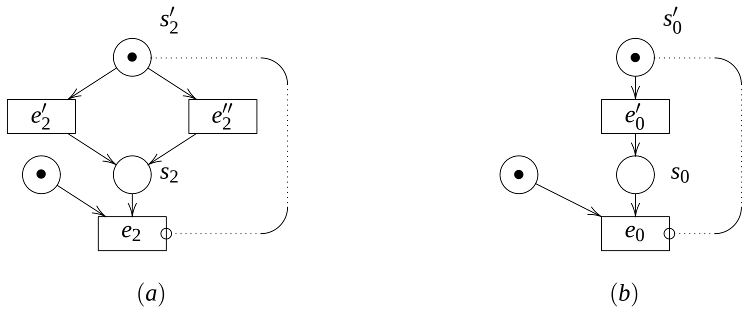


Figure 13: (a) Part of $\mathscr{M}_i(I_2)$ and (b) Part of $\mathscr{M}_i(I_0)$

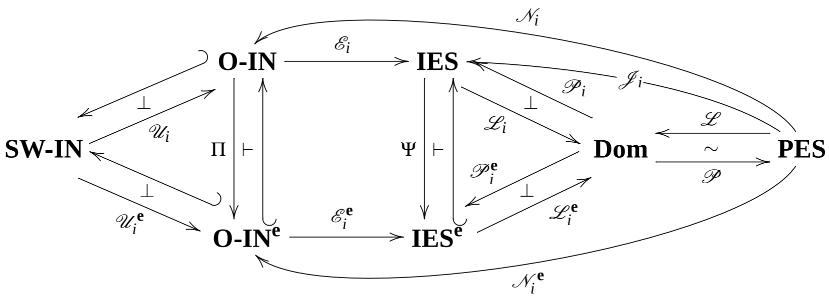
i-net morphism $\mathscr{M}_i(f) : \mathscr{M}_i(I_2) \to \mathscr{M}_i(I_0)$, there must be places $s'_2 \in e'^{\bullet}_2$ and $s_2 \in {}^{\bullet}e_2$ such that $\mathscr{M}_i(f)(s_2, s_0)$ and $\mathscr{M}_i(f)(s'_2, s_0)$. Therefore $\mathscr{M}_i(f)(e'_2) < \mathscr{M}_i(f)(e_2)$ and thus, since $\mathscr{M}_i(f)$ is an occurrence i-net morphism, necessarily $e'_2 < e_2$ or $e_2 \# e'_2$ in $\mathscr{M}_i(I_2)$ and thus in $\mathscr{E}_i(\mathscr{M}_i(I_2))$. Hence in both cases we would reach a contradiction with the assumption that $\mathscr{E}_i(\mathscr{M}_i(I_2)) \simeq I_2$.

At a more intuitive level, imagine to construct an i-net for an inhibitor event structure by following Winskel's idea of saturating the IES with places in order to induce the same relations among events as in the IES. Fig. 13 represents fragments of the nets $\mathscr{M}_i(I_2)$ and $\mathscr{M}_i(I_0)$ which we would obtain for event structures I_2 and I_0. Observe that the causal dependency $e'_0 < e_0$ is induced both by place $s_0 \in e'^{\bullet}_0 \cap {}^{\bullet}e_0$ and by means of an inhibitor arc, i.e., through the marked place $s'_0 \in {}^{\bullet}e'_0 \cap {}^{\circledcirc}e_0$. Correspondingly, the functoriality of $\mathscr{M}_i$ would require the net $\mathscr{M}_i(I_2)$ to include the places s_2 and s'_2 (since the IES-morphism $f : I_2 \to I_0$, defined by $f(e_2) = e_0$ and $f(e'_2) = f(e''_2) = e'_0$, must be "extendable" to a i-net morphism $\mathscr{M}_i(f) : \mathscr{M}_i(I_2) \to \mathscr{M}_i(I_0)$). However $\mathscr{M}_i(I_2)$ cannot be defined in this way since backward conflicts (several transitions in the pre-set of a place, like s_2 in $\mathscr{M}_i(I_2)$) are not allowed in occurrence i-nets.

Observe also that the naïve solution of widening the category of occurrence i-net to include also nets with backward conflicts (which, by analogy with the flow nets of [7], could be called *flow i-nets*) does not work, as one can easily check.

6 Conclusions

We have provided a coreflective concurrent semantics for Petri nets with read and inhibitor arcs. The proposed constructions, which generalise Winskel's work on safe ordinary nets and the work in [6] on contextual nets, are summarised in the diagram below (where unnamed functors are inclusions).



The paper singles out two distinct notions of occurrence i-net: ordinary occurrence i-nets, where some events might be non-executable, and executable occurrence i-nets, where some additional conditions ensure the firability of any transition. Correspondingly two different unfolding constructions are provided which associate to each semi-weighted inhibitor net an occurrence inhibitor net. The unfoldings can be naturally abstracted to an IES, having the transitions of the net as events, and thus, by exploiting a coreflection between **IES** and **Dom**, to a domain (or, equivalently, to a prime event structure). Both constructions (with or without non-executable events) lead to the same domain.

The coreflection between occurrence nets and prime event structures does not factorize through **IES**, namely, the functor from **PES** to the category of occurrence i-nets cannot be expressed as the composition of functors one from **PES** to IES's, and the other from IES's to occurrence i-nets.

In the paper we hinted at the relationship between IES's and other event structure models proposed in the literature. It can be easily seen that IES's properly generalise prime [41], asymmetric [6], (extended) bundle event structures [25] and prime event structures with possible events [35]. Instead IES's and flow event structures [7] (with possible flow [35]), although strictly related, are, in a sense, not comparable since there are IES's whose sets of configurations cannot be described by a flow event structure and vice versa.

Inhibitor event structures are also related to *event automata* [35], a class of automata where states are sets of events and the transition relation specifies which events can occur in a certain state. Although not explicitly worked out in this paper, it is easy to see that given an IES we can obtain a corresponding event automaton via a functorial construction which takes the partial order of configurations, forgetting about the history of events, namely identifying different configurations which involve the same set of events.

This connection between IES's and event automata suggests also the possibility of comparing our model with other event based models proposed in the literature as generalisations of the family of configurations of event structures, like configuration structures [38] and Chu spaces [18]. In particular it could be interesting to try to give a logical view of IES's, in the style of the presentation of event structures as propositional theories in [38]. To this end, also the logical approach to causality of [17] could provide some interesting hints. Some similarities can be found also with local event structures [20], where, as in the case of IES's, the enabling of events is not required to be monotonic. However a direct comparison appears difficult to carry out since local event structures explicitly represent configurations and concurrent enabling of sets of events, while IES's give an intensional description of such notions by means of the DE-relation. Probably also in this case one could try a comparison at the level of corresponding event automata.

A semantics for inhibitor nets, based on a generalisation of Mazurkiewicz traces, has been developed in [21]. Such paper assumes a notion of enabling different from ours, allowing for the concurrent firing of steps where a token is generated in the inhibitor set. Consequently concurrent steps may not be serializable and this is the reason why the simultaneity (independence) relation of Mazurkiewicz traces is not sufficiently expressive, and one must consider also a serializability relation which explicitly says if two simultaneous events are serializable and in which order. Along the same line, more recently a process semantics for inhibitor nets (possibly unbounded and with weighted arcs) has been developed [22, 23]. Understanding if, despite the different notions of enabling, a relationship can be established with our work is left as a matter of future investigation. We also conjecture that, keeping our notion of enabling, Mazurkiewicz trace theory could be successfully applied to extract a PES from an inhibitor net and that the domain of configurations of such a PES would be isomorphic to the prime algebraic domain obtained through our unfolding construction.

Acknowledgements. We are grateful to the anonymous referees for their useful and constructive comments on the submitted version of this paper.

A Full proofs of results in the paper.

A.1 Categories of i-nets

Proposition 4. *[(composition of i-net morphisms)] The class of i-net morphisms is closed under composition.*

Proof . Let $h_0 : N_0 \to N_1$ and $h_1 : N_1 \to N_2$ be two i-net morphisms. Their composition $h_1 \circ h_0$ obviously satisfies conditions (1) and (2.a)-(2.c) of Definition 3, since these are exactly the defining conditions of c-net morphisms which are known to be closed under composition.

Finally, $h_1 \circ h_0$ satisfies also condition (2.d). In fact, for any transition t in N_0:

$$\begin{aligned}
&[\![h_{1S} \circ h_{0S}]\!]^{-1}({}^{\circledcirc}h_{1T} \circ h_{0T}(t)) = \\
&\quad = [\![h_{0S}]\!]^{-1}([\![h_{1S}]\!]^{-1}({}^{\circledcirc}h_{1T}(h_{0T}(t)))) \\
&\quad \subseteq [\![h_{0S}]\!]^{-1}({}^{\circledcirc}h_{0T}(t)) && \text{[since } h_1 \text{ is a morphism]} \\
&\quad \subseteq {}^{\circledcirc}t && \text{[since } h_0 \text{ is a morphism]}
\end{aligned}$$

Proposition 20. *Let N be an executable occurrence i-net and let $M \subseteq S$. Then $conc(M)$ iff there exists a reachable marking M' such that $M \subseteq M'$.*

Proof . ($\Rightarrow$) By definition of concurrency (Definition 19), there is an assignment ρ such that $\lfloor M \rfloor_{\rho}$ is finite and $\nearrow_{\rho}$ is acyclic on $\lfloor M \rfloor_{\rho}$. Therefore there is an enumeration of transitions $t^{(1)}, \ldots, t^{(k)}$ in $\lfloor M \rfloor_{\rho}$ compatible with $\nearrow_{\rho}^{+}$. Let us show by induction on k that

$$m = M^{(0)}\,[t^{(1)}\rangle\, M^{(1)}\,[t^{(2)}\rangle\, \ldots\, M^{(k-1)}\,[t^{(k)}\rangle\, M^{(k)} \supseteq M$$

($k = 0$) Obvious.

($k > 0$) By construction $t^{(k)}$ is $\nearrow_{\rho}$-maximal $\lfloor M \rfloor_{\rho}$. Take $M'' = M - t^{(k)\bullet} + {}^{\bullet}t^{(k)}$. It is easy to show that $conc(M'')$ (with the same assignment ρ) and $\lfloor M'' \rfloor_{\rho} = \{t^{(1)}, \ldots, t^{(k-1)}\}$. Hence by inductive hypothesis

$$m = M^{(0)}\,[t^{(1)}\rangle\, M^{(1)}\,[t^{(2)}\rangle\, \ldots\, M^{(k-1)}\,[t^{(k-1)}\rangle\, M^{(k-1)} \supseteq M''$$

Now, showing that $t^{(k)}$ is enabled at $M^{(k-1)}$ we can conclude. To this aim, observe that clearly ${}^{\bullet}t^{(k)} \subseteq M^{(k-1)}$. Moreover $\underline{t^{(k)}} \subseteq M^{(k-1)}$ and ${}^{\circledcirc}t^{(k)} \cap M^{(k-1)} = \emptyset$, as otherwise $t^{(k)}$ would not be $\nearrow_{\rho}$-maximal in $\lfloor M \rfloor_{\rho}$.

Therefore $t^{(k)}$ is enabled at $M^{(k-1)}$ and we can extend the firing sequence above to

$$m = M^{(0)}\,[t^{(1)}\rangle\, M^{(1)}\,[t^{(2)}\rangle\, \ldots\, M^{(k-1)}\,[t^{(k-1)}\rangle\, M^{(k-1)}\,[t^{(k)}\rangle\, M^{(k)}$$

with $M^{(k)} = M^{(k-1)} - {}^{\bullet}t^{(k)} + t^{(k)\bullet} \supseteq M'' - {}^{\bullet}t^{(k)} + t^{(k)\bullet} = M$.

($\Leftarrow$) The thesis follows from an inductive reasoning on the number of firings leading from the initial marking m to M'. □

A.2 Basic results on IES's

Proposition 33. *Let I_0 and I_1 be* IES*'s and let $f : I_0 \to I_1$ be an* IES*-morphism. For any $e_0, e_0' \in E_0$,*

1. *if $f(e_0) < f(e_0')$ then $\exists A_0.\ e_0 \in A_0 < e_0'$ or $e_0 \# e_0'$;*
2. *if $f(e_0) \nearrow f(e_0')$ then $e_0 \nearrow e_0'$.*

Proof .

1. Let $f(e_0) < f(e_0')$, namely $\{f(e_0)\} < f(e_0')$. By condition (2) in the definition of IES-morphisms, there exists $A_0 \subseteq f^{-1}(\{f(e_0)\})$ such that $A_0 < e_0'$. Now, if $e_0 \in A_0$ the desired property is proved. Otherwise for each $e_0'' \in A_0$, $e_0'' \neq e_0$ and, by construction $f(e_0'') = f(e_0)$. Hence by condition (1) in the definition of IES-morphism, it must be $e_0 \# e_0''$ for each $e_0'' \in A_0$. Hence, by rule (#2), we conclude $e_0 \# e_0'$.

2. Let $f(e_0) \nearrow f(e_0')$. Then, by definition of IES, $\vdash\!\circ\, (\{f(e_0')\}, f(e_0), \emptyset)$. By condition (3) in the definition of IES-morphism there must exist $a_0 \subseteq \{e_0'\}$ and $A_0 \subseteq f^{-1}(\emptyset) = \emptyset$ such that $\vdash\!\circ\, (a_0, e_0, A_0)$. Therefore, if $a_0 = \{e_0'\}$ then $\vdash\!\circ\, (\{e_0'\}, e_0, \emptyset)$ and thus, by rule ($\nearrow 1$), we conclude $e_0 \nearrow e_0'$. If instead $a_0 = \emptyset$ then $\vdash\!\circ\, (\emptyset, e_0, \emptyset)$ and thus, by rule (< 1), $\emptyset < e_0$. Hence, by rule (#2), we deduce $\#\{e_0, e_0'\}$ and thus $e_0 \nearrow e_0'$ by ($\nearrow 3$). □

Proposition 34. *The* IES*-morphisms are closed under composition.*

Proof. Let $f_0 : I_0 \to I_1$ and $f_1 : I_1 \to I_2$ be IES-morphisms. We want to show that their composition $f_1 \circ f_0$ still satisfies conditions (1)-(3) of Definition 32.

1. Let $e_0, e_0' \in E_0$ be events such that $e_0 \neq e_0'$ and $f_1(f_0(e_0)) = f_1(f_0(e_0'))$. If $f_0(e_0) = f_0(e_0')$ then, being f_0 a morphism, $e_0 \# e_0'$. Otherwise, since also f_1 is a morphism, $f_0(e_0) \# f_0(e_0')$ and thus, by rule $(\nearrow 3)$, $f_0(e_0) \nearrow f_0(e_0') \nearrow f_0(e_0)$. Hence, by Proposition 33.(2), it must hold that $e_0 \nearrow e_0' \nearrow e_0$, which in turn, by rule (#1) allows us to deduce $e_0 \# e_0'$.

2. Consider $A_2 \subseteq E_2$ and $e_0 \in E_0$ such that $A_2 < f_1(f_0(e_0))$. Since f_1 is an IES-morphism there exists $A_1 \subseteq f_1^{-1}(A_2)$ such that $A_1 < f_0(e_0)$. By using again condition (2) in the definition of IES-morphism, applied to f_0, we obtain the existence of $A_0 \subseteq f_0^{-1}(A_1)$ satisfying $A_0 < e_0$. We conclude observing that $A_0 \subseteq f_0^{-1}(A_1) \subseteq f_0^{-1}(f_1^{-1}(A_2)) = (f_1 \circ f_0)^{-1}(A_2)$.

3. Let us assume $\vdash\!\circ\, (\{f_1(f_0(e_0'))\}, f_1(f_0(e_0)), A_2)$. Since f_1 is an IES-morphism there exist $A_1 \subseteq f_1^{-1}(A_2)$ and $a_1 \subseteq \{f_0(e_0')\}$ such that $\vdash\!\circ\, (a_1, f_0(e_0), A_1)$. We can distinguish two cases according to the form of a_1.

   - If $a_1 = \emptyset$ and thus $A_1 < f_0(e_0)$, since f_0 is an IES-morphism, there will be $A_0 \subseteq f_0^{-1}(A_1)$ such that $A_0 < e_0$. By definition of IES this implies $\vdash\!\circ\, (\emptyset, e_0, A_0)$. Moreover $A_0 \subseteq f_0^{-1}(A_1) \subseteq f_0^{-1}(f_1^{-1}(A_2))$ and thus condition (3) is satisfied.
   - If $a_1 = \{f_0(e_0')\}$ and thus $\vdash\!\circ\, (\{f_0(e_0')\}, f_0(e_0), A_1)$ reasoning as above, but using point (3) in the definition of morphism, we deduce the existence of $A_0 \subseteq f_0^{-1}(A_1) \subseteq f_0^{-1}(f_1^{-1}(A_2))$ and $a_0 \subseteq \{e_0'\}$ such that $\vdash\!\circ\, (a_0, e_0, A_0)$, thus satisfying condition (3).

□

Lemma 37. *Let $I_i = \langle E_i, \vdash\!\circ_i \rangle$ $(i \in \{0,1\})$ be pre-*IES*'s satisfying condition (1) of Definition 31, let $\overline{I_i} = \langle E_i, \vdash\!\circ_i^s \rangle$, and let $<_i$, $\nearrow_i$ and $\#_i$ be the relations of causality, asymmetric conflict and conflict in I_i. Let $f : E_0 \to E_1$ be a partial function such that for each $e_0, e_0' \in E_0$ and $A_1 \subseteq E_1$:*

1. $f(e_0) = f(e_0') \wedge e_0 \neq e_0' \quad \Rightarrow \quad e_0 \#_0 e_0'$;
2. $\vdash\!\circ_1(\emptyset, f(e_0), A_1) \quad \Rightarrow \quad \exists A_0 \subseteq f^{-1}(A_1).\ A_0 <_0 e_0$;
3. $\vdash\!\circ_1(f(e_0'), f(e_0), \emptyset) \quad \Rightarrow \quad e_0 \nearrow_0 e_0'$;
4. $\vdash\!\circ_1(\{f(e_0')\}, f(e_0), A_1) \wedge A_1 \neq \emptyset \ \Rightarrow\ \exists A_0 \subseteq f^{-1}(A_1). \exists a_0 \subseteq \{e_0'\}.\ \vdash\!\circ_0^s(a_0, e_0, A_0)$.

Then $f : \overline{I_0} \to \overline{I_1}$ is an IES*-morphism.*

Proof. We first show that f satisfies the following properties:

a. $A_1 <_1 f(e_0) \quad \Rightarrow \quad \exists A_0 \subseteq f^{-1}(A_1).\ A_0 <_0 e_0$;

b. $f(e_0) \nearrow_1 f(e_0') \quad \Rightarrow \quad e_0 \nearrow_0 e_0'$.

c. $\#_1 f(A_0) \quad \Rightarrow \quad \#_0 A_0$.

The three points are proved simultaneously by induction on the height of the derivation of the judgement, involving the relations $<_1$, $\nearrow_1$ and $\#_1$, which appears in the premise of each implication and by cases on the form of the judgement.

a. *Judgement* $A_1 <_1 f(e_0)$.
We distinguish various subcases according to the last rule used in the derivation:

(<1) Let the last rule be

$$\frac{\vdash\!\circ_1(\emptyset, f(e_0), A_1) \quad \#_p A_1}{A_1 <_1 f(e_0)} \ (<1)$$

In this case, since $\vdash\!\circ_1(\emptyset, f(e_0), A_1)$, we immediately conclude by using point (2) in the hypotheses.

(<2) Let the last rule be

$$\frac{A_1' <_1 f(e_0) \quad \forall e_1 \in A_1'.\, A_{e_1} <_1 e_1 \quad \#_p(\cup\{A_{e_1} \mid e_1 \in A_1'\})}{(\cup\{A_{e_1} \mid e_1 \in A_1'\}) <_1 f(e_0)} \ (<2)$$

By inductive hypothesis from $A_1' <_1 f(e_0)$ we deduce that

$$\exists A_0 \subseteq f^{-1}(A_1').\, A_0 <_0 e_0 \qquad (\dagger)$$

Now, for all $e_0' \in A_0$, by (†), $f(e_0') \in A_1'$. Therefore, by the second premise of the rule above, $A_{f(e_0')} <_1 f(e_0')$, and thus, by inductive hypothesis, there exists $A_{e_0'} \subseteq f^{-1}(A_{f(e_0')})$ such that $A_{e_0'} <_0 e_0'$. Finally, $\cup\{A_{e_0'} \mid e_0' \in A_0\}$ is pairwise conflicting. In fact if $e_0^1, e_0^2 \in \cup\{A_{e_0'} \mid e_0' \in A_0\}$ with $e_0^1 \neq e_0^2$, we have $f(e_0^1), f(e_0^2) \in \bigcup_{e_1 \in A_1'} A_{e_1}$, which is pairwise conflicting. Therefore $f(e_0^1) = f(e_0^2)$ or $f(e_0^1) \#_1 f(e_0^2)$ and, by using point (1) in the hypotheses in the first case, and by inductive hypothesis in the second case, we conclude $e_0^1 \#_0 e_0^2$.
By using the facts proved so far we can apply rule (<2) as follows:

$$\frac{A_0 <_0 e_0 \quad \forall e_0' \in A_0.\, A_{e_0'} <_0 e_0' \quad \#_p(\cup\{A_{e_0'} \mid e_0' \in A_0\})}{(\cup\{A_{e_0'} \mid e_0' \in A_0\}) <_0 e_0} \ (<2)$$

This concludes the proof of this case since

$$\begin{aligned} &\cup\{A_{e_0'} \mid e_0' \in A_0\} \subseteq \\ &\subseteq \cup\{f^{-1}(A_{f(e_0')}) \mid e_0' \in A_0\} \\ &\subseteq \{f^{-1}(A_{e_1}) \mid e_1 \in A_1'\} \\ &= f^{-1}(\cup\{A_{e_1} \mid e_1 \in A_1'\}) \end{aligned}$$

b. *Judgement* $f(e_0) \nearrow_1 f(e_0')$.
We distinguish various subcases according to the last rule used in the derivation:

$(\nearrow 1)$ Let the last rule be

$$\frac{\vdash\!\circ_1(\{f(e_0')\}, f(e_0), \emptyset)}{f(e_0) \nearrow_1 f(e_0')} \ (\nearrow 1)$$

From $\vdash\!\circ_1(\{f(e'_0)\}, f(e_0), \emptyset)$, by point (3) in the hypotheses, we immediately have that $e_0 \nearrow_0 e'_0$.

$(\nearrow 2)$ Let the last rule be

$$\frac{f(e_0) \in A_1 <_1 f(e'_0)}{f(e_0) \nearrow_1 f(e'_0)} \quad (\nearrow 2)$$

By inductive hypothesis there exists $A_0 \subseteq f^{-1}(A_1)$ such that $A_0 <_0 e'_0$.

For all $e''_0 \in A_0$, we have $f(e''_0) \in A_1$. Thus recalling that, since $A_1 <_1 f(e'_0)$, the set A_1 is pairwise conflicting, it follows that $f(e''_0) = f(e_0)$ or $f(e''_0) \#_1 f(e_0)$. By using point (1) of the hypotheses in the first case and the inductive hypothesis in the second case, we can conclude that for all $e''_0 \in A_0$, $e_0 = e''_0$ or $e_0 \#_0 e''_0$.

Consequently there are two possibilities. One is that $e_0 = e''_0 \in A_0$ for some $e''_0 \in A_0$, which allows us to conclude since $A_0 <_0 e'_0$. The other one is that $e_0 \#_0 e''_0$ for all $e''_0 \in A_0$. Thus, by rule (#2), we can derive that $\#_0\{e_0, e'_0\}$, and therefore $e_0 \nearrow_0 e'_0$ by rule $(\nearrow 3)$.

$(\nearrow 3)$ Let the last rule be

$$\frac{\#_1\{f(e_0), f(e'_0)\}}{f(e_0) \nearrow_1 f(e'_0)} \quad (\nearrow 3)$$

In this case by inductive hypothesis $\#_0\{e_0, e'_0\}$ and therefore, by rule $(\nearrow 3)$, $e_0 \nearrow_0 e'_0$.

c. *Judgement* $\#_1 f(A_0)$.

We distinguish various subcases according to the last rule used in the derivation:

(#1) Let the last rule be

$$\frac{f(e_0^{(0)}) \nearrow_1 \ldots \nearrow_1 f(e_0^{(n)}) \nearrow_1 f(e_0^{(0)})}{\#_1\{f(e_0^{(0)}), \ldots, f(e_0^{(n)})\}} \quad (\#1)$$

s

where $A_0 = \{e_0^{(0)}, \ldots, e_0^{(n)}\}$. By inductive hypothesis $e_0^{(0)} \nearrow_0 \ldots \nearrow_0 e_0^{(n)} \nearrow_0 e_0^{(0)}$, and therefore $\# A_0$.

(#2) Let the last rule be

$$\frac{A_1 <_1 f(e_0) \quad \forall e_1 \in A_1.\ \#_1(f(A'_0) \cup \{e_1\})}{\#_1(f(A'_0) \cup \{f(e_0)\})} \quad (\#2)$$

where $A_0 = A'_0 \cup \{e_0\}$.

By inductive hypothesis, from $A_1 <_1 f(e_0)$ it follows that

$$\exists A''_0 \subseteq f^{-1}(A_1).\ A''_0 <_0 e_0 \qquad (\dagger)$$

Now, for all $e'_0 \in A''_0$, by (†), $f(e'_0) \in A_1$. Therefore, by the second premise of the rule above, $\#_1(f(A'_0) \cup \{f(e'_0)\})$, namely $\#_1 f(A'_0 \cup \{e'_0\})$. Thus, by inductive hypothesis, $\#_0(A'_0 \cup \{e'_0\})$ for all $e'_0 \in A''_0$. Recalling that $A''_0 <_0 e_0$, by using rule (#2), we obtain

$$\frac{A''_0 <_0 e_0 \quad \forall e'_0 \in A''_0.\ \#_0(A'_0 \cup \{e'_0\})}{\#_0(A'_0 \cup \{e_0\})} \quad (\#2)$$

which is the desired result.

This completes the proof of the properties (a), (b) and (c).

It is now easy to conclude that $f : \overline{I_0} \to \overline{I_1}$ is a IES-morphism. Let $\overline{I_i} = \langle E_i, \mapsto^s_i \rangle$ for $i \in \{1,2\}$. Conditions (1) and (2) of the definition of IES-morphism (Definition 32) are clearly satisfied. In fact, by Proposition 36 the relations of causality and conflict in I_i and $\overline{I_i}$ coincide, and thus the mentioned conditions coincide with point (1) in the hypotheses and point (a) proved above.

Hence it remains to verify condition (3) of Definition 32, that is

$$\mapsto^s_1(\{f(e_0')\}, f(e_0), A_1) \quad \Rightarrow \quad \exists A_0 \subseteq f^{-1}(A_1).\ \exists a_0 \subseteq \{e_0'\}.\ \mapsto^s_0(a_0, e_0, A_0).$$

Suppose that $\mapsto^s_1(\{f(e_0')\}, f(e_0), A_1)$. If $A_1 \neq \emptyset$, by definition of $\overline{I_i}$, it must be the case that $\mapsto_1(\{f(e_0')\}, f(e_0), A_1)$ and thus the thesis trivially holds by point (4) in the hypotheses. If instead $A_1 = \emptyset$ then, by rule $(\nearrow 1)$, $f(e_0) \nearrow_1 f(e_0')$. Hence, by point (b) proved above, $e_0 \nearrow_0 e_0'$ and therefore $\mapsto^s_0(\{e_0'\}, e_0, \emptyset)$, which satisfies the desired condition. □

A.3 Algebraic properties of the domain of configurations of an IES

Proposition 41. *Let $\langle C_i, \hookrightarrow_{C_i} \rangle$ for $i \in \{1,2\}$ be configurations of an* IES *I.*

1. *If $e, e' \in C_1 \cap C_2$ and $e \hookrightarrow_{C_1} e'$ then $e \hookrightarrow_{C_2} e'$ or $e' \hookrightarrow^*_{C_2} e$.*
2. *If $C_1 = C_2$ and $\hookrightarrow^*_{C_1} \subseteq \hookrightarrow^*_{C_2}$ then $\hookrightarrow_{C_1} = \hookrightarrow_{C_2}$, namely the two configurations coincide.*

Proof.

1. Let $e, e' \in C_1 \cap C_2$ with $e \hookrightarrow_{C_1} e'$. By definition of choice, it follows that $\mapsto_{C_1}(\{e'\}, e, A)$ or $\mapsto_{C_1}(a, e', A')$, with $e \in A'$. Assume that $\mapsto_{C_1}(\{e'\}, e, A)$ and thus $\mapsto(\{e'\}, e, A'')$ with $A = A'' \cap C_1$ (the other case can be treated in a similar way). Since $e, e' \in C_2$, $\mapsto_{C_2}(\{e'\}, e, A'' \cap C_2)$, and thus, by definition of choice, also C_2 must choose among the two possible orders of executions, namely $e \hookrightarrow_{C_2} e'$ or $e'' \hookrightarrow_{C_2} e$ for $e'' \in A'' \cap C_2$. In the second case, since by definition of IES $e' < e''$, by Proposition 39.(2), we have $e' \hookrightarrow_{C_2} e''$ and thus $e' \hookrightarrow^*_{C_2} e$.
2. If $e \hookrightarrow_{C_1} e'$, by point (1), $e \hookrightarrow_{C_2} e'$ or $e' \hookrightarrow^*_{C_2} e$. But the second possibility cannot arise, since $e \hookrightarrow_{C_1} e'$ implies $e \hookrightarrow^*_{C_1} e'$ and thus $e \hookrightarrow^*_{C_2} e'$. Vice versa, if $e \hookrightarrow_{C_2} e'$, by point (1), $e \hookrightarrow_{C_1} e'$ or $e' \hookrightarrow^*_{C_1} e$. Again the second possibility cannot arise, otherwise we would have $e' \hookrightarrow^*_{C_2} e$, contradicting the acyclicity of $\hookrightarrow_{C_2}$.

Lemma 44. *Let $X \subseteq Conf(I)$ be a pairwise compatible set of configurations of an* IES *I and let $C_1, C_2 \in X$. Then*

1. *if $e \hookrightarrow^*_{C_1} e'$ and $e' \in C_2$ then $e \in C_2$ and $e \hookrightarrow^*_{C_2} e'$;*
2. *if $e \in C_1 \cap C_2$ then $C_1[\![e]\!] = C_2[\![e]\!]$;*

3. *$C_1 \sqcap C_2 = C_1 \cap C_2$, with $\hookrightarrow_{C_1 \sqcap C_2} = \hookrightarrow_{C_1} \cap \hookrightarrow_{C_2}$;*

4. *the least upper bound of X exists, and it is given by*

$$\bigsqcup X = \langle \bigcup_{C \in X} C, \bigcup_{C \in X} \hookrightarrow_C \rangle.$$

Proof .

1. Let us first suppose that $e \hookrightarrow_{C_1} e'$ and $e' \in C_2$. Let $C \in X$ be an upper bound for C_1 and C_2, which exists since X is pairwise compatible. From $C_1 \sqsubseteq C$, by definition of extension, we have that $e, e' \in C$ and $e \hookrightarrow_C e'$. Recalling that $C_2 \sqsubseteq C$ and $e' \in C_2$ we deduce $e \in C_2$. Since $e, e' \in C_2 = C_2 \cap C$ and $e \hookrightarrow_C e'$, by Proposition 41.(1), it must be $e \hookrightarrow_{C_2} e'$ or $e' \hookrightarrow^*_{C_2} e$. The second possibility cannot arise, otherwise we should have $e' \hookrightarrow^*_C e$, contradicting the acyclicity of $\hookrightarrow_C$. Hence we can conclude $e \hookrightarrow_{C_2} e'$.

   In the general case in which $e \hookrightarrow^*_{C_1} e'$ the desired property is easily derived via an inductive reasoning using the above argument.

2. Immediate consequence of point (1).

3. To show that $\hookrightarrow_{C_1 \sqcap C_2} = \hookrightarrow_{C_1} \cap \hookrightarrow_{C_2}$ is a choice for $C_1 \cap C_2$, the only non trivial point is the proof of condition (1) of Definition 38. Suppose that $\vdash\!\circ_{C_1 \cap C_2} (a, e, A)$, namely $\vdash\!\circ (a, e, A')$ with $a \subseteq C_1 \cap C_2$ and $A = A' \cap (A_1 \cap A_2)$. Hence $\vdash\!\circ_{C_1} (a, e, A' \cap C_1)$ and thus either $e \hookrightarrow_{C_1} e'$ for $e' \in a$ or $e'' \hookrightarrow_{C_1} e$ with $e'' \in A' \cap C_1$. Being C_1 and C_2 compatible, by Lemma 44.(1) it must be $e \hookrightarrow_{C_2} e'$, or $e'' \in A' \cap C_2$ and $e'' \hookrightarrow_{C_2} e$, respectively. Therefore, as desired, $e \hookrightarrow_{C_1 \sqcap C_2} e'$ or $e'' \in A$ with $e'' \hookrightarrow_{C_1 \sqcap C_2} e$.

   Hence $C_1 \cap C_2$ is a configuration. Moreover, it is the greatest lower bound of C_1 and C_2 as one can check via a routine verification using Lemma 44.(1).

4. Let us verify that $\hookrightarrow_{\bigsqcup X} = \bigcup_{C \in X} \hookrightarrow_C$ is a choice for $\bigcup X$. First, it is easy to see that $\hookrightarrow_{\bigsqcup X} \subseteq choices(\bigcup X)$.

   As for condition (1) of the definition of choice, suppose that $\vdash\!\circ_{\bigcup X} (a, e, A)$, namely $\vdash\!\circ (a, e, A')$ with $a \subseteq \bigcup X$ and $A = A' \cap \bigcup X$. Since $a, \{e\} \subseteq \bigcup X$ we can find $C, C' \in X$ such that $a \subseteq C$ and $e \in C'$. Moreover, being X pairwise compatible, there is $C'' \in X$, upper bound of C and C', containing both a and e. Therefore $\vdash\!\circ_{C''} (a, e, A' \cap C'')$, and thus by definition of choice $e \hookrightarrow_{C''} e'$ for $e' \in a$ or $e'' \hookrightarrow_{C''} e$ for $e'' \in A' \cap C''$. It follows that, as desired, $e \hookrightarrow_{\bigsqcup X} e'$ or ($e'' \in \bigcup X$ and) $e'' \hookrightarrow_{\bigsqcup X} e$.

   The relation $\hookrightarrow_{\bigsqcup X}$ is acyclic since Lemma 44.(1) implies that a cycle of $\hookrightarrow_{\bigsqcup X}$ in $\bigcup X$ should be entirely inside a single configuration $C \in X$. Furthermore it is easily seen that given an event $e \in \bigcup X$, $(\bigcup X)[\![e]\!] = C[\![e]\!]$, for any $C \in X$ such that $e \in C$. Therefore $(\bigcup X)[\![e]\!]$ is surely finite.

   Hence $\hookrightarrow_{\bigsqcup X}$ is a choice and thus $\bigsqcup X$ is a configuration. A routine verification, using Lemma 44.(1) allows one to conclude that $\bigcup X$ is the least upper bound of X. □

Lemma 46. *Let $f : I_0 \to I_1$ be an* IES*-morphism and let $\langle C_0, \hookrightarrow_0 \rangle \in Conf(I_0)$. Then the pair $\langle C_1, \hookrightarrow_1 \rangle$ with $C_1 = f(C_0)$ and $\hookrightarrow_1 = f(\hookrightarrow_0) \cap choices(f(C_0))$, namely the unique choice relation on C_1 included in $f(\hookrightarrow_{C_0})$, is a configuration in I_1. Moreover the function $f^* : Conf(I_0) \to$*

$Conf(I_1)$ which associates to each configuration C_0 the configuration C_1 defined as above, is a domain morphism.

Proof. To prove that $\hookrightarrow_1$ is a choice for $f(C_0)$ and thus $\langle f(C_0), \hookrightarrow_1 \rangle$ is a configuration, first observe that $\hookrightarrow_1 \subseteq choices(C_1)$ by definition.

Let us verify the validity of condition (1) in the definition of choice (Definition 38). Assume that $\vdash\!\circ_{f(C_0)}(a_1, f(e_0), A_1)$. This means that $\vdash\!\circ_1(a_1, f(e_0), A_1')$ with $a_1 \subseteq f(C_0)$ and $A_1 = A_1' \cap f(C_0)$. We distinguish two cases according to the shape of a_1:

- If $a_1 = \emptyset$, and thus $A_1' < f(e_0)$, by condition (2) in the definition of IES-morphism it follows that there exists $A_0 \subseteq f^{-1}(A_1')$ such that $A_0 < e_0$. Since $e_0 \in C_0$, by Proposition 39.(1), $A_0 \cap C_0$ is non-empty (precisely, it is a singleton). Take $e_0'' \in A_0 \cap C_0$. By rule $(\nearrow 2)$, $e_0'' \nearrow e_0$ and thus, by Proposition 39.(2), we have $e_0'' \hookrightarrow_0 e_0$. Hence, by construction, $f(e_0'') \hookrightarrow_1 f(e_0)$. Notice that $f(e_0'') \in A_1' \cap f(C_0) = A_1$.
- If $a_1 = \{f(e_0')\}$, then by condition (3) in the definition of IES-morphism we can find $a_0 \subseteq \{e_0'\}$ and $A_0 \subseteq f^{-1}(A_1')$ such that $\vdash\!\circ_0(a_0, e_0, A_0)$.

  If $a_0 = \emptyset$ we proceed as in the previous case. If instead $a_0 = \{e_0'\}$ then, by definition of choice $e_0 \hookrightarrow_0 e_0'$ or $e_0'' \hookrightarrow_0 e_0$ for $e_0'' \in A_0$. Therefore $f(e_0) \hookrightarrow_1 f(e_0')$ or $f(e_0'') \hookrightarrow_1 f(e_0)$ (and observe that $f(e_0'') \in A_1$).

As for condition (2), to show that $\hookrightarrow_1$ is acyclic, first observe that a IES-morphism is injective on a configuration. In fact, if $e_0, e_0' \in C_0$ and $f(e_0) = f(e_0')$ then $e_0 = e_0'$ or $e_0 \# e_0'$. But, by Proposition 39.(3), the second possibility cannot arise. Now, if there were a cycle of $\hookrightarrow_1$ then, by the above observation and by definition of $\hookrightarrow_1$, a cycle should have been already present in $\hookrightarrow_0$, contradicting the hypothesis that C_0 is a configuration.

Finally, observe that also condition (3) holds, since by an analogous reasoning, the finitariness of the choice in C_0 implies the finitariness of the choice in $f(C_0)$.

Let us show that $f^* : Conf(I_0) \to Conf(I_1)$ is a morphism in **Dom**.

- *If C and C' are compatible then $f^*(C \sqcap C') = f^*(C) \sqcap f^*(C')$.*
  Recalling how the greatest lower bound of configurations is computed (see Lemma 44.(3)), we have that

  $$f^*(C \sqcap C') = \langle f(C \cap C'), f(\hookrightarrow_C \cap \hookrightarrow_{C'}) \cap choices(f(C \cap C')) \rangle,$$

  while

  $$\begin{aligned} & f^*(C) \sqcap f^*(C') = \\ & \quad = \langle f(C), f(\hookrightarrow_C) \cap choices(f(C)) \rangle \sqcap \langle f(C'), f(\hookrightarrow_{C'}) \cap choices(f(C')) \rangle \\ & \quad = \langle f(C) \cap f(C'), f(\hookrightarrow_C) \cap f(\hookrightarrow_{C'}) \cap choices(f(C)) \cap choices(f(C')) \rangle \end{aligned}$$

  Observe that f is injective on $C \cup C'$ since C and C' have an upper bound C'', and, as already observed, f is injective on configurations. By using this fact, we can deduce that $f(C) \cap f(C') = f(C \cap C')$, $f(\hookrightarrow_C) \cap f(\hookrightarrow_{C'}) = f(\hookrightarrow_C \cap \hookrightarrow_{C'})$. Moreover it is easy to see that $choices(C \cap C') = choices(C) \cap choices(C')$ holds in general. Therefore we conclude that $f^*(C \sqcap C') = f^*(C) \sqcap f^*(C')$.

- $f^*(\bigsqcup X) = \bigsqcup f^*(X)$*, for* $X \subseteq Conf(I_0)$ *pairwise compatible*.
  Keeping in mind the characterisation of the least upper bound given in Lemma 44.(4), we obtain

$$\begin{aligned}
&\bigsqcup f^*(X) = \\
&\quad = \langle \bigcup\{f(C) \mid C \in X\}, \bigcup\{f(\hookrightarrow_C) \cap choices(f(C)) \mid C \in X\}\rangle \\
&\quad = \langle f(\bigcup X), f(\bigcup\{\hookrightarrow_C \mid C \in X\}) \cap choices(f(\bigcup X))\rangle \\
&\quad = f^*(\langle \bigcup X, \bigcup\{\hookrightarrow_C \mid C \in X\}\rangle) \\
&\quad = f^*(\bigsqcup X)
\end{aligned}$$

  To understand the second passage observe that

$$\begin{aligned}
&\bigcup\{f(\hookrightarrow_C) \cap choices(f(C)) \mid C \in X\} \subseteq && \text{[by set-theoretical properties]} \\
&\quad \subseteq \bigcup\{f(\hookrightarrow_C) \mid C \in X\} \cap \bigcup\{choices(f(C)) \mid C \in X\} && \text{[by definition of } choices\text{]} \\
&\quad \subseteq f(\bigcup\{\hookrightarrow_C \mid C \in X\}) \cap choices(f(\bigcup X))
\end{aligned}$$

  Therefore Proposition 41.(2) and the equality $\bigcup\{f(C) \mid C \in X\} = f(\bigcup X)$ allow us to conclude.

- $C \prec C'$ *implies* $f^*(C) \preceq f^*(C')$.
  This property immediately follows from the observation that, as in the case of AES's, $C \prec C'$ iff $C \sqsubseteq C'$ and $|C' - C| = 1$.

□

Theorem 48. *The functor* $\mathscr{P}_i : \mathbf{Dom} \to \mathbf{IES}$ *is left adjoint to* $\mathscr{L}_i : \mathbf{IES} \to \mathbf{Dom}$*. The counit of the adjunction at an* IES *I is the function* $\varepsilon_I : \mathscr{P}_i \circ \mathscr{L}_i(I) \to I$*, mapping each history of an event e into the event e itself, i.e.,* $\varepsilon_I(C[\![e]\!]) = e$*, for all* $C \in Conf(I)$ *and* $e \in C$*.*

Proof. Let I be an IES and let $\varepsilon_I : \mathscr{P}_i(\mathscr{L}_i(I)) \to I$ be the function defined as $\varepsilon_I(C[\![e]\!]) = e$, for all $C \in Conf(I)$ and $e \in C$. Let us prove that ε_I is a well-defined IES-morphism by showing that ε_I satisfies conditions (1)-(3) of Definition 32.

1. $\varepsilon_I(C[\![e]\!]) = \varepsilon_I(C'[\![e']\!]) \wedge C[\![e]\!] \neq C'[\![e']\!] \quad \Rightarrow \quad C[\![e]\!] \# C'[\![e']\!]$.
   Assume that $\varepsilon_I(C[\![e]\!]) = \varepsilon_I(C'[\![e']\!])$, namely $e = e'$, and $C[\![e]\!] \neq C'[\![e']\!]$. By Lemma 44.(2) it follows that there is no upper bound for $\{C, C'\}$. In fact, if there were an upper bound C'' then necessarily $C[\![e]\!] = C''[\![e]\!] = C'[\![e]\!]$. Hence $e \# e'$.

2. $A_1 < \varepsilon_I(C[\![e]\!]) \quad \Rightarrow \quad \exists A_0 \subseteq \varepsilon_I^{-1}(A_1).\ A_0 < C[\![e]\!]$.
   Let us assume $A_1 < \varepsilon_I(C[\![e]\!]) = e$. Since $e \in C$, by Proposition 39.(1), $A_1 \cap C = \{e'\}$ for some e'. Moreover, since $e' \in A_1 < e$, by rule $(\nearrow 2)$, $e' \nearrow e$ and thus, by Proposition 39.(2) and the definition of history, $e' \in C[\![e]\!]$.

   By point (1) of Lemma 44, one easily derives that $C[\![e']\!] \sqsubseteq C[\![e]\!]$. Therefore, according to the definition of $\mathscr{P}_i$, $C[\![e']\!] < C[\![e]\!]$ and since $e' \in A_1$, $\{C[\![e']\!]\} \subseteq \varepsilon_I^{-1}(A_1)$.

3. $\vdash\!\circ (\{\varepsilon_I(C'[\![e']\!])\}, \varepsilon_I(C[\![e]\!]), A_1) \;\Rightarrow\; \exists A_0 \subseteq \varepsilon_I^{-1}(A_1).\ \exists a_0 \subseteq \{C'[\![e']\!]\}.\ \vdash\!\circ (a_0, C[\![e]\!], A_0)$.
Assume $\vdash\!\circ (\{\varepsilon_I(C'[\![e']\!])\}, \varepsilon_I(C[\![e]\!]), A_1)$, namely

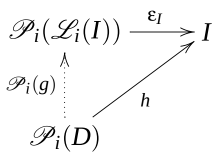
$$\vdash\!\circ (\{e'\}, e, A_1).$$

If $\neg(C[\![e]\!] \uparrow C'[\![e']\!])$ then, by definition of $\mathscr{P}_i$, $C[\![e]\!] \# C'[\![e']\!]$ and thus $C[\![e]\!] \nearrow C'[\![e']\!]$. Hence $\vdash\!\circ (\{C'[\![e']\!]\}, C[\![e]\!], \emptyset)$, which clearly satisfies the desired condition.

Suppose, instead, that $C[\![e]\!] \uparrow C'[\![e']\!]$. We distinguish two subcases:

- If $e' \in C[\![e]\!]$ then $A_1 \cap C[\![e]\!] \neq \emptyset$. Indeed, being $C[\![e]\!]$ a configuration, $A_1 \cap C[\![e]\!]$ must be a singleton $\{e''\}$. As above, by Lemma 44.(2), $C[\![e'']\!] \sqsubseteq C[\![e]\!]$ and thus, by definition of $\mathscr{P}_i$, $C[\![e'']\!] < C[\![e]\!]$. Therefore $\vdash\!\circ (\emptyset, C[\![e]\!], \{C[\![e'']\!]\})$, which allows us to conclude, since $e'' \in A_1$ implies $\{C[\![e'']\!]\} \subseteq \varepsilon_I^{-1}(A_1)$.
- Assume $e' \notin C[\![e]\!]$. Consider a configuration C'', upper bound of $C[\![e]\!]$ and $C'[\![e']\!]$, which exists by assumption. Since $e, e' \in C''$ it must be $e \hookrightarrow_{C''} e'$. In fact, otherwise there would be $e'' \in C'' \cap A_1$ and $e'' \hookrightarrow_{C''} e$. But then, by Lemma 44.(1), $e'' \in C[\![e]\!]$, an thus, being $e' < e''$, we would have $e' \in C[\![e]\!]$, contradicting the hypothesis.
  Therefore, by Lemma 44.(1), $e \in C'[\![e']\!]$, and thus $C[\![e]\!] \sqsubseteq C'[\![e']\!]$, implying $C[\![e]\!] < C'[\![e']\!]$. Hence $C[\![e]\!] \nearrow C'[\![e']\!]$, and therefore $\vdash\!\circ (\{C'[\![e']\!]\}, C[\![e]\!], \emptyset)$.

We have to show that given any domain $(D, \sqsubseteq)$ and IES-morphism $h : \mathscr{P}_i(D) \to I$, there is a unique domain morphism $g : D \to \mathscr{L}_i(I)$ such that the following diagram commutes:



The morphism $g : D \to \mathscr{L}_i(I)$ can be defined as follows. Given $d \in D$, observe that $C_d = \langle Pr(d), \sqsubset_{Pr(d)} \rangle$ is a configuration of $\mathscr{P}_i(D)$, where $\sqsubset_{Pr(d)} = \sqsubset \cap (Pr(d) \times Pr(d))$. Therefore we can define

$$g(d) = h^*(C_d).$$

The fact that $h^*(C_d)$ is a configuration in I and thus an element of $\mathscr{L}_i(I)$, follows from Lemma 46.

Moreover g is a domain morphism. In fact it is

- *$\preceq$-preserving*. By prime algebraicity, $d, d' \in D$, with $d \prec d'$ then $Pr(d') - Pr(d) = \{p\}$, for some $p \in Pr(D)$. Thus

  $$\begin{aligned} & g(d') - g(d) = \\ & \quad = h^*(Pr(d')) - h^*(Pr(d)) \\ & \quad \subseteq \{h(p)\} \end{aligned}$$

  Therefore $|g(d') - g(d)| \leq 1$ and, since it is easy to see that $g(d) \sqsubseteq g(d')$, we conclude $g(d) \preceq g(d')$.

- *Additive*. Let $X \subseteq D$ be a pairwise compatible set. Then

$$g(\bigsqcup X) = h^*(\langle C_X, \hookrightarrow_{C_X} \rangle) = \langle h(C_X), h(\hookrightarrow_{C_X}) \cap choices(h(C_X)) \rangle$$

where $C_X = Pr(\bigsqcup X) = \bigcup_{x \in X} Pr(x)$ and $\hookrightarrow_{C_X} = \sqsubset_{C_X}$. On the other hand

$$\begin{aligned}
&\textstyle\bigsqcup_{x\in X} g(x) = \\
&\quad = \textstyle\bigsqcup_{x\in X} h^*(\langle Pr(x), \sqsubset_{Pr(x)} \rangle) \\
&\quad = \langle \textstyle\bigcup_{x\in X} h(Pr(x)), \bigcup_{x\in X} (h(\sqsubset_{Pr(x)}) \cap choices(h(Pr(x)))) \rangle \\
&\quad = \langle h(C_X), \textstyle\bigcup_{x\in X} (h(\sqsubset_{Pr(x)}) \cap choices(h(Pr(x)))) \rangle
\end{aligned}$$

Now, the choice relation of the configuration above is included in the choice of the configuration $g(\bigsqcup X)$, namely

$$\textstyle\bigcup_{x\in X} (h(\sqsubset_{Pr(x)}) \cap choices(h(Pr(x)))) \subseteq h(\hookrightarrow_{C_X}) \cap choices(C_X)$$

Thus by using Proposition 41.(2) we can conclude that $g(\bigsqcup X) = \bigsqcup_{x\in X} g(x)$.

- *Stable*. Let $d, d' \in D$ with $d \uparrow d'$, then:

$$g(d \sqcap d') = h^*(\langle C, \hookrightarrow_C \rangle) = \langle h(C), h(\hookrightarrow_C) \cap choices(h(C)) \rangle,$$

where $C = Pr(d \sqcap d') = Pr(d) \cap Pr(d')$ and $\hookrightarrow_C = \sqsubset_C$. Moreover

$$\begin{aligned}
&g(d) \sqcap g(d') = \\
&\quad = \langle h(Pr(d)), h(\sqsubset_{Pr(d)}) \cap choices(h(Pr(d))) \rangle \\
&\qquad \sqcap \langle h(Pr(d')), h(\sqsubset_{Pr(d')}) \cap choices(h(Pr(d'))) \rangle
\end{aligned}$$

Now, since $d \uparrow d'$ it is easy to see that h is injective on $Pr(d) \cup Pr(d')$ and therefore the set of events of $g(d) \sqcap g(d')$ is

$$h(Pr(d)) \cap h(Pr(d')) = h(Pr(d) \cap Pr(d')) = h(C),$$

namely it coincides with the set of events of $g(d \sqcap d')$.

By a similar argument, $h(\sqsubset_{Pr(d)}) \cap h(\sqsubset_{Pr(d')}) = h(\sqsubset_{Pr(d) \cap Pr(d')}) = h(\sqsubset_C)$. Moreover, reasoning as in the proof of Lemma 46, we have,

$$\begin{aligned}
&choices(h(Pr(d))) \cap choices(h(Pr(d'))) \\
&\quad = choices(h(Pr(d)) \cap h(Pr(d'))) \quad [\text{since } choices(X \cap Y) = choices(X) \cap choices(Y)] \\
&\quad = choices(h(Pr(d) \cap Pr(d'))) \quad [\text{by injectivity of } h \text{ on } C] \\
&\quad = choices(h(C))
\end{aligned}$$

and we are able to conclude that also the choice relation in $g(d) \sqcap g(d')$ is the same as in $g(d \sqcap d')$. In fact

$$h(\sqsubset_{Pr(d)}) \cap h(\sqsubset_{Pr(d')}) \cap choices(h(Pr(d))) \cap choices(h(Pr(d')))$$
$$= h(\sqsubset_C) \cap choices(h(Pr(d) \cap Pr(d'))) \quad \text{[by injectivity of } h \text{ on } C \text{ and remark above]}$$
$$= h(\hookrightarrow_C) \cap choices(h(C))$$

The rest of the proof essentially relies on a general result which holds of any domain morphism $f : D \to \mathscr{L}_i(I)$ having as target the domain of configurations of an IES: for all $p \in Pr(D)$, $| f(p) - \bigcup f(Pr(p) - \{p\}) | \leq 1$ and

$$\mathscr{P}_i(f)(p) = \begin{cases} \perp & \text{if } f(p) - \bigcup f(Pr(p) - \{p\}) = \emptyset \\ f(p)[\![e]\!] & \text{if } f(p) - \bigcup f(Pr(p) - \{p\}) = \{e\} \end{cases}$$

Exploiting such result, the fact that morphism g defined as above makes the diagram commute and its uniqueness follow as easy consequences. □

A.4 Removing non-executable events from an IES

Proposition 52. *Let I_0 and I_1 be IES's and let $f : I_0 \to I_1$ be an IES-morphism. Then $\Psi(f) : \Psi(I_0) \to \Psi(I_1)$, defined as in Definition 51, is an IES-morphism. Hence Ψ is a well-defined functor.*

Proof. We start observing that for any IES I and for any $e, e' \in \psi(E)$ and $A \subseteq E$

F1. $e \nearrow_I e' \quad \Rightarrow \quad e \nearrow_{\Psi(I)} e'$;

F2. $A <_I e \quad \Rightarrow \quad (A \cap \psi(E)) <_{\Psi(I)} e$

F3. $\#_I A \,\wedge\, A \subseteq \psi(E) \quad \Rightarrow \quad \#_{\Psi(I)} A$.

Now, notice that

$$f(\psi(E_0)) \subseteq \psi(E_1) \tag{†}$$

and thus the restriction $f_{|\psi(E_0)} : \psi(E_0) \to \psi(E_1)$ is a well-defined function. In fact, if $e_0 \in \psi(E_0)$ then $e_0 \in C_0$ for some configuration $C_0 \in Conf(I_0)$. Hence, if defined, $f(e_0) \in f(C_0)$ and, by Lemma 46, $f^*(C_0)$ is a configuration of I_1. Thus $f(e_0) \in \psi(E_1)$.

For $i \in \{0,1\}$, let us denote by $\vdash\!\circ_i$, $<_i$, $\nearrow_i$ and $\#_i$ the relations in I_i, and by $\vdash\!\circ_{\psi_i}$, $<_{\psi_i}$, $\nearrow_{\psi_i}$ and $\#_{\psi_i}$ the relations in $\langle \psi(E_i), \vdash\!\circ_{\psi(E_i)} \rangle$, the pre-IES which, when saturated, gives the IES $\Psi(I_i)$. To show that $\Psi(f) : \Psi(I_0) \to \Psi(I_1)$ is an IES-morphism we verify that $\Psi(f) : \langle \psi(E_0), \vdash\!\circ_{\psi(E_0)} \rangle \to \langle \psi(E_1), \vdash\!\circ_{\psi(E_1)} \rangle$ satisfies conditions (1)-(4) of Lemma 37, namely

1. $\Psi(f)(e_0) = \Psi(f)(e_0') \,\wedge\, e_0 \neq e_0' \quad \Rightarrow \quad e_0 \#_{\psi_0} e_0'$;
2. $\vdash\!\circ_{\psi_1}(\emptyset, \Psi(f)(e_0), A_1) \quad \Rightarrow \quad \exists A_0 \subseteq \Psi(f)^{-1}(A_1).\ A_0 <_{\psi_0} e_0$;
3. $\vdash\!\circ_{\psi_1}(\{\Psi(f)(e_0')\}, \Psi(f)(e_0), \emptyset) \quad \Rightarrow \quad e_0 \nearrow_{\psi_0} e_0'$;
4. $\vdash\!\circ_{\psi_1}(\{\Psi(f)(e_0')\}, \Psi(f)(e_0), A_1) \,\wedge\, A_1 \neq \emptyset \quad \Rightarrow$
$\exists A_0 \subseteq \Psi(f)^{-1}(A_1).\ \exists a_0 \subseteq \{e_0'\}.\ \vdash\!\circ_{\psi_0}(a_0, e_0, A_0)$.

To lighten the notation let f' denote $\Psi(f)$, i.e., the restriction $f_{|\psi(E_0)}$.

1. If $f'(e_0) = f'(e'_0)$ and $e_0 \neq e'_0$, since $f : I_0 \to I_1$ is an IES-morphism, it must be the case that $e_0 \#_0 e'_0$. Hence, by Fact (F3) above, $e_0 \#_{\psi_0} e'_0$.

2. Assume that $\vdash\!\circ_{\psi_1}(\emptyset, f'(e_0), A_1)$. By definition of $\Psi(I_1)$, recalling that $f'(e_0) = f(e_0)$, we have $\vdash\!\circ_1(\emptyset, f(e_0), A'_1)$, with $A_1 = A'_1 \cap \psi(E_1)$. Since, by definition of IES, $\#_p A'_1$, we can apply rule (< 1), thus obtaining

$$\frac{\vdash\!\circ_1(\emptyset, f(e_0), A'_1) \quad \#_p A'_1}{A'_1 <_1 f(e_0)} \ (< 1)$$

   By definition of morphism, there exists $A'_0 \subseteq f^{-1}(A'_1)$ such that $A'_0 <_0 e_0$. If we define $A_0 = A'_0 \cap \psi(E_0)$ then, by Fact (F1) above, $A_0 <_{\psi_0} e_0$ and, by the property (†) above, $A_0 \subseteq f'^{-1}(A_1)$.

3. Assume that $\vdash\!\circ_{\psi_1}(\{f'(e'_0)\}, f'(e_0), \emptyset)$. By definition of $\vdash\!\circ_{\psi_1}$ and recalling that f' is the restriction of f, it must be the case that $\vdash\!\circ_1(\{f(e'_0)\}, f(e_0), A_1)$ with $A_1 \cap \psi(E_1) = \emptyset$. Hence, by definition of morphism, there exist $a_0 \subseteq \{e'_0\}$ and $A_0 \subseteq f^{-1}(A_1)$ such that $\vdash\!\circ_0(a_0, e_0, A_0)$. Since $A_1 \cap \psi(E_1) = \emptyset$, we deduce that $A_0 \cap \psi(E_0) = \emptyset$. Moreover, recalling that $e_0 \in \psi(E_0)$, namely it is executable, necessarily $a_0 = \{e'_0\}$. Therefore $\vdash\!\circ_{\psi_0}(\{e'_0\}, e_0, \emptyset)$, and thus $e_0 \nearrow_{\psi_0} e'_0$.

4. Assume that $\vdash\!\circ_{\psi_1}(\{f'(e'_0)\}, f'(e_0), A_1)$ with $A_1 \neq \emptyset$. Then, by definition of $\vdash\!\circ_{\psi_1}$, we must have

$$\vdash\!\circ_1(\{f(e'_0)\}, f'(e_0), A'_1)$$

   where $A_1 = A'_1 \cap \psi(E_1)$. By definition of IES-morphism, there must exist $A'_0 \subseteq f^{-1}(A'_1)$ and $a_0 \subseteq \{e'_0\}$ such that $\vdash\!\circ_0(a_0, e_0, A'_0)$.

   If we define $A_0 = A'_0 \cap \psi(E_0)$, then by definition of $\vdash\!\circ_{\psi_0}$, we have $\vdash\!\circ_{\psi_0}(a_0, e_0, A_0)$ and, by the property (†) proved above, $A_0 \subseteq f'^{-1}(A_1)$. □

A.5 Event structure semantics for i-nets

Proposition 56. *Let N_0 and N_1 be occurrence i-nets and let $h : N_0 \to N_1$ be an i-net morphism. Then $h_T : I_{N_0} \to I_{N_1}$ is a IES-morphism.*

Proof. For $i \in \{0, 1\}$, let $<_i$, $\nearrow_i$ and $\#_i$ be the relations of causality, asymmetric conflict and conflict in the pre-IES $I^p_{N_i} = \langle E_i, \vdash\!\circ^p \rangle$. We show that $h_T : I^p_0 \to I^p_1$ satisfies conditions (1)-(4) in the hypotheses of Lemma 37 and thus h_T is an IES-morphism between the corresponding "saturated" IES's.

1. $h_T(t_0) = h_T(t'_0) \wedge t_0 \neq t'_0 \quad \Rightarrow \quad t_0 \#_0 t'_0$.
   This property can be proved exactly as for ordinary nets.

2. $\vdash\!\circ^{p}_{1}(\emptyset, h_T(t_0), A_1) \quad \Rightarrow \quad \exists A_0 \subseteq h_T^{-1}(A_1).\ A_0 <_0 t_0$.
Let us assume $\vdash\!\circ^{p}_{1}(\emptyset, h_T(t_0), A_1)$. By the definition of $\vdash\!\circ^{p}_{1}$ we can have

(a) $A_1 = \{t_1\}$ and $t_1{}^{\bullet} \cap {}^{\bullet}h_T(t_0) \neq \emptyset$.
Consider $s_1 \in t_1{}^{\bullet} \cap {}^{\bullet}h_T(t_0)$. By Lemma 59 there must exist $s_0 \in {}^{\bullet}t_0$ such that $h_S(s_0, s_1)$, and $t'_0 \in T_0$ such that $h_T(t'_0) = t_1$ and $s_0 \in t'_0{}^{\bullet}$. By definition of $\vdash\!\circ^{p}_{0}$, if we define $A_0 = \{t'_0\}$, it follows that $\vdash\!\circ^{p}_{0}(\emptyset, t_0, A_0)$, and thus by rule (< 1), $A_0 < t_0$. Recalling that $t'_0 \in h_T^{-1}(t_1)$ and thus $A_0 \subseteq h_T^{-1}(A_1)$ we conclude.

(b) $A_1 = \{t_1\}$ and $t_1{}^{\bullet} \cap \underline{h_T(t_0)} \neq \emptyset$.
Analogous to case (a).

(c) $\exists s_1 \in {}^{\circledcirc}h_T(t_0).\ {}^{\bullet}s_1 = \emptyset \ \wedge\ s_1{}^{\bullet} = A_1$.
Since ${}^{\bullet}s_1 = \emptyset$, namely s_1 is in the initial marking m_1 of N_1, by definition of i-net morphism, there exists a unique $s_0 \in m_0$ such that $h_S(s_0, s_1)$. Again, by definition of i-net morphism, from $s_1 \in {}^{\circledcirc}h_T(t_0)$ and $h_S(s_0, s_1)$ it follows that $s_0 \in {}^{\circledcirc}t_0$. Hence $\vdash\!\circ^{p}_{0}({}^{\bullet}s_0, t_0, s_0{}^{\bullet})$, namely, recalling that $s_0 \in m_0$,

$$\vdash\!\circ^{p}_{0}(\emptyset, t_0, s_0{}^{\bullet}).$$

Therefore, by rule (< 1), we have $s_0{}^{\bullet} <_0 t_0$. Observe that, by the condition (2.a) in the definition of i-net morphisms, $h_T(s_0{}^{\bullet}) \subseteq s_1{}^{\bullet}$ and, since $h_S(s_0, s_1)$, necessarily h is defined on each $t'_0 \in s_0{}^{\bullet}$. Thus $s_0{}^{\bullet} \subseteq h_T^{-1}(s_1{}^{\bullet})$ concluding the proof for this case.

3. $\vdash\!\circ^{p}_{1}(\{h_T(t'_0)\}, h_T(t_0), \emptyset) \quad \Rightarrow \quad t_0 \nearrow_0 t'_0$.
By definition of $\vdash\!\circ^{p}_{1}$, we can have

(a) $({}^{\bullet}h_T(t_0) \cup \underline{h_T(t_0)}) \cap {}^{\bullet}h_T(t'_0) \neq \emptyset$.
Let $s_1 \in ({}^{\bullet}h_T(t_0) \cup \underline{h_T(t_0)}) \cap {}^{\bullet}h_T(t'_0)$. If s_1 is in the initial marking then, by the definition of i-net morphisms, one easily deduces that there exists a unique place $s_0 \in S_0$ such that $h_S(s_0, s_1)$ and moreover $s_0 \in ({}^{\bullet}t_0 \cup \underline{t_0}) \cap {}^{\bullet}t'_0$. Therefore, by definition, $\vdash\!\circ^{p}_{0}(\{t'_0\}, t_0, \emptyset)$ and thus, by rule $(\nearrow 1)$, $t_0 \nearrow_0 t'_0$.

Suppose instead that $s_1 \notin m_1$. If $({}^{\bullet}t_0 \cup \underline{t_0}) \cap {}^{\bullet}t'_0 \neq \emptyset$ then we conclude as above. Otherwise, one easily deduces that $t_0 \#_0 t'_0$, and therefore, by rule $(\nearrow 3)$, we can conclude $t_0 \nearrow_0 t'_0$.

(b) $\exists s_1 \in h_T(t'_0)^{\bullet} \cap {}^{\circledcirc}h_T(t_0) \ \wedge\ s_1{}^{\bullet} = \emptyset$.
By condition (2.c) in the definition of i-net morphism (Definition 3), there must be $s_0 \in t'_0{}^{\bullet}$ such that $h_S(s_0, s_1)$. By condition (2.d) in the same definition, $s_0 \in {}^{\circledcirc}t_0$. Observing that necessarily $s_0{}^{\bullet} = \emptyset$, we conclude $\vdash\!\circ^{p}_{0}(\{t'_0\}, t_0, \emptyset)$ and thus $t_0 \nearrow_0 t'_0$.

4. $\vdash\!\circ^{p}_{1}(\{h_T(t'_0)\}, h_T(t_0), A_1) \wedge A_1 \neq \emptyset \quad \Rightarrow \quad \exists A_0 \subseteq h_T^{-1}(A_1).\ \exists a_0 \subseteq \{t'_0\}.\ \vdash\!\circ^{p}_{0}(a_0, t_0, A_0)$.
Assume $\vdash\!\circ^{p}_{1}(\{h_T(t'_0)\}, h_T(t_0), A_1)$ and $A_1 \neq \emptyset$. Thus, by definition of $\vdash\!\circ^{p}_{1}$ there is a place $s_1 \in {}^{\circledcirc}h_T(t_0) \cap h_T(t'_0)^{\bullet}$ such that $A_1 = s_1{}^{\bullet}$. Hence there is $s_0 \in t'_0{}^{\bullet}$ such that $h_S(s_0, s_1)$. By condition (2.a) in the definition of i-net morphism $h_T(s_0{}^{\bullet}) \subseteq s_1{}^{\bullet} = A_1$ and necessarily h_T is defined on each $t''_0 \in s_0{}^{\bullet}$. Therefore

$$s_0{}^{\bullet} \subseteq h_T^{-1}(A_1).$$

Since $s_1 \in {}^{\circledcirc}h_T(t_0)$ and $h_S(s_0, s_1)$, by condition (2.d) in the definition of i-net morphism, $s_0 \in {}^{\circledcirc}t_0$. Hence we conclude that, as desired, $\vdash\!\circ_{N_0}(\{t'_0\}, t_0, s_0{}^{\bullet})$. $\square$

Lemma 60. *Let P be a* PES*, let N_0 be an occurrence i-net and let $h_T : \mathscr{J}_i(P) \to \mathscr{E}_i(N_0)$ be an* IES*-morphisms. Then there exists a unique h_S such that $h = \langle h_T, h_S \rangle : \mathscr{N}_i(P) \to N_0$ is an i-net morphism.*

Proof . Consider the contextual net $\mathscr{R}_{ic}(\mathscr{N}_i(P))$, obtained from $\mathscr{N}_i(P)$ by removing the inhibitor arcs. Then there exists a *unique* h_S such that $h = \langle h_T, h_S \rangle : \mathscr{R}_{ic}(\mathscr{N}_i(P)) \to \mathscr{R}_{ic}(N_0)$ is a contextual net morphism. The relation h_S is defined by taking the conditions of Lemma 59 specialised to the net $\mathscr{N}_i(P)$, that is, for all $s = \langle x, A, B \rangle \in S$ and $s_0 \in S_0$:

$$
\begin{aligned}
h_S(s, s_0) \qquad \text{iff} \quad & ((x = \emptyset \ \wedge \ s_0 \in m_0) \ \vee \ (x = \{t\} \ \wedge \ s_0 \in h_T(t)^{\bullet})) \\
& \wedge B = h_T^{-1}(s_0{}^{\bullet}) \cap \lceil x \rceil \\
& \wedge A = h_T^{-1}(\underline{s_0}) \cap \lceil x \rceil
\end{aligned}
$$

This can be proved along the same lines of Theorem 7.3 in [6].

Therefore, to conclude the validity of the thesis we only need to prove that h, seen as a morphism $h = \langle h_T, h_S \rangle : \mathscr{N}_i(P) \to N_0$, is a well-defined i-net morphism. To this aim, observe that $h : \mathscr{R}_{ic}(\mathscr{N}_i(P)) \to \mathscr{R}_{ic}(N_0)$ is a c-net morphism and thus it satisfies conditions (1), (2.a)-(2.c) of Definition 3. Hence it remains only to verify the validity of condition (2.d), i.e., that for all $e \in T$, $h_S^{-1}({}^{\circledcirc}h_T(e)) \subseteq {}^{\circledcirc}e$. Let $s = \langle x, A, B \rangle \in S$ and assume $s \in h_S^{-1}({}^{\circledcirc}h_T(e))$, namely that there exists $s_0 \in {}^{\circledcirc}h_T(e)$ such that $h_S(s, s_0)$. We distinguish two cases

$(x = \emptyset)$ In this case, in $\mathscr{E}_i(N_0)$ we have $\vdash\!\circ\, (\emptyset, h_T(e), s_0{}^{\bullet})$ and thus $s_0{}^{\bullet} < h_T(e)$. Since h_T is an IES-morphism, there exists $X \subseteq h_T^{-1}(s_0{}^{\bullet})$ such that $X < e$. By definition of h_S we have $h_T^{-1}(s_0{}^{\bullet}) = B$ and thus, by definition of $\mathscr{N}_i$, $e \in {}^{\circledcirc}s$, namely $s \in {}^{\circledcirc}e$

$(x = \{e'\})$ In this case ${}^{\bullet}s = \{e'\}$. Hence, by Lemma 59,

$$
{}^{\bullet}s_0 = h_T({}^{\bullet}s) = \{h_T(e')\}
$$

and thus $\vdash\!\circ\, (\{h_T(e')\}, h_T(e), s_0{}^{\bullet})$ in $\mathscr{E}_i(N_0)$. Therefore, by definition of IES-morphism, there exist $y \subseteq \{e'\}$ and $X \subseteq h_T^{-1}(s_0{}^{\bullet})$ such that $\vdash\!\circ\, (y, e, X)$ in $\mathscr{J}_i(P)$. Since P is a PES we have two possibilities:

i. $X = \{e''\}$, $y = \emptyset$, and thus $e'' < e$.
   Since $e'' \in h_T^{-1}(s_0{}^{\bullet})$, we have $h_T(e') < h_T(e'')$ and thus, by Proposition 33, $e' < e''$ or $e' \# e''$. In the first case $e'' \in B$ and thus $e \in {}^{\circledcirc}s$, while, in the second case, $e' \# e$, and thus $\vdash\!\circ\, (\{e'\}, e, \emptyset)$, implying (since $\emptyset \subseteq B$) that $e \in {}^{\circledcirc}s$.

ii. $X = \emptyset$.
   Since trivially $X \subseteq B$, by definition of $\mathscr{N}_i$ we have $e \in {}^{\circledcirc}s$. □

Lemma 61. *For any* PES *P, the identity over the events $\rho_P : \mathscr{J}_i(P) \to \mathscr{E}_i(\mathscr{N}_i(P))$ is an* IES*-isomorphism.*

Proof . We first observe that η_P is a well-defined IES-morphism. To this aim we prove that the identity, seen as a mapping from $\mathscr{J}_i(P)$ to the pre-IES associated to $\mathscr{N}_i(P)$ (whose DE-relation is denoted as $\vdash\!\circ_N$) satisfies the conditions of Lemma 37. Condition (1) trivially holds, while (2)-(4) are discussed below, where the subscript P is used to refer to the dependency relations of $\mathscr{J}_i(P)$.

2. $\mapsto_N(\emptyset, e, A) \quad \Rightarrow \quad \exists A' \subseteq A.\ A' <_P e.$

Let $\mapsto_N(\emptyset, e, A)$. We distinguish two possibilities. If $A = \{e'\}$ and $e'^\bullet \cap ({}^\bullet e \cup \underline{e}) \neq \emptyset$ in $\mathscr{N}_i(P)$, then $e' <_P e$. Otherwise, there is place s in $\mathscr{N}_i(P)$ such that $e \in {}^{\circledcirc}s$ and $A = s^\bullet$. Thus, by definition of $\mathscr{N}_i$, there is $e' \in A$ such that $e' <_P e$.

3. $\mapsto_N(\{e'\}, e, \emptyset) \quad \Rightarrow \quad e \nearrow_P e'.$

Let $\mapsto_N(\{e'\}, e, \emptyset)$. This triple is generated in two cases. The first one is that $({}^\bullet e \cup \underline{e}) \cap {}^\bullet e' \neq \emptyset$ in the net $\mathscr{N}_i(P)$ and thus $e \nearrow_P e'$. Otherwise there must exist $s \in {}^{\circledcirc}e$, with ${}^\bullet s = \{e'\}$ and $s^\bullet = \emptyset$. Hence, by definition of I (see Definition 58), $e \nearrow_P e'$.

4. $\mapsto_N(\{e'\}, e, A) \wedge A \neq \emptyset \ \Rightarrow\ \exists A' \subseteq A. \exists a \subseteq \{e'\}.\ \mapsto_P(a, e, A').$

Let $\mapsto_N(\{e'\}, e, A)$ and $A \neq \emptyset$. Therefore there exists a place s in $\mathscr{N}_i(P)$, with ${}^\bullet s = \{e'\}$, $e \in {}^{\circledcirc}s$ and $A = s^\bullet$. Hence, by definition of I (see Definition 58), there are two possibilities:

- $\exists e'' \in x.\ e \nearrow e''$. Since $x = \{e'\}$ this implies $e \nearrow e'$ and thus $\mapsto_P(\{e'\}, e, \emptyset)$.
- $\exists e'' \in A.\ e'' < e$. Hence $\mapsto_P(\emptyset, e, \{e''\})$.

Observe that in both cases we can conclude the existence of $A' \subseteq s^\bullet = A$ (possibly empty) and $a \subseteq \{e'\}$ such that $\mapsto_P(a, e, A')$.

A similar reasoning shows that the identity on events is a morphism also from $\mathscr{E}_i(\mathscr{N}_i(P))$ to P. Hence ρ_P is an isomorphism. $\square$

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