From “Stochastic Calculus of Variations on Wiener space” to “Stochastic Calculus of Variations on Poisson space”.

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Malliavin's derivative: "Calculus of Variations approach"

$\Omega = C_0(0,T)$ $\qquad$ $F \in L^2(\Omega)$ (a *functional* on Wiener space)

Cameron-Martin space **CM**: $h \in$**CM** if

$h(t) = \int_0^t \dot{h}(s)\,\mathrm{d}s\,, \quad \dot{h} \in L^2(0,T)$

and $\|h\|_{CM} = \|\dot{h}\|_{L^2}\,.$

Suppose that $\exists\, Z_s \in L^2(\Omega \times [0,T])$ such that

$$\lim_{\varepsilon \to 0} \frac{F(\omega + \epsilon h) - F(\omega)}{\epsilon} = \int_0^T Z_s(\omega)\dot{h}(s)\,\mathrm{d}s$$

then F is **derivable** (in Malliavin's sense) and $Z_s = D_s F$ (more generally $D_h F = \int_0^T D_s F\ \dot{h}(s)\,\mathrm{d}s$).

With this definition, D is like a **Fréchet** derivative, but only along the directions in **CM**. Why?

Girsanov's theorem: if

$$\frac{d\mathbf{P}^*}{d\mathbf{P}} = \exp\Big(\int_0^T \dot{h}(s) dW_s - \frac{1}{2}\int_0^T \dot{h}^2(s)\, ds\Big) = L_T$$

law of $\big(W_{(.)} + h(.)\big)$ under $\mathbf{P}$ = law of $W_{(.)}$ under $\mathbf{P}^*$

(recall that on the canonical space $W_t(\omega) = \omega(t)$).

Introducing

$$\frac{\mathrm{d}\mathbf{P}^\epsilon}{\mathrm{d}\mathbf{P}} = \exp\left(\epsilon \int_0^T \dot{h}(s)\mathrm{d}W_s - \frac{\epsilon^2}{2}\int_0^T \dot{h}^2(s)\,\mathrm{d}s\right) = L_T^\epsilon$$

we have

$$\mathbb{E}\Big[\frac{F(\omega+\epsilon h) - F(\omega)}{\epsilon}\Big] = \mathbb{E}\Big[F(\omega)\,\frac{L_T^\epsilon - 1}{\epsilon}\Big]$$

since $\lim_{\epsilon\to 0} \frac{L_T^\epsilon - 1}{\epsilon} = \int_0^T \dot{h}(s)\,\mathrm{d}W_s$

we obtain the **integration by parts formula**

$$\mathbb{E}\Big[\int_0^T D_s F\ \dot{h}(s)\,\mathrm{d}s\Big] = \mathbb{E}\Big[F\ \int_0^T \dot{h}(s)\,\mathrm{d}W_s\Big]$$

($\dot{h}(s)$ can be replaced by $H_s \in L^2\big(\Omega \times [0,T]\big)$ adapted)

Intuitively: Malliavin's calculs is the analysis of the **variations of the paths along the directions supported by Girsanov's theorem**.

More generally: for $k \in L^2\big(0,T\big)$, define $W(k) = \int_0^T k(s)\,\mathrm{d}W_s$ (Wiener's integral) and define **smooth functional**

$$F = \phi\big(W(k_1), \ldots, W(k_n)\big)$$

(ϕ smooth). We obtain easily

$$D_s F = \sum_{i=1}^{n} \frac{\partial \phi}{\partial x_i}\big(W(k_1), \ldots, W(k_n)\big)\, k_i(s)$$

The operator $D : \mathcal{S} \subset L^2(\Omega) \to L^2(\Omega \times [0,T])$ ($\mathcal{S}$ space of smooth functionals) is **closable** (by the **integration by parts formula**) (from now on we consider the closure).

The adjoint operator $D^* = \delta : L^2(\Omega \times [0,T])$ is called **divergence** or **Skorohod integral** and D^* **restricted** to the **adapted** processes **coincides with Ito's integral**.

This is **equivalent** to the **Clark-Ocone-Karatzas** formula: if F is derivable

$$F = \mathbb{E}[F] + \int_0^T \mathbb{E}\left[D_s F \middle| \mathcal{F}_s\right] \mathrm{d}W_s$$

Very important is the so-called **Chain rule**:

$$D_s\,\phi\big(F^1,\ldots,F^n\big) = \sum_{i=1}^{n} \frac{\partial \phi}{\partial x_i}\big(\cdots\big)D_s\,F^i$$

(if $\phi : \mathbb{R}^n \to \mathbb{R}$ is derivable in the **classic** sense and $F^1,\ldots,F^n$ in the Malliavin's sense).

Summing up:

- integration by parts formula
- D^* restricted to adapted processes coincides with Ito's integral
- Clark-Ocone-Karatzas formula
- chain rule

A remark: Skorohod (anticipating) integral **is not an integral** (limit of Riemann's sums).

Intuitively

$$\int_0^T H_s \, \mathrm{d}X_s = \lim \sum_i H_{t_i} \big(X_{t_{i+1}} - X_{t_i} \big)$$

Formula: if H_s is adapted and F derivable

$$\int_0^T \big(F H_s \big) \, \delta W_s = F \int_0^T H_s \, \mathrm{d}W_s - \int_0^T D_s F \, H_s \, \mathrm{d}s$$

Malliavin derivative in Chaos Expansion.

An introductory example: alternative description on the space $H^{1,2}(0, 2\pi)$.

$f \in L^2(0, 2\pi)$ can be written

$$f = a_0 + \sum_{k \geq 1} \left(a_k \cos kx + b_k \sin\, kx \right) \qquad \sum_k |a_k|^2 + |b_k|^2 < +\infty$$

If there is a finite number of terms

$$f' = \sum_k \left(k\, b_k \cos kx \, - k\, a_k \sin\, kx \right)$$

Therefore f is **derivable** (in weak sense) and $f' \in L^2(0, 2\pi)$ if

$$\sum_k k^2 \left(|a_k|^2 + |b_k|^2\right) < +\infty \qquad \text{and} \qquad f' = \sum_k k\left(b_k \cos kx - a_k \sin kx\right)$$

- short and easy definition of (weak) derivative and of the space $H^{1,2}(0, 2\pi)$;
- the **meaning** of derivative is hidden.

Wiener Chaos Expansion

$S_n = \left\{0 < t_1 < \cdots < t_n < T\right\}$, given $f \in S_n$

$$J_n(f) = \int_{]0,T]} \mathrm{d}W_{t_n} \int_{]0,t_n]} \mathrm{d}W_{t_{n-1}} \cdots \int_{]0,t_2]} f(t_1, \ldots, t_n)\, \mathrm{d}W_{t_1}$$

$$\mathbb{E}\Big[J_n(f)^2\Big] = \|f\|^2_{L^2(S_n)}$$

If $C_n =$ image of $L^2(S_n)$ by J_n, we have $L^2\big(\Omega\big) = C_0 \oplus C_1 \oplus C_2 \ldots$

If $\widetilde{L}^2([0,T]^n)$ is the subspace of **symmetric** functions of $\widetilde{L}^2([0,T]^n)$, **define**:

$$I_n(f) = n! \int \cdots \int_{S_n} f(\cdots)\, \mathrm{d}W_{t_n} \cdots \mathrm{d}W_{t_1}$$

we have $\mathbb{E}\left[I_n(f)^2\right] = n!\, \|f\|^2_{L^2([0,T]^n)}$.

$F \in L^2(\Omega)$ can be written $F = \sum_{n\geq 0} I_n(f_n)$ with $\sum_{n\geq 0} n!\, \|f_n\|^2_{L^2} < +\infty$

By **direct calculus**

$$D_t\, I_n\big(f_n(t_1,\ldots,t)\big) = n\, I_{n-1}\big(f_n(t_1,\ldots,t_{n-1},t)\big)$$

We can define

$D_t\, F = \sum_{n\geq 1} n\, I_{n-1}\big(\ldots\big)$ provided that $\sum_{n\geq 1} n\, n!\, \|f_n\|^2_{L^2} < +\infty$

A similar characterization can be given for **Skorohod integral**.

With this approach:

- concise and more elementary definitions of Malliavin's derivative and divergence
- some proof are easier, some more complicated (e.g. *"chain rule"*)
- the idea of **derivative** is hidden

A good result with this approach: **Energy identity for Skorohod integral** (Nualart, Pardoux, Shigekawa)

$$\mathbb{E}\Big[\Big(\int_0^T Z_s\,\delta W_s\Big)^2\Big] = \mathbb{E}\Big[\int_0^T Z_s^2\,\mathrm{d}s + \int_0^T\int_0^T (D_t Z_s + D_s Z_t)\,\mathrm{d}s\,\mathrm{d}t\Big]$$

Other approaches: discretization (Ocone, Mallavin–Thalmaier), weak derivation ...

Main applications of Malliavin calculus:

- Clark–Ocone–Karatzas formula (explicit characterization of the integrand)

- Regularity of the law of some r.v. (solutions of S.D.E.)

- Sensitivity analysis in Mathematical Finance (Monte Carlo weights for the Greek's)

An idea of *"sensitivity analysis"* (Fournié, Lasry, Lebuchoux, Lions, Touzi [99], and F.L.L.L. [01]):

$$\frac{\partial}{\partial\zeta}\mathbb{E}\Big[f\big(F^\zeta\big)\Big] = \mathbb{E}\Big[f'\big(F^\zeta\big)\ \partial_\zeta F^\zeta\Big] =$$

$$= \mathbb{E}\Big[\frac{D_w\,\big[f\big(F^\zeta\big)\big]}{D_w\,F^\zeta}\,\partial_\zeta F^\zeta\Big] = \mathbb{E}\Big[f\big(F^\zeta\big)\ D_w^*\Big(\frac{\partial_\zeta F^\zeta}{D_w F^\zeta}\Big)\Big]\,.$$

The *"weight"* $W = D_w^*\Big(\frac{\partial_\zeta F^\zeta}{D_w F^\zeta}\Big)$ is independent of f (and not unique).

In order to extend to more general situations (from **diffusion** models to **jump–diffusion** models), we need:

- an **integration by parts** formula
- **chain rule**.

Plain Poisson process

Let P_t be a **Poisson process** with jump times $\tau_1 < \tau_2 < \ldots$

($\sigma_i = \tau_i - \tau_{i-1}$ are independent exponential density) and $N_t = (P_t - t)$ the *compensated* Poisson.

Point of view of Chaos Expansion:

Starting from

$$\widetilde{J}_n(f) = \int_{]0,T]} \mathrm{d}N_{t_n} \int_{]0,t_n[} \mathrm{d}N_{t_{n-1}} \cdots \int_{]0,t_2[} f(t_1, \ldots, t_n)\, \mathrm{d}N_{t_1}$$

A similar theory, based on *chaotic representation*, can be developed w.r.t. N_t (Lokka, Oksendal and ...)

- similar definition of **derivative** D^c and **Skorohod integral**

- $(D^c)^*$ coincides with ordinary stochastic integrals on predictable processes

- Clark–Ocone–Karatzas formula

A serious drawback: the chain rule **is not satisfied**.

In fact, the *"chaotic"* derivative satisfies the formula

$$D_t^c\big(FG\big) = F\,D_t^c G + G\,D_t^c F + D_t^c F\,D_t^c G$$

(Chain rule is (*morally*) equivalent to the formula

$D_t(FG) = FD_tG + GD_tF$).

An alternative point of view: Variations on the paths

(via Girsanov theorem)

Given $h(t) = \int_0^t \dot{h}(s)\,\mathrm{d}s$, $\dot{h} \in L^2(0,T)$ and $\dot{h}$ uniformly bounded from below, consider a **perturbed** probability

$$\frac{\mathrm{d}\mathbf{P}^\epsilon}{\mathrm{d}\mathbf{P}} = L_T^\epsilon = \exp\Big(-\epsilon \int_0^T \dot{h}(s)\mathrm{d}s\Big) \prod_{s \le T} \Big(1 + \epsilon \dot{h}(s) \Delta P_s\Big)$$

Let $\alpha_\epsilon(t) = \int_0^t \big(1 + \epsilon\, \dot{h}(r)\big)\,\mathrm{d}r$ (a **variation** on time):

law of $P_{\alpha_\epsilon(.)}$ under P = law of $P_{(.)}$ under P^ϵ

Similar definition for derivative of a **Poisson** functional:

$$\lim_{\varepsilon \to 0} \frac{F(P_{\alpha_\epsilon .}) - F(P_.)}{\epsilon}$$

Since $\lim_{\epsilon \to 0} \frac{L_T^\epsilon - 1}{\epsilon} = \int_0^T \dot{h}(s)\, \mathrm{d}N_s$ we obtain the **integration by parts formula**.

Some differences with Gaussian case: only a **deterministic perturbation** is allowed, (the integration by parts formula is less immediate).

On **smooth functionals** of the form

$$F = \phi(\tau_1, \ldots, \tau_n)$$

we obtain by a direct calculus

$$D_t^v \, \phi(\tau_1, \ldots, \tau_n) = -\sum_{i=1}^{n} \frac{\partial \phi}{\partial x_i}(\ \ldots\ ) I_{[0,\tau_i]}(t)$$

Good properties: $(D^v)^*$ concides with stochastic integrals for predictable processes (**Clark–Ocone–Karatzas**), the **chain rule** is satisfied.

A drawback: the analysis of divergence is more complicated (w.r.t. chaotic point of view)

A serious drawback: P_T is not derivable (not in the domain of the operator D^v)!

$$P_T = \sum_{i \geq 1} I_{[0,T]}(\tau_i)$$

is not a **smooth function** of the jump times.

A remark: the domains of the operators D^c and D^v are completely different. Typical derivable functionals are:

- stochastic integrals $\int_0^T h(s)\,\mathrm{d}N_s$ (or iterated stoch. int.) for the operator D^c;

- *smooth* functions $\phi\big(\tau_1, \ldots, \tau_n\big)$ for the operator D^v.

The **“variations”** point of view was investigated in some papers by **Privault** with a different approach (**Bouleau–Hirsch**, who started by proving Clark–Ocone-Karatzas formula). A similar approach is in **Elliot–Tsoi ”93**.

Privault obtained sensitivity results for models of the kind

$$dS_t = S_{t-}\Big(m(t)\,\mathrm{d}t + \sum_{j=1}^{n} \alpha_j\,\mathrm{d}P_t^j\Big)$$

$(P^1, \dots, P^n)$ independent Poisson processes, for **Asian** options of the form $\int_0^T f(t, S_t)\,\mathrm{d}t$.

Compound Poisson processes

$$X_t = \sum_{j \le P_t} U_j - \lambda\, t\, \mathbb{E}\big[U_j\big] = \int\!\!\int_{[0,t]\times\mathbb{R}} x\, \mathsf{d}\big(\mu - \nu\big)$$

P_t Poisson process with intensity λ, $U_1, U_2, \ldots$ i.i.d.

$$\mu = \sum_n \epsilon_{(\tau_n, U_n)} \quad ; \quad \nu(\omega, \mathsf{d}t, \mathsf{d}x) = \lambda\, \mathsf{d}t\, \mathsf{d}F(x)$$

Chaotic expansion approach developed by **Léon et coll.** (2002), **Oksendal and coll.** (many papers) with attention to anticipative calculus, anticipative Ito's formulae ...

Variations on the paths

Two possibilities: variations on **jump times** and on **jump amplitude** (supported by Girsanov's theorem)

Variations on jump times.

Integration by parts formula

$$\mathbb{E}\Big[\int_0^T D_s^t F \; H_s \, \mathrm{d}s\Big] = \mathbb{E}\Big[F\Big(\int_0^T H_s \, \mathrm{d}N_s\Big)\Big]$$

(No hope for a Clark-Ocone-Karatzas formula)

Variations on jump amplitude.

This is the **good point of view**, and it was investigated by **Bismut**, **Bass–Cranston**, **Jacod–Bichteler–Pellaumail** under a restriction: $\mathrm{d}F(x)$ is the Lebesgue measure under a suitable open interval E.

Their results can be extended to the case $\mathrm{d}F(x) = f(x)\,\mathrm{d}x$ (where the *"density"* f is continuous and strictly positive on an open interval $E =]a, b[$).

Methods:

- look at the process X_t in the form $\int\int_{[0,t]\times\mathbb{R}} x \,\mathrm{d}\big(\mu-\nu\big)$
- use Girsanov theorem for **random measures**
- consider s.d.e. with respect to random measures.

Integration by parts formula

$$\mathbb{E}\Big[\int\int_{[0,T]\times E} D^j_{(s,x)}F\ H(s,x)\,\mathrm{d}s\mathrm{d}F(x)\Big] = \mathbb{E}\Big[F\Big(\int\int_{[0,T]\times E} H\,\mathrm{d}(\mu-\nu)\Big)\Big]$$

A remark: some papers extend **sensitivity analysis** to jump-diffusion models by using the **chaotic** approach. How is it possible?

Idea: if $F(\omega,\omega')$ (ω in the Wiener space, ω' in Poisson space), we have

$$D^\omega \phi\big(F(\omega,\omega')\big) = \phi'\big(F\big)\, D^\omega F(\omega,\omega')$$

(where D^ω is the derivative w.r.t. Wiener component, ω' is only a parameter).

Davis–Johannson (2006) under a *separability* assumption, **Teichmann–Forster–Lutkebohmert** (2007) under more general hypothesis.

Separability assumption:

$$S_t = f\Big(X_t^c\,,\, X_t^d\Big)$$

where X^c satisfies an equation

$$dX_t^c = X_t^c\Big(m(t)\,\mathrm{d}t + \sigma_t^c\,\mathrm{d}W_t\Big)$$

and X_t^d satisfies a similar equation on the Poisson space.

Bavouzet–Messaoud uses *integration by parts w.r.t. jump amplitude*, but only after discretization.

These methods seems not convenient for **more general Lévy processes**.

Happy Belated Birthday

Wolfgang !