FLAT MITTAG-LEFFLER MODULES OVER COUNTABLE RINGS

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Abstract. We show that over any ring, the double Ext-orthogonal class to all flat Mittag-Leffler modules contains all countable direct limits of flat Mittag-Leffler modules. If the ring is countable, then the double orthogonal class consists precisely of all flat modules and we deduce, using a recent result of Šaroch and Trlifaj, that the class of flat Mittag-Leffler modules is not precovering in $\text{Mod-}R$ unless $R$ is right perfect.

Introduction

The notion of a Mittag-Leffler module was introduced by Raynaud and Gruson [10], who used the concept to prove a conjecture due to Grothendieck that the projectivity of infinitely generated modules over commutative rings is a local property. This is a crucial step for defining and working with infinitely generated vector bundles, as considered by Drinfeld in [3], where we also refer for more explanation.

The main step behind this geometrically motivated result is a completely general characterization of projective modules over any (in general non-commutative) ring $R$. Namely, one can show (consult [3]) that an $R$-module $M$ satisfies the following three conditions:

1. $M$ is flat,
2. $M$ is Mittag-Leffler,
3. $M$ is a direct sum of countably generated modules.

As mentioned by Drinfeld, the proof of projectivity of a given module might be non-constructive even in very simple cases, because it requires the Axiom of Choice. This applies for instance to the ring $\mathbb{Q}$ of rational numbers and the $\mathbb{Q}$-module $\mathbb{R}$ of real numbers. The main trouble there is condition (3). Thus, one might consider replacing projective modules by flat Mittag-Leffler modules (these are called “projective modules with a human face” in a preliminary version of [3]).

However, a surprising result in [5, §5] indicates that if one is interested in homological algebra, this might not be a good idea at all. Namely, the class of flat Mittag-Leffler abelian groups does not provide for precovers (sometimes also called right approximations). In the present paper, we...
use recent results due to Šaroch and Trlifaj [11] to show this is a much more general phenomenon and applies to many geometrically interesting examples. Namely, we prove in Theorem 6 that the class of flat Mittag-Leffler $R$-modules over a countable ring $R$ is precovering if and only if $R$ is a right perfect ring. Note that in that case the classes of projective modules, flat Mittag-Leffler modules and flat modules coincide, so the flat Mittag-Leffler precovers are just the projective ones.

1. Preliminaries

In this paper, $R$ will always be an associative, not necessarily commutative ring with a unit. If not specified otherwise, a module will stand for a right $R$-module. We will denote by $D$ the class of all modules which are flat and satisfy the Mittag-Leffler condition in the sense of [10, 8]:

**Definition 1.** $M$ is called a Mittag-Leffler module if the canonical morphism $\rho : M \otimes_R \prod_{i \in I} Q_i \rightarrow \prod_{i \in I} M \otimes_R Q_i$ is injective for each family $(Q_i \mid i \in I)$ of left $R$-modules.

A crucial closure property of the class $D$ has been obtained in [8]:

**Proposition 2.** [8, Proposition 2.2] Let $R$ be a ring and $(F_i, \eta_{ji} : F_i \rightarrow F_j)$ be a direct system of modules from $D$ indexed by $(I, \leq)$. Assume that for each increasing chain $(i_n \mid n < \omega)$ in $I$, the module $\lim_{n < \omega} F_{i_n}$ belongs to $D$. Then $M = \lim_{i \in I} F_i$ belongs to $D$.

Let us look more closely at countable chains of modules and their limits. Recall that given a sequence of morphisms

$$F_0 \xrightarrow{\eta_0} F_1 \xrightarrow{\eta_1} F_2 \xrightarrow{\eta_2} F_3 \rightarrow \ldots$$

we have a short exact sequence

$$\eta : 0 \rightarrow \bigoplus_{n < \omega} F_n \xrightarrow{\varphi} \bigoplus_{n < \omega} F_n \rightarrow \lim_{n < \omega} F_n \rightarrow 0,$$

such that $\varphi$ is defined by $\varphi_{i_n} = \iota_n - \iota_{n+1} \iota_m$, where $\iota_n : F_n \rightarrow \bigoplus F_m$ are the canonical inclusions. Note the following simple fact:

**Lemma 3.** Given a chain $F_0 \xrightarrow{\eta_0} F_1 \xrightarrow{\eta_1} F_2 \rightarrow \ldots$ of morphisms as above and a number $n_0 < \omega$, the middle term of $(\ast)$ decomposes as

$$\bigoplus_{n < \omega} F_n = \varphi \left( \bigoplus_{m < n_0} F_m \right) \oplus \left( \bigoplus_{m \geq n_0} F_m \right).$$

**Proof.** Note that the module $\varphi(\bigoplus_{m < n_0} F_m)$ is generated by elements of the form

$$(0, \ldots, 0, x_m, -u_m(x_m), 0 \ldots) \in \bigoplus_{n < \omega} F_n,$$

where $m < n_0$ and $x_m \in F_m$. It follows easily that one can uniquely express each $y = (y_n) \in \bigoplus_{n < \omega} F_n$ as $y = z + w$, where $z \in \varphi(\bigoplus_{m < n_0} F_m)$ and $w \in \bigoplus_{m \geq n_0} F_m$. Namely, we take $z = (y_0, \ldots, y_{n_0-1}, y_{n_0^l} 0, 0, \ldots)$ with $-y_{n_0} = u_{n_0-1}(y_{n_0-1}) + u_{n_0-1}u_{n_0-2}(y_{n_0-2}) + \cdots + u_{n_0-1}u_{n_0-2} \cdots u_1u_0(y_0)$ and $w = y - z \in \bigoplus_{m \geq n_0} F_m$. \qed
We will also need a few simple results concerning infinite combinatorics, starting with a well known lemma.

**Lemma 4.** For any cardinal $\mu$ there is a cardinal $\lambda \geq \mu$ such that $\lambda^{\aleph_0} = 2^\lambda$.

**Proof.** We refer for instance to [7, Lemma 3.1]. For the reader’s convenience, we recall how to construct $\lambda$. We put $\mu_0 = \mu$ and for each $n < \omega$ inductively construct $\mu_{n+1} = 2^{\mu_n}$. Then $\lambda = \sup_{n<\omega} \mu_n$ has the required property, see for example [9, p. 50, fact (6.21)].

The next lemma deals with a construction of a large family of “almost disjoint” maps $f : \omega \to \lambda$. The result is well known in the literature and it has many different proofs. We refer for instance to [2, Lemma 2.3] or [4, Proposition II.5.5].

**Lemma 5.** Let $\lambda$ be an infinite cardinal. Then there is a subset $J \subseteq \lambda^\omega$ of cardinality $\lambda^{\aleph_0}$ such that for any pair of distinct maps $f, g : \omega \to \lambda$ of $J$, the set formed by the $x \in \omega$ on which the values $f(x)$ and $g(x)$ coincide is a finite initial segment of $\omega$.

**Proof.** Consider the tree $T$ of the finite sequences of elements of $\lambda$, i.e. $T = \{ t : n \to \lambda \mid n < \omega \}$. Since $\lambda$ is infinite, we have $|T| = |\bigcup_{n<\omega} \lambda^n| = \lambda$, so there is a bijection $b : T \to \lambda$. For every map $f : \omega \to \lambda$ denote by $A_f : \omega \to T$ the induced map which sends $n < \omega$ to $f \upharpoonright n \in T$. Clearly, if $f$, $g$ are two different maps in $\lambda^\omega$, the values of $A_f$ and $A_g$ coincide only on a finite initial segment of $\omega$. Now, we can put $J = \{ b \circ A_f \mid f \in \lambda^\omega \}$. □

## 2. Main result

Now we are in a position to state our main result, which is inspired by [8, §5]. It will be proved by using a cardinal argument similar to the one in [2, Proposition 2.5]. Note that the result sharpens [11, Theorem 2.9] by removing the additional set-theoretical assumption of Singular Cardinal Hypothesis, and also [8, Corollaries 7.6 and 7.7] by removing the assumption that $\mathcal{D}$ is closed under products.

Regarding the notation and terminology, given a class $\mathcal{C} \subseteq \text{Mod-}R$, we put $\mathcal{C}^\perp = \{ M \in \text{Mod-}R \mid \text{Ext}_R^1(C, M) = 0 \}$ and dually $\perp \mathcal{C} = \{ M \in \text{Mod-}R \mid \text{Ext}_R^1(M, C) = 0 \}$. Recall that a module is called cotorsion if it cannot be non-trivially extended by a flat module.

We recall also the notion of a precover, or sometimes called right approximation. If $\mathcal{X}$ is any class of modules and $M \in \text{Mod-}R$, a homomorphism $f : X \to M$ is called an $\mathcal{X}$-precover of $M$ if $X \in \mathcal{X}$ and for every homomorphism $f' \in \text{Hom}_R(X', M)$ with $X' \in \mathcal{X}$ there exists a homomorphism $g : X' \to X$ such that $f' = fg$. The class $\mathcal{X}$ is called precovering if each $M \in \text{Mod-}R$ admits an $\mathcal{X}$-precover.

**Theorem 6.** Let $R$ be a ring and $\mathcal{D}$ be the class of all flat Mittag-Leffler right $R$-modules. Given any countable chain

$$F_0 \xrightarrow{\alpha_0} F_1 \xrightarrow{\alpha_1} F_2 \xrightarrow{\alpha_2} F_3 \xrightarrow{\ldots}$$

of morphisms such that $F_n \in \mathcal{D}$ for all $n < \omega$, we have $\lim_{\text{lim}} F_n \in \perp(\mathcal{D}^\perp)$. If, moreover, $R$ is a countable ring, then the following hold:
(1) $D^\perp$ is precisely the class of all cotorsion modules.

(2) $D$ is a precovering class in $\text{Mod-}R$ if and only if $R$ is right perfect.

**Proof.** Assume we have a countable direct system $(F_n, u_n)$ as above, put $F = \varinjlim F_n$, and fix a module $C \in D^\perp$. We must prove that $\text{Ext}_R^1(F, C) = 0$.

Let us fix an infinite cardinal $\lambda$, depending on $C$, such that we have $\lambda \geq |\text{Hom}_R(F_n, C)|$ for each $n < \omega$ and $\lambda^\omega = 2^\lambda$; we can do this using Lemma 4. Applying Lemma 5, we find a subset $J \subseteq \omega_1$ of cardinality $2^\lambda$ such that the values of each pair $f, g : \omega \to \lambda$ of distinct elements of $J$ coincide only on a finite initial segment of $\omega$. We claim that there is a short exact sequence of the form

$$0 \to P \to E \to F(2^\lambda) \to 0$$

such that $E \in D$ and $|\text{Hom}_R(P, C)| \leq 2^\lambda$.

Let us construct such a sequence. First, denote for each $\alpha < \lambda$ by $F_n, \alpha$ a copy of $F_n$, and by $P$ the direct sum $\bigoplus_{n<\omega} F_n^{(\lambda)}$ taken over all pairs $(n, \alpha)$ such that $n < \omega$ and $\alpha = f(n)$ for some $f \in J$. Note that $P$ is a summand in $\bigoplus_{n<\omega} F_n^{(\lambda)}$, so we have

$$|\text{Hom}_R(P, C)| \leq |\text{Hom}_R\left(\bigoplus_{n<\omega} F_n^{(\lambda)}, C\right)| \leq \prod_{n<\omega} |\text{Hom}_R(F_n, C)|^\lambda \leq \lambda^{\omega \times \lambda} = 2^\lambda.$$

Next, we will construct $E$. Given $f \in J$, let

$$\iota_f : \bigoplus_{n<\omega} F_n \to P$$

be the split inclusion which sends each $F_n$ to $F_{n, f(n)}$. Using the short exact sequence $(\ast)$ from page 2, we can extend $P$ by $F$ via the following pushout diagram:

$$\vcenter{\xymatrix{ & \bigoplus_{n<\omega} F_n \ar[r]^-{\varphi} \ar[d]_-{\iota_f} & \bigoplus_{n<\omega} F_n \ar[r] \ar[d]_-{\theta_f} & F \ar[d] \ar[r] & 0 \\
0 \ar[r] & P \ar[r]^{\subseteq} & E_f \ar[r] & F \ar[r] & 0 }} \quad (\Delta)$$

Now, we can put these extensions for all $f \in J$ together. Namely, let $\sigma : P^{(J)} \to P$ be the summing map and consider the pushout diagram:

$$\vcenter{\xymatrix{ & \bigoplus_{f \in J} E_f \ar[r]^{\rho} \ar[d]_-{\sigma} & F^{(J)} \ar[d]_-{\pi} \ar[r] & 0 \\
0 \ar[r] & P \ar[r] & E \ar[r] & F^{(J)} \ar[r] & 0 }}$$

For each $g \in J$, the composition of the canonical inclusion $\nu_g : E_g \to \bigoplus_{f \in J} E_f$ with the morphism $\pi$ yields a monomorphism $E_g \to E$. In fact, if $y \in E_g$ is such that $\pi \nu_g(y) = 0$, then $\rho(\nu_g(y)) = 0$, hence the exact sequence $\varepsilon_g$ gives that $y$ is in the image of $P$ and the composition of the canonical embedding $\mu_g : P \to P^{(J)}$ with the morphism $\sigma$ is a monomorphism. From now on we shall without loss of generality view these monomorphisms $E_g \to E$ as inclusions.
To prove the existence of \((\dagger)\), it suffices to show that \(E \in \mathcal{D}\) in \(\varepsilon\). To this end, denote for any subset \(S \subseteq J\) by \(M_S\) the module

\[
M_S = \sum_{f \in S} \text{Im } \vartheta_f \quad (\subseteq E, \text{ see diagram } (\Delta))
\]

Then the family \((M_S \mid S \subseteq J \& |S| \leq \aleph_0)\) with obvious inclusions forms a direct system and we claim that its union is the whole of \(E\). Indeed, it is straightforward to check, using diagram \((\Delta)\) and the construction of the embeddings \(E_g \subseteq E\), that \(E = P + \sum_{f \in J} \text{Im } \vartheta_f\). Further, the left hand square of diagram \((\Delta)\) is a pull-back, which implies \(P \cap \sum_{f \in J} \text{Im } \vartheta_f = \text{Im } \iota_f\) and \(P \cap \text{Im } \vartheta_f \supseteq \sum_{f \in J} \text{Im } \iota_f = P\).

Thus, \(E = \sum_{f \in J} \text{Im } \vartheta_f\) and the claim is proved.

Moreover, the union of any chain \(M_{S_0} \subseteq M_{S_1} \subseteq M_{S_2} \subseteq \ldots\) from the direct system belongs to the direct system again. Therefore, if we prove that \(M_S \in \mathcal{D}\) for each countable \(S \subseteq J\), it will follow from Proposition 2 that \(E \in \mathcal{D}\). Our task is then reduced to prove the following lemma:

**Lemma 7.** With the notation as above, the following hold:

1. Given \(S \subseteq T \subseteq J\) with \(S\) and \(T\) finite and such that \(|T| = |S| + 1\), the inclusion \(M_S \subseteq M_T\) splits and there is \(n_0 < \omega\) such that \(M_T / M_S \cong \bigoplus_{m \geq n_0} F_m\).
2. Given a countable subset \(S \subseteq J\), the module \(M_S\) is isomorphic to a countable direct sum with each summand isomorphic to some \(F_n\), \(n < \omega\). In particular, \(M_S \in \mathcal{D}\).

**Proof.** Let us focus on (1) since (2) is an immediate consequence. Denote by \(f : \omega \to \lambda\) the single element of \(T \setminus S\), and let \(n_0 < \omega\) be the smallest number such that \(f(n_0) \neq g(n_0)\) for each \(g \in S\).

We claim that the following are satisfied by the construction:

\[
M_S \cap \text{Im } \vartheta_f = \iota_f \left( \bigoplus_{m < n_0} F_m \right) = \vartheta_f \circ \varphi \left( \bigoplus_{m < n_0} F_m \right).
\]

The second equality holds simply because \(\vartheta_f \circ \varphi = \iota_f \) by diagram \((\Delta)\). For the first, note that \(M_S \cap \text{Im } \vartheta_f\) as a submodule of \(E\), is contained in \(P\). Since \(P \cap \text{Im } \vartheta_f = \text{Im } \iota_f\), we have

\[
M_S \cap \text{Im } \vartheta_f = \left( \sum_{g \in S} \text{Im } \vartheta_g \right) \cap \text{Im } \iota_f = \left( \sum_{g \in S} \text{Im } \iota_g \right) \cap \text{Im } \iota_f = \iota_f \left( \bigoplus_{m < n_0} F_m \right),
\]

by the construction of \(P\). This proves the claim.

Invoking Lemma 3, we further deduce that

\[
\text{Im } \vartheta_f = \iota_f \left( \bigoplus_{m < n_0} F_m \right) \oplus \vartheta_f \left( \bigoplus_{m \geq n_0} F_m \right).
\]
In particular, the inclusion $M_S \cap \text{Im } \vartheta_f \subseteq \text{Im } \vartheta_f$ splits and so does the inclusion $M_S \subseteq M_S + \text{Im } \vartheta_f = M_T$. Moreover, we have the isomorphisms

\[ M_T/M_S = (M_S + \text{Im } \vartheta_f)/M_S \cong \text{Im } \vartheta_f/(M_S \cap \text{Im } \vartheta_f) \cong \bigoplus_{m \geq n_0} F_m, \]

which finishes the proof of the lemma.

Having established the existence of (†) such that $E \in \mathcal{D}$ and $|\text{Hom}_R(P, C)| \leq 2^\lambda$, let us apply $\text{Hom}_R(-, C)$ on (†). Since $C \in \mathcal{D}^\perp$ by assumption, we get an exact sequence

\[ \text{Hom}_R(P, C) \to \text{Ext}_R^1(F^{(2^\lambda)}, C) \to 0. \]

Suppose now that $\text{Ext}_R^1(F, C) \neq 0$. Then we would have $|\text{Ext}_R^1(F^{(2^\lambda)}, C)| \geq 2^\lambda$, which would contradict the fact that $|\text{Hom}_R(P, C)| \leq 2^\lambda$. Hence $\text{Ext}_R^1(F, C) = 0$ as desired.

To finish the proof of Theorem 6, suppose $R$ is a countable ring. Since each $F \in \mathcal{D}$ is flat, $\mathcal{D}^\perp$ contains all cotorsion modules. On the other hand, if $C$ is not cotorsion, there is a countable flat module $F$ such that $\text{Ext}_R^1(F, C) \neq 0$; see for instance [6, Theorems 4.1.1 and 3.2.9]. By the first part of Theorem 6, we know that $F \in \perp(\mathcal{D}^\perp)$, so $C \notin \mathcal{D}^\perp$. Hence $\mathcal{D}^\perp$ consists precisely of cotorsion modules.

The fact that $\mathcal{D}$ is not precovering unless $R$ is right perfect (and $\mathcal{D}$ is then the class of projective modules) follows directly from [11, Theorem 2.10]. This finishes the proof of Theorem 6.

Remark 8. The proof of Theorem 6 is to some extent constructive. Namely, if $R$ is a countable ring and $C$ is a module which is not cotorsion, the theorem gives us a recipe how to construct $E \in \mathcal{D}$ such that $\text{Ext}_R^1(E, C) \neq 0$, and it allows us to estimate the size of $E$ based on the size of $C$. Note that if $R$ is non-perfect, the size of $E$ must grow with the size of $C$. This is because for any set $S \subseteq \mathcal{D}$, we have $\perp(S^\perp) \subseteq \mathcal{D} \subsetneq \text{Flat-}R$ by [1, Proposition 1.9] and [6, Corollary 3.2.3].

References


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