$\mathbb{Z}$-graded Lie Superalgebras of Infinite Depth and Finite Growth

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Abstract

In 1998 Victor Kac classified infinite-dimensional $\mathbb{Z}$-graded Lie superalgebras of finite depth. We construct new examples of infinite-dimensional Lie superalgebras with a $\mathbb{Z}$-gradation of infinite depth and finite growth and classify $\mathbb{Z}$-graded Lie superalgebras of infinite depth and finite growth under suitable hypotheses.

Mathematics Subject Classification (2000): 17B65 (primary), 17B70 (secondary).

Introduction

Simple finite-dimensional Lie superalgebras were classified by V.G. Kac in [K2]. In the same paper Kac classified the finite-dimensional, $\mathbb{Z}$-graded Lie superalgebras under the hypotheses of irreducibility and transitivity.

The classification of infinite-dimensional, $\mathbb{Z}$-graded Lie superalgebras of finite depth is also due to V.G. Kac [K3] and is deeply related to the classification of linearly compact Lie superalgebras. We recall that finite depth implies finite growth.

This naturally leads to investigate infinite-dimensional, $\mathbb{Z}$-graded Lie superalgebras of infinite depth and finite growth. The hypothesis of finite growth is central to the problem; indeed, it is well known that it is not possible to classify $\mathbb{Z}$-graded Lie algebras (and thus Lie superalgebras) of any growth (see [K1, M]). The only known examples of infinite-dimensional, $\mathbb{Z}$-graded Lie superalgebras of finite growth and infinite depth are given by contragredient Lie superalgebras which were classified by V.G. Kac in [K2] in the case of finite dimension and by J.W. van de Leur in the general case [vdL]. Contragredient Lie superalgebras, as well as Kac-Moody Lie algebras, have a $\mathbb{Z}$-gradation of infinite depth and growth equal to 1, due to their periodic structure.

We construct three new examples of infinite-dimensional Lie superalgebras with a consistent $\mathbb{Z}$-gradation of infinite depth and finite growth, and we realize them as covering superalgebras of finite-dimensional Lie superalgebras. It turns out that if $\mathcal{G}$ is an irreducible, simple Lie superalgebra generated by its local part, with a consistent $\mathbb{Z}$-gradation, and if we assume that $\mathcal{G}_0$ is simple and that $\mathcal{G}_1$ is an irreducible $\mathcal{G}_0$-module which is not contragredient to $\mathcal{G}_{-1}$, then

$\mathcal{G}$ is isomorphic to one of these three algebras (Theorem 3.1) and its growth is therefore equal to 1.

So far, any known example of a $\mathbb{Z}$-graded Lie superalgebra of infinite depth and finite growth is, up to isomorphism, either a contragredient Lie superalgebra or the covering superalgebra of a finite-dimensional Lie superalgebra. Since the aim of this paper is analyzing $\mathbb{Z}$-graded Lie superalgebras of infinite depth, we shall not describe the cases of finite depth which can be found in [K2, K3].

Let $\mathcal{G}$ be a $\mathbb{Z}$-graded Lie superalgebra. Suppose that $\mathcal{G}_0$ is a simple Lie algebra and that $\mathcal{G}_{-1}$ and $\mathcal{G}_1$ are irreducible $\mathcal{G}_0$-modules and are not contragredient. Let F_Λ be a highest weight vector of $\mathcal{G}_{-1}$ of weight Λ and let E_M be a lowest weight vector of $\mathcal{G}_1$ of weight M. Since $\mathcal{G}_{-1}$ and $\mathcal{G}_1$ are not contragredient, the sum $\Lambda + M$ is a root of $\mathcal{G}_0$, and, without loss of generality, we may assume that it is a negative root, i.e. $\Lambda + M = -\alpha$ for some positive root α. The paper is based on the analysis of the relations between the $\mathcal{G}_0$-modules $\mathcal{G}_{-1}$ and $\mathcal{G}_1$. It is organized in three sections: Section 1 contains some basic definitions and fundamental results in the general theory of Lie superalgebras. In Section 2 the main hypotheses on the Lie superalgebra $\mathcal{G}$ are introduced. Section 2.1 is devoted to the case $(\Lambda, \alpha) = 0$. Since Λ is a dominant weight, in this section the rank of $\mathcal{G}_0$ is assumed to be greater than 1. The hypothesis $(\Lambda, \alpha) = 0$ always holds for $\mathbb{Z}$-graded Lie superalgebras of finite depth (see [K2, Lemma 4.1.4] and [K3, Lemma 5.3]) but if the Lie superalgebra $\mathcal{G}$ has infinite depth weaker restrictions on the weight Λ are obtained (compare, for example, Lemma 4.1.3 in [K2] with Lemma 1.14 in this paper).

In Section 2.2 we examine the case $(\Lambda, \alpha) \neq 0$. In the finite-depth case this hypothesis may not occur (cf. [K3, Lemma 5.3]). It turns out that, under this hypothesis, $\mathcal{G}_0$ has necessarily rank one (cf. Theorem 2.17) namely it is isomorphic to $sl(2)$. Besides, a strong restriction on the possible values of (Λ, α) is obtained (cf. Corollary 2.12) so that $\mathcal{G}_{-1}$ is necessarily isomorphic either to the adjoint module of $sl(2)$ or to the irreducible $sl(2)$-module of dimension 2.

Finally, Section 3 is devoted to the construction of the examples and to the classification theorem.

Throughout the paper the base field is assumed to be algebraically closed and of characteristic zero.

Acknowledgments

I would like to express my gratitude to Professor Victor Kac for introducing me to the subject and for dedicating me so much of his time.

1 Basic definitions and main results

1.1 Lie superalgebras

Definition 1.1 *A superalgebra is a $\mathbb{Z}_2$-graded algebra $A = A_{\bar{0}} \oplus A_{\bar{1}}$; $A_{\bar{0}}$ is called the even part of A and $A_{\bar{1}}$ is called the odd part of A.*

Definition 1.2 *A Lie superalgebra is a superalgebra $\mathcal{G} = \mathcal{G}_{\bar{0}} \oplus \mathcal{G}_{\bar{1}}$ whose product $[\cdot, \cdot]$ satisfies the following axioms:*

(i) $[a,b] = -(-1)^{deg(a)deg(b)}[b,a]$;
(ii) $[a,[b,c]] = [[a,b],c] + (-1)^{deg(a)deg(b)}[b,[a,c]]$.

Definition 1.3 *A $\mathbb{Z}$-grading of a Lie superalgebra $\mathcal{G}$ is a decomposition of $\mathcal{G}$ into a direct sum of finite-dimensional $\mathbb{Z}_2$-graded subspaces $\mathcal{G} = \oplus_{i\in\mathbb{Z}}\mathcal{G}_i$ for which $[\mathcal{G}_i, \mathcal{G}_j] \subset \mathcal{G}_{i+j}$. A $\mathbb{Z}$-grading is said to be consistent if $\mathcal{G}_{\bar{0}} = \oplus\mathcal{G}_{2i}$ and $\mathcal{G}_{\bar{1}} = \oplus\mathcal{G}_{2i+1}$.*

Remark 1.4 By definition, if $\mathcal{G}$ is a $\mathbb{Z}$-graded Lie superalgebra, then $\mathcal{G}_0$ is a subalgebra of $\mathcal{G}$ and $[\mathcal{G}_0, \mathcal{G}_i] \subset \mathcal{G}_i$; therefore the restriction of the adjoint representation to $\mathcal{G}_0$ induces linear representations of it on the subspaces $\mathcal{G}_i$.

Definition 1.5 *A $\mathbb{Z}$-graded Lie superalgebra $\mathcal{G}$ is called irreducible if $\mathcal{G}_{-1}$ is an irreducible $\mathcal{G}_0$-module.*

Definition 1.6 *A $\mathbb{Z}$-graded Lie superalgebra $\mathcal{G} = \oplus_{i\in\mathbb{Z}}\mathcal{G}_i$ is called transitive if for $a \in \mathcal{G}_i$, $i \geq 0$, $[a, \mathcal{G}_{-1}] = 0$ implies $a = 0$, and bitransitive if, in addition, for $a \in \mathcal{G}_i$, $i \leq 0$, $[a, \mathcal{G}_1] = 0$ implies $a = 0$.*

Let $\hat{G}$ be a $\mathbb{Z}_2$-graded space, decomposed into the direct sum of $\mathbb{Z}_2$-graded subspaces, $\hat{G} = \mathcal{G}_{-1} \oplus \mathcal{G}_0 \oplus \mathcal{G}_1$. Suppose that whenever $|i+j| \leq 1$ a bilinear operation is defined: $\mathcal{G}_i \times \mathcal{G}_j \to \mathcal{G}_{i+j}$, $(x,y) \mapsto [x,y]$, satisfying the axiom of anticommutativity and the Jacobi identity for Lie superalgebras, provided that all the commutators in this identity are defined. Then $\hat{G}$ is called a *local* Lie superalgebra.

If $\mathcal{G} = \oplus_{i\in\mathbb{Z}}\mathcal{G}_i$ is a $\mathbb{Z}$-graded Lie superalgebra then $\mathcal{G}_{-1} \oplus \mathcal{G}_0 \oplus \mathcal{G}_1$ is a local Lie superalgebra which is called the *local part* of $\mathcal{G}$. The following proposition holds:

Proposition 1.7 *[K2] Two bitransitive $\mathbb{Z}$-graded Lie superalgebras are isomorphic if and only if their local parts are isomorphic.*

Definition 1.8 *A Lie superalgebra is called simple if it contains no nontrivial ideals.*

Proposition 1.9 *[K2] If in a simple $\mathbb{Z}$-graded Lie superalgebra $\mathcal{G} = \oplus_{i\in\mathbb{Z}}\mathcal{G}_i$ the subspace $\mathcal{G}_{-1} \oplus \mathcal{G}_0 \oplus \mathcal{G}_1$ generates $\mathcal{G}$ then $\mathcal{G}$ is bitransitive.*

1.2 On the growth of $\mathcal{G}$

Definition 1.10 *Let $\mathcal{G} = \oplus_{i\in\mathbb{Z}}\mathcal{G}_i$ be a $\mathbb{Z}$-graded Lie superalgebra. The limit*

$$r(\mathcal{G}) = \lim_{n\to\infty} ln(\sum_{i=-n}^{n} \dim \mathcal{G}_i)/ln(n)$$

is called the growth of $\mathcal{G}$. If $r(\mathcal{G})$ is finite then we say that $\mathcal{G}$ has finite growth.

Let us fix some notation. Given a semisimple Lie algebra L, by $V(\omega)$ we shall denote its finite-dimensional highest weight module of highest weight ω. ω_i will be the fundamental weights. It is well known that if λ is a weight of a finite-dimensional representation of L and β is a nonzero root of L, then the set of weights of the form $\lambda + s\beta$ forms a continuous string: $\lambda - p\beta, \lambda - (p-1)\beta, \dots, \lambda - \beta, \lambda, \lambda + \beta, \dots, \lambda + q\beta$, where p and q are nonnegative integers and $p - q = 2(\lambda, \beta)/(\beta, \beta)$. Let us put $2(\lambda, \beta)/(\beta, \beta) = \lambda(h_\beta)$. The numbers $\lambda(h_{\alpha_i})$, for a fixed basis of simple roots α_i, are called the *numerical marks* of the weight λ.

For any positive root β of L we shall denote by e_β a root vector of L corresponding to β.

Lemma 1.11 *[K1] Let L be a Lie algebra containing elements $H \neq 0$, E_i, F_i, $i = 1, 2$, connected by the equations $[E_i, F_j] = \delta_{ij} H$, $[H, E_1] = aE_1$, $[H, E_2] = bE_2$, $[H, F_1] = -aF_1$, $[H, F_2] = -bF_2$, where $a \neq -b$, $b \neq -2a$, and $a \neq -2b$, then the growth of L is infinite.*

Lemma 1.12 *[K1] Let $L = \oplus L_i$ be a graded Lie algebra, where L_0 is semisimple. Assume that there exist weight vectors x_λ and x_μ corresponding to the weights λ and μ of the adjoint representation of L_0 on L, and a root vector e_γ, corresponding to the root γ of L_0, which satisfy the following relations:*

$$[x_\mu, x_\lambda] = e_\gamma,$$

$$[x_\lambda, e_{-\gamma}] = 0 = [x_\mu, e_\gamma],$$

$$\lambda(h_\gamma) \neq -1, \;\; (\lambda, \gamma) \neq 0.$$

Then the growth of L is infinite.

Lemma 1.13 *Let $\mathcal{G}$ be a consistent, $\mathbb{Z}$-graded Lie superalgebra and suppose that $\mathcal{G}_0$ is a semisimple Lie algebra. Let E_i, F_i ($i = 1, 2$) be odd elements and H a non zero element in $\mathcal{G}_{\bar{0}}$ such that:*

$$[E_i, F_j] = \delta_{ij} H, \qquad [H, E_i] = a_i E_i, \qquad [H, F_i] = -a_i F_i, \tag{1}$$

where $a_1 \neq -a_2$, $a_1 \neq -2a_2$ and $a_2 \neq -2a_1$. Then the growth of $\mathcal{G}$ is infinite.

Proof. Suppose first that $a_1 \neq 0 \neq a_2$. Then the elements $\tilde{E}_1 = {a_1}^{-1/2}[E_1, E_1]$, $\tilde{E}_2 = {a_2}^{-1/2}[E_2, E_2]$, $\tilde{F}_1 = {a_1}^{-1/2}[F_1, F_1]$, $\tilde{F}_2 = {a_2}^{-1/2}[F_2, F_2]$, $K = -4H$ satisfy the hypotheses of Lemma 1.11 in the Lie algebra $\mathcal{G}_{\bar{0}}$. Thus, the growth of $\mathcal{G}_{\bar{0}}$ is infinite and we get the thesis.

If, let us say, $a_1 \neq 0$, $a_2 = 0$ then the elements $E_1' = [E_1, E_1]$, $E_2' = [E_1, E_2]$, $F_1' = -(4a_1)^{-1}[F_1, F_1]$, $F_2' = a_1^{-1}[F_1, F_2]$, H satisfy the hypotheses of Lemma 1.11 in $\mathcal{G}_{\bar{0}}$, thus we conclude. □

Lemma 1.14 *Let $\mathcal{G} = \oplus \mathcal{G}_i$ be a $\mathbb{Z}$-graded, consistent Lie superalgebra and suppose that $\mathcal{G}_0$ is a semisimple Lie algebra. Assume that there exist odd elements x_λ and x_μ that are weight vectors of the adjoint representation of $\mathcal{G}_0$ on $\mathcal{G}$ of* *weight λ and μ respectively, and a root vector $e_{-\delta}$ of $\mathcal{G}_0$, connected by the relations:*

$$\begin{cases} [x_\lambda, x_\mu] = e_{-\delta} \\ [x_\lambda, e_\delta] = [x_\mu, e_{-\delta}] = 0 \end{cases}$$

with $2(\lambda,\delta) \neq (\delta,\delta)$, $(\lambda,\delta) \neq 0$ and $(\lambda,\delta) \neq (\delta,\delta)$. Then the growth of $\mathcal{G}$ is infinite.

Proof. We choose a root vector e_δ in $\mathcal{G}_0$ such that $[e_\delta, e_{-\delta}] = h_\delta$ and consider the following elements:

$$E_1 = [e_\delta, x_\mu]$$

$$E_2 = [[[x_\mu, e_\delta], e_\delta], e_\delta]$$

$$F_1 = x_\lambda$$

$$F_2 = -1/6\lambda(h_\delta)^{-1}(\lambda(h_\delta) - 1)^{-1}[[x_\lambda, e_{-\delta}], e_{-\delta}]$$

$$H = h_\delta.$$

By a direct computation it is easy to check that E_i, F_i, H satisfy the hypotheses of Lemma 1.13 with $a_1 = (\mu+\delta)(h_\delta) = -\lambda(h_\delta)$, $a_2 = (\mu + 3\delta)(h_\delta) = -\lambda(h_\delta) + 4$. By Lemma 1.13 the growth of $\mathcal{G}$ is therefore infinite. □

We can reformulate Lemma 1.14 as follows:

Corollary 1.15 *Suppose that $\mathcal{G}$ is a Lie superalgebra of finite growth. Let x_λ, x_μ, $e_{-\delta}$ be as in Lemma 1.14. Then one of the following holds:*

$$(i)(\lambda,\delta) = 0,$$

$$(ii)(\lambda,\delta) = (\delta,\delta),$$

$$(iii)(\lambda,\delta) = 1/2(\delta,\delta).$$

Theorem 1.16 *[K1] Let $L = \oplus L_i$ be a $\mathbb{Z}$-graded Lie algebra with the following properties:*

a) the Lie algebra L_0 has no center;

b) the representations ϕ_{-1} and ϕ_1 of L_0 on L_{-1} and L_1 are irreducible;

c) $[L_{-1}, L_1] \neq 0$;

d) $\Lambda + M = -\alpha$ where Λ is the highest weight of ϕ_{-1}, M is the lowest weight of ϕ_1 and α is a positive root of L_0;

e) the representations ϕ_{-1} and ϕ_1 are faithful;

f) the growth of L is finite.

Then L_0 is isomorphic to one of the Lie algebras A_n or C_n, ϕ_{-1} is the corresponding standard representation and α is the highest root of L_0.

In the following sl_n, sp_n and so_n will denote the standard representations of the corresponding Lie algebras.

Corollary 1.17 *Let $\mathcal{G} = \oplus \mathcal{G}_i$ be a Lie superalgebra with a consistent $\mathbb{Z}$-gradation. Suppose that $\mathcal{G}_0$ is simple. Suppose that there exist a highest weight vector x in $\mathcal{G}_{-2}$ of weight $\lambda \neq 0$ and a lowest weight vector y in $\mathcal{G}_2$ of weight μ such that $[x, y] \neq 0$ and $\lambda + \mu = -\rho$ for a positive root ρ of $\mathcal{G}_0$. Then, if the growth of $\mathcal{G}$ is finite, $\mathcal{G}_0$ is isomorphic to one of the Lie algebras A_n or C_n, ρ is the highest root of $\mathcal{G}_0$ and $\mathcal{G}_{-2}$ is the standard $\mathcal{G}_0$-module.*

Proof. It follows from Theorem 1.16. $\square$

2 Main results

In this section we will consider an irreducible, consistent, simple $\mathbb{Z}$-graded Lie superalgebra $\mathcal{G}$ generated by its local part, and we will always suppose that $\mathcal{G}$ has finite growth. Besides, we will assume that $\mathcal{G}_0$ is a simple Lie algebra and that $\mathcal{G}_1$ is an irreducible $\mathcal{G}_0$-module which is not contragredient to $\mathcal{G}_{-1}$. Let us fix a Cartan subalgebra $\mathcal{H}$ of $\mathcal{G}_0$ and the following notation: let F_Λ be a highest weight vector of $\mathcal{G}_{-1}$ of weight Λ (dominant weight) and let E_M be a lowest weight vector of $\mathcal{G}_1$ of weight M. As shown in [K2, Proposition 1.2.10], it turns out that $[F_\Lambda, E_M] = e_{-\alpha}$, where $\alpha = -(\Lambda + M)$ is a nonzero root of $\mathcal{G}_0$ and $e_{-\alpha}$ is a root vector in $\mathcal{G}_0$ corresponding to $-\alpha$. Interchanging, if necessary, $\mathcal{G}_k$ with $\mathcal{G}_{-k}$ we can assume that α is a positive root. Indeed, by transitivity, $[F_\Lambda, E_M] \neq 0$ and for any $t \in \mathcal{H}$ we have:

$$[t, [F_\Lambda, E_M]] = (\Lambda + M)(t)[F_\Lambda, E_M].$$

Notice that $\Lambda + M \neq 0$ since the representations of $\mathcal{G}_0$ on $\mathcal{G}_{-1}$ and $\mathcal{G}_1$ are not contragredient.

Remark 2.1 Under the above assumptions, $-M = \Lambda + \alpha$ is a dominant weight. Therefore $(\Lambda + \alpha, \beta) \geq 0$ for every positive root β of $\mathcal{G}_0$.

Lemma 2.2 *Under the above hypotheses, $[E_M, E_M] = 0$ and $[E_M, [e_\rho, E_M]] = 0$ for every positive root ρ.*

Proof. We have $[F_\Lambda, [E_M, E_M]] = 2[e_{-\alpha}, E_M] = 0$ since E_M is a lowest weight vector. Transitivity and irreducibility imply $[E_M, E_M] = 0$. Now, since E_M is odd, for every positive root ρ we have:

$$[E_M, [e_\rho, E_M]] = [[E_M, e_\rho], E_M] = -[E_M, [e_\rho, E_M]]$$

therefore $[E_M, [e_\rho, E_M]] = 0$. $\square$

2.1 Case $(\Lambda, \alpha) = 0$

In this paragraph we suppose $(\Lambda, \alpha) = 0$. If Λ is zero then the depth of $\mathcal{G}$ is finite. Therefore we suppose that Λ is not zero. This implies that the rank of $\mathcal{G}_0$ is greater than one.

Remark 2.3 Let $\mathcal{G}$ be a bitransitive, irreducible $\mathbb{Z}$-graded Lie superalgebra. If $(\Lambda, \alpha) = 0$ then the vectors $[F_\Lambda, F_\Lambda]$ and $[[F_\Lambda, e_{-\rho}], F_\Lambda]$ are zero for every positive root ρ.

Proof. Once we have shown that $[F_\Lambda, F_\Lambda] = 0$, we proceed as in Lemma 2.2 and conclude that $[[F_\Lambda, e_{-\rho}], F_\Lambda] = 0$ for every positive root ρ. Since $[[F_\Lambda, F_\Lambda], E_M] = 2[F_\Lambda, e_{-\alpha}] = 0$, we conclude by bitransitivity. $\square$

Lemma 2.4 *α is the highest root of one of the parts of the Dynkin diagram of $\mathcal{G}_0$ into which it is divided by the numerical marks of Λ.*

Proof. Suppose by contradiction that α is not the highest root of one of the parts of the Dynkin diagram of $\mathcal{G}_0$ into which it is divided by the numerical marks of Λ. Then there exists a simple root β such that $(\Lambda, \beta) = 0$ and $\alpha + \beta$ is a root. This gives a contradiction because: $0 = [[e_{-\beta}, F_\Lambda], E_M] = [e_{-\beta}, [F_\Lambda, E_M]] = e_{-\beta-\alpha} \neq 0$. $\square$

Lemma 2.5 *If Λ has at least two numerical marks then, for any numerical mark γ, we have:*

$$(\Lambda + \alpha, \gamma) = 0.$$

Proof. From Lemma 2.4 we know that α is the highest root of one of the parts of the Dynkin diagram of $\mathcal{G}_0$ into which it is divided by the numerical marks of Λ. Therefore we can choose a numerical mark β such that $\alpha + \beta$ is a root. Now suppose that γ is a numerical mark, $\gamma \neq \beta$, such that $(\Lambda + \alpha, \gamma) \neq 0$.

Notice that γ and β are not subroots of α, since $(\Lambda, \gamma) \neq 0$ and $(\Lambda, \beta) \neq 0$, therefore $\gamma(h_\alpha) \leq 0$, $\beta(h_\alpha) < 0$.

Consider the following vectors:

$$x := [[[F_\Lambda, e_{-\beta}], e_{-\gamma}], F_\Lambda]$$
$$y := [[[E_M, e_\alpha], e_\gamma], E_M].$$

First of all we want to show that x is a highest weight vector in $\mathcal{G}_{-2}$. By Remark 2.3, since β and γ are simple roots, it is sufficient to show that $x \neq 0$. In fact, $[e_\gamma, [x, E_M]] = (\Lambda + \alpha)(h_\gamma)[F_\Lambda, [e_{-\alpha}, e_{-\beta}]] \neq 0$.

Now let us prove that y is a lowest weight vector in $\mathcal{G}_2$. First $y \neq 0$, indeed:

$$[y, F_\Lambda] = (2 - \gamma(h_\alpha))[E_M, e_\gamma]$$

which is different from 0 since $\gamma(h_\alpha) \leq 0$ and by the assumption $(\Lambda + \alpha, \gamma) \neq 0$.

We now compute the commutators $[y, e_{-\alpha_k}]$ for any simple root α_k. If $\alpha_k = \gamma$ then, by Lemma 2.2, $[y, e_{-\alpha_k}] = 0$, since $\alpha - \gamma$ is not a root. If $\alpha_k \neq \gamma$,

$[y, e_{-\alpha_k}] = [[[E_M, e_{\alpha-\alpha_k}], e_\gamma], E_M]$, and this can be shown to be zero using the transitivity of $\mathcal{G}$.

Notice that $[x, y] = (2 - \gamma(h_\alpha))(\Lambda + \alpha)(h_\gamma)e_{-\alpha-\beta}$. By Theorem 1.16 we get a contradiction since $\alpha + \beta$ cannot be the highest root of $\mathcal{G}_0$. As a consequence, $(\Lambda + \alpha, \gamma) = 0$. In particular, $\alpha + \gamma$ is a root and we can repeat the same argument interchanging β and γ in order to get $(\Lambda + \alpha, \beta) = 0$. □

Corollary 2.6 *If $\mathcal{G}_0$ is of type A_n, B_n, C_n, F_4, G_2 then Λ has at most two numerical marks; if $\mathcal{G}_0$ is of type D_n, E_6, E_7, E_8 then Λ has at most three numerical marks.*

Proof. Immediate from Lemma 2.5. □

Lemma 2.7 *If Λ has only one numerical mark β then either $(\Lambda + \alpha, \beta) = 0$ or $\Lambda(h_\beta) = 1$.*

Proof. Suppose both $(\Lambda + \alpha, \beta) \neq 0$ and $\Lambda(h_\beta) > 1$, and define

$$x := [[[F_\Lambda, e_{-\beta}], e_{-\beta}], F_\Lambda]$$

$$y := [[[E_M, e_\alpha], e_\beta], E_M].$$

Then x is a highest weight vector in $\mathcal{G}_{-2}$ and y is a lowest weight vector in $\mathcal{G}_2$. Besides, $[x, y] = 2(2 - \beta(h_\alpha))(\Lambda + \alpha)(h_\beta)e_{-\alpha-\beta}$. By Theorem 1.16, $\mathcal{G}_0$ is either of type A_n or of type C_n, $\alpha + \beta$ is the highest root of $\mathcal{G}_0$ and $\mathcal{G}_{-2}$ is its elementary representation. It is easy to show that these conditions cannot hold. □

Proposition 2.8 *Let β be a positive root such that:*
- $\alpha + \beta$ is a root;
- $\alpha - \beta$ is not a root;
- $2\alpha + \beta$ is not a root.
Then either $(\Lambda + \alpha, \beta) = 0$ or $\Lambda(h_\beta) = 1$.

Proof. Let us first make some remarks:

(a) Since $\beta + \alpha$ is a root but $\beta + 2\alpha$ and $\beta - \alpha$ are not, we have that $\beta(h_\alpha) = -1$. It follows that $\alpha + \beta$ and β are roots of the same length.

(b) Since $\beta - (\alpha + \beta)$ is a root and $\beta - 2(\alpha + \beta)$ is not, then $\beta(h_{\alpha+\beta}) \leq 1$.
Now suppose that $\Lambda(h_\beta) > 1$, which implies $[F_\Lambda, e_{-\beta}] \neq 0$.
Let $x_\mu = E_M$ and $x_\lambda = [F_\Lambda, e_{-\beta}]$. We have:

$$[x_\lambda, x_\mu] = e_{-\alpha-\beta}$$

$$[e_{-\alpha-\beta}, x_\mu] = 0 = [x_\lambda, e_{\alpha+\beta}].$$

Therefore, by Lemma 1.14, we deduce that the difference $\Lambda(h_\beta) - \beta(h_{\alpha+\beta})$ is equal to 0, 1, or 2. In particular, $2 \leq \Lambda(h_\beta) \leq 3$ and $0 \leq \beta(h_{\alpha+\beta}) \leq 1$. We therefore distinguish the following two cases:

CASE A: $\beta(h_{\alpha+\beta}) = 0$, i.e. $\alpha + 2\beta$ is a root, $2\alpha + 3\beta$ is not, and $\Lambda(h_\beta)$=2.

In this case $(\beta,\beta)=-(\beta,\alpha)$ and $(\Lambda,\beta)=(\beta,\beta)$ therefore $(\Lambda+\alpha,\beta)=0$ which concludes the proof in this case.

CASE B: $\beta(h_{\alpha+\beta})=1$, i.e. $\alpha+2\beta$ is not a root, and $\Lambda(h_\beta)$ is either 2 or 3. In this case $\beta(h_\alpha)=-1=\alpha(h_\beta)$, therefore $(\Lambda+\alpha,\beta)\neq 0$. The two cases $\Lambda(h_\beta)=2$ and $\Lambda(h_\beta)=3$ need to be analyzed separately.

(i) $\Lambda(h_\beta)=2$
Let us define the following elements:

$$x_\lambda=[[[F_\Lambda,e_{-\beta}],e_{-\beta}],F_\Lambda]$$

$$x_\mu=[[[E_M,e_{\alpha+\beta}],e_\beta],E_M].$$

Then $[x_\lambda,x_\mu]=6e_{-\alpha}$, $[x_\lambda,e_\alpha]=0$ since $\alpha-\beta$ is not root, and $[x_\mu,e_{-\alpha}]=0$ since $(\Lambda+\alpha)(h_\beta)=1$, thus $[[E_M,e_\beta],e_\beta]=0$. Then we find a contradiction to Lemma 1.12 applied to the Lie algebra $\mathcal{G}_{\bar{0}}$, since $\mathcal{G}$ was assumed to have finite growth. Indeed, using the same notation as in Lemma 1.12, we have: $\lambda(h_\gamma)=-\lambda(h_\alpha)=-(2\Lambda-2\beta)(h_\alpha)=2\beta(h_\alpha)=-2$.

(ii) $\Lambda(h_\beta)=3$
Let us define the following elements:

$$E_1=1/8[[E_M,e_{\alpha+\beta}],[E_M,e_{\alpha+\beta}]]$$

$$F_1=[[F_\Lambda,e_{-\beta}],[F_\Lambda,e_{-\beta}]]$$

$$E_2=1/64[[[E_M,e_{\alpha+\beta}],e_\beta],[[E_M,e_{\alpha+\beta}],e_\beta]]$$

$$F_2=[[[F_\Lambda,e_{-\beta}],e_{-\beta}],[[F_\Lambda,e_{-\beta}],e_{-\beta}]]$$

$$H=h_{\alpha+\beta}=h_\alpha+h_\beta$$

Then the hypotheses of Lemma 1.11 are satisfied with $a_1=-4$ and $a_2=-2$, and this leads to a contradiction. □

In the following, for what concerns simple Lie algebras, we will use the same notation as in [H, §11, §12]. In particular we shall adopt the same enumeration of the vertices in the Dynkin diagrams and refer to the bases of simple roots described by Humphreys [H].

Lemma 2.9 *Let M be the lowest weight of the $\mathcal{G}_0$-module $\mathcal{G}_1$.*

(i) Let $z:=[[E_M,e_{\alpha+\beta}],[e_\gamma,E_M]]$, where β and γ are positive roots of $\mathcal{G}_0$ such that $[E_M,e_\beta]=0$, $\alpha+\beta+\gamma$ is not a root, $\beta+\gamma$ is not a root and $\gamma-\alpha$ is a negative root. Then $[z,F_\Lambda]=0$.

(ii) Let β and ρ be positive roots such that $\alpha+\beta$ and $\beta+\rho$ are positive roots, $\alpha+\beta+\rho$ is not a root, $\rho-\alpha$ is a negative root. If $(M,\beta)=0$ and $(M,\rho)\neq 0$, then the vector $[[E_M,e_{\alpha+\beta}],[e_\rho,E_M]]$ is non-zero.

(iii) Let β and ρ be as in (ii) and let α_k be a simple root of $\mathcal{G}_0$. Suppose, in addition, that either $\rho+\beta-\alpha_k$ is not a root or $(M,\rho+\beta-\alpha_k)=0$. Then $[[[E_M,e_{\alpha+\beta-\alpha_k}],[e_\rho,E_M]],F_\Lambda]=0$.

(iv) If ρ is a positive root such that $\alpha+\rho$ is not a root, $\rho-\alpha$ is a negative root, $(M,\rho)\neq 0$ and $\rho(h_\alpha)=1$, then $[[[E_M,e_\alpha],[e_\rho,E_M]],F_\Lambda]=0$.

Proof. The proof consists of simple direct computations. □

Theorem 2.10 *Let $\mathcal{G}$ be an irreducible, simple, $\mathbb{Z}$-graded Lie superalgebra of finite growth, generated by its local part. Suppose that $\mathcal{G}_0$ is simple, that the $\mathbb{Z}$-gradation of $\mathcal{G}$ is consistent and that $(\Lambda, \alpha) = 0$. If $\mathcal{G}$ has infinite depth then one of the following holds:*

- *$\mathcal{G}_0$ is of type A_3, $\mathcal{G}_{-1}$ is its adjoint module, $\mathcal{G}_1 = V(2\omega_2)$;*
- *$\mathcal{G}_0$ is of type B_n $(n \geq 2)$, $\mathcal{G}_{-1}$ is its adjoint module, $\mathcal{G}_1 = V(2\omega_1)$;*
- *$\mathcal{G}_0$ is of type C_n $(n \geq 3)$, $\mathcal{G}_{-1} \cong \Lambda_0^2 sp_{2n}$, $\mathcal{G}_1$ is its adjoint module;*
- *$\mathcal{G}_0$ is of type D_n $(n \geq 4)$, $\mathcal{G}_{-1}$ is its adjoint module, $\mathcal{G}_1 = V(2\omega_1)$.*

Proof. Let us analyze all the possible cases. Corollary 2.6 states that if $\mathcal{G}_0$ is of type A_n, B_n, C_n, F_4 or G_2 then Λ might have one or two numerical marks while if $\mathcal{G}_0$ is of type D_n, E_6, E_7 or E_8 then Λ might also have three numerical marks. Using Lemma 2.5 one can easily see that if $\mathcal{G}_0$ is not of type A_n then the hypothesis that Λ has at least two numerical marks contradicts Proposition 2.8. It follows that if $\mathcal{G}_0$ is not of type A_n then Λ has exactly one numerical mark and this numerical mark satisfies Lemma 2.7.

Using Remark 2.1 we immediately exclude the following possibilities, for which the weight M is not antidominant:
- $\mathcal{G}_0$ of type B_n $(n \geq 2)$, $\mathcal{G}_{-1} = V(\omega_n)$;
- $\mathcal{G}_0$ of type C_n $(n \geq 3)$, $\mathcal{G}_{-1} = V(\omega_1)$;
- $\mathcal{G}_0$ of type C_n $(n \geq 3)$, $\mathcal{G}_{-1} = V(\omega_i)$ with $2 \leq i \leq n-1$, $\alpha = 2\alpha_{i+1} + \ldots + 2\alpha_{n-1} + \alpha_n$;
- $\mathcal{G}_0$ of type F_4, $\mathcal{G}_{-1} = V(\omega_3)$, $\alpha = \alpha_1 + \alpha_2$;
- $\mathcal{G}_0$ of type F_4, $\mathcal{G}_{-1} = V(\omega_4)$;
- $\mathcal{G}_0$ of type G_2, $\mathcal{G}_{-1} = V(2\omega_1)$;
- $\mathcal{G}_0$ of type G_2, $\mathcal{G}_{-1} = V(\omega_1)$ (simplest representation).

Proposition 2.8 allows us to rule out the cases summarized in Table 1, where we describe the irreducible modules $\mathcal{G}_{-1}$ and $\mathcal{G}_1$ through their highest weights and indicate the positive root β used in Proposition 2.8.

On the other hand, Corollary 1.17 allows us to rule out the cases summarized in Table 2, where the vectors x and y used in Corollary 1.17 are indicated, and where the columns denoted by $\mathcal{G}_{-1}$ and $\mathcal{G}_1$ contain the highest weights of these $\mathcal{G}_0$-modules. In order to show that the vectors x and y in Table 2 are highest and lowest weight vectors in the $\mathcal{G}_0$-modules $\mathcal{G}_{-2}$ and $\mathcal{G}_2$ respectively, one can use the bitransitivity of $\mathcal{G}$ and, where needed, Lemma 2.9.

For the remaining cases let us point out what follows: suppose that $\mathcal{G}_{-2}$ contains a highest weight vector x of weight λ and that $\mathcal{G}_2$ contains a lowest weight vector y of weight $-\lambda$ such that $[x,y] \neq 0$. Then the irreducible submodules $\bar{\mathcal{G}}_{-2}$ and $\bar{\mathcal{G}}_2$ generated respectively by x and y are dual $\mathcal{G}_0$-modules and the Lie subalgebra of $\mathcal{G}_{\bar{0}}$ with local part $\bar{\mathcal{G}}_{-2} \oplus \mathcal{G}_0 \oplus \bar{\mathcal{G}}_2$ is an affine Kac-Moody algebra which will be denoted by $\mathcal{A}$.

Using the classification of affine Kac-Moody algebras we therefore exclude the cases in Table 3, where we indicate the highest weight vector x of $\mathcal{G}_{-2}$, the

$\mathcal{G}_0$	$\mathcal{G}_{-1}$	$\mathcal{G}_1$	α	β	
B_n	ω_i	$\omega_1+\omega_{i-1}$	$\alpha_1+\ldots+\alpha_{i-1}$	$\alpha_{i-1}+2\alpha_i+\ldots+2\alpha_n$	$3\leq i\leq$
B_n	$2\omega_n$	$\omega_1+\omega_{n-1}$	$\alpha_1+\ldots+\alpha_{n-1}$	$\alpha_{n-1}+2\alpha_n$	$n>$
C_n	ω_i	$\omega_1+\omega_{i-1}$	$\alpha_1+\ldots+\alpha_{i-1}$	$\alpha_{i-1}+2\alpha_i+\ldots+2\alpha_{n-1}+\alpha_n$	$3\leq i\leq$
D_n	ω_i	$\omega_1+\omega_{i-1}$	$\alpha_1+\ldots+\alpha_{i-1}$	$\alpha_{i-1}+2\alpha_i+\ldots+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n$	$3\leq i\leq$
E_6	ω_3	$\omega_1+\omega_2$	$\alpha_2+\alpha_4+\alpha_5+\alpha_6$	$\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5$	
E_6	ω_5	$\omega_2+\omega_6$	$\alpha_1+\alpha_2+\alpha_3+\alpha_4$	$\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6$	
E_6	ω_4	$\omega_5+\omega_6$	$\alpha_1+\alpha_3$	$\alpha_2+\alpha_3+2\alpha_4+\alpha_5$	
E_6	ω_4	$2\omega_2$	α_2	$\alpha_1+\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6$	
E_7	ω_3	$2\omega_1$	α_1	$\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7$	
E_7	ω_3	$\omega_2+\omega_7$	$\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7$	$\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5$	
E_7	ω_2	$\omega_1+\omega_7$	$\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7$	$\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6$	
E_7	ω_4	$\omega_1+\omega_3$	$\alpha_1+\alpha_3$	$\alpha_2+\alpha_3+2\alpha_4+\alpha_5$	
E_7	ω_4	$2\omega_2$	α_2	$\alpha_1+\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6$	
E_7	ω_4	$\omega_5+\omega_7$	$\alpha_5+\alpha_6+\alpha_7$	$\alpha_2+\alpha_3+2\alpha_4+\alpha_5$	
E_7	ω_5	$\omega_1+\omega_2$	$\alpha_1+\alpha_2+\alpha_3+\alpha_4$	$\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6$	
E_7	ω_5	$\omega_6+\omega_7$	$\alpha_6+\alpha_7$	$\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6$	
E_7	ω_6	ω_3	$\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5$	$\alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7$	
E_8	ω_1	ω_7	$\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+2\alpha_7+\alpha_8$	$2\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7+\alpha_8$	
E_8	ω_3	$\omega_2+\omega_8$	$\alpha_2+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8$	$\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5$	
E_8	ω_2	$\omega_1+\omega_8$	$\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8$	$\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6$	
E_8	ω_k	$\omega_{k+1}+\omega_8$	$\alpha_{k+1}+\ldots+\alpha_8$	$\alpha_2+\alpha_3+2\alpha_4+\ldots+2\alpha_k+\alpha_{k+1}$	$4\leq k$
E_8	ω_7	$2\omega_8$	α_8	$2\alpha_1+3\alpha_2+4\alpha_3+6\alpha_4+5\alpha_5+4\alpha_6+3\alpha_7+\alpha_8$	
F_4	ω_2	$2\omega_1$	α_1	$\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4$	
F_4	ω_2	$\omega_3+\omega_4$	$\alpha_3+\alpha_4$	$\alpha_1+2\alpha_2+2\alpha_3+\alpha_4$	
F_4	$2\omega_3$	$\omega_1+\omega_2$	$\alpha_1+\alpha_2$	$\alpha_2+2\alpha_3$	
F_4	ω_3	$2\omega_4$	α_4	$\alpha_1+2\alpha_2+3\alpha_3+\alpha_4$	
F_4	$2\omega_4$	ω_2	$\alpha_1+2\alpha_2+2\alpha_3$	$\alpha_1+\alpha_2+2\alpha_3+2\alpha_4$	
G_2	$3\omega_1$	$2\omega_2$	α_2	$3\alpha_1+\alpha_2$	

Table 1

$\mathcal{G}_0$	$\mathcal{G}_{-1}$	$\mathcal{G}_1$	x	y	
A_n	ω_s	$\omega_{n-s+2} + \omega_n$	$[[F_\Lambda, e_{-\alpha_s}], [e_{-\alpha_{s-1}-\alpha_s-\alpha_{s+1}}, F_\Lambda]]$	$[[E_M, e_{\alpha+\alpha_s}], [e_{\alpha_{s-1}}, E_M]]$	$n \geq$
A_n	ω_s	$\omega_1 + \omega_{n-s}$	$[[F_\Lambda, e_{-\alpha_s}], [e_{-\alpha_{s-1}-\alpha_s-\alpha_{s+1}}, F_\Lambda]]$	$[[E_M, e_{\alpha+\alpha_s}], [e_{\alpha_{s+1}}, E_M]]$	$n \geq 5,$
A_n	$\omega_s + \omega_t$	$\omega_{n-s} + \omega_{n-t+2}$	$[[F_\Lambda, e_{-\alpha_s}], [e_{-\alpha_t}, F_\Lambda]]$	$[[E_M, e_{\alpha+\alpha_s}], [e_{\alpha_{s+1}}, E_M]]$	$s \geq 1,$
B_n	ω_s	ω_{s+2}	$[[F_\Lambda, e_{-\alpha_s}], [e_{-\alpha_{s-1}-\alpha_s-\alpha_{s+1}}, F_\Lambda]]$	$[[E_M, e_{\alpha+\alpha_s}], [e_{\alpha_{s+1}+\alpha_{s+2}}, E_M]]$	$2 \leq$
B_n	ω_{n-2}	$2\omega_n$	$[[F_\Lambda, e_{-\alpha_{n-2}}], [e_{-\alpha_{n-3}-\alpha_{n-2}-\alpha_{n-1}}, F_\Lambda]]$	$[[E_M, e_{\alpha+\alpha_{n-2}}], [e_{\alpha_{n-1}+\alpha_n}, E_M]]$	
B_n	ω_{n-1}	$2\omega_n$	$[[F_\Lambda, e_{-\alpha_{n-1}}], [e_{-\alpha_{n-2}-\alpha_{n-1}-\alpha_n}, F_\Lambda]]$	$[[E_M, e_{\alpha_n}], [e_{\alpha_{n-1}+2\alpha_n}, E_M]]$	
C_n	$2\omega_i$	$2\omega_{i+1}$	$[[F_\Lambda, e_{-\alpha_i}], [e_{-\alpha_i}, F_\Lambda]]$	$[[E_M, e_{\alpha+\alpha_i}], [e_{\alpha_{i+1}}, E_M]]$	$1 \leq$
C_n	ω_n	$\omega_1 + \omega_{n-1}$	$[[F_\Lambda, e_{-\alpha_{n-1}-\alpha_n}], [e_{-\alpha_{n-1}-\alpha_n}, F_\Lambda]]$	$[[E_M, e_{\alpha+\alpha_n}], [e_{\alpha_{n-1}}, E_M]]$	
D_n	ω_i	ω_{i+2}	$[[F_\Lambda, e_{-\alpha_i}], [e_{-\alpha_{i-1}-\alpha_i-\alpha_{i+1}}, F_\Lambda]]$	$[[E_M, e_{\alpha+\alpha_i}], [e_{\alpha_{i+1}+\alpha_{i+2}}, E_M]]$	$2 \leq$
D_n	ω_{n-3}	$\omega_{n-1} + \omega_n$	$[[F_\Lambda, e_{-\alpha_{n-3}}], [e_{-\alpha_{n-4}-\alpha_{n-3}-\alpha_{n-2}}, F_\Lambda]]$	$[[E_M, e_{\alpha+\alpha_{n-3}}], [e_{\alpha_{n-2}+\alpha_{n-1}}, E_M]]$	
D_n	ω_n	$\omega_1 + \omega_n$	$[[F_\Lambda, e_{-\alpha_n}], [e_{-\alpha_{n-3}-2\alpha_{n-2}-\alpha_{n-1}-\alpha_n}, F_\Lambda]]$	$[[E_M, e_{\alpha+\alpha_n}], [e_{\alpha_{n-2}+\alpha_{n-1}}, E_M]]$	
D_n	ω_{n-1}	$\omega_1 + \omega_{n-1}$	$[[F_\Lambda, e_{-\alpha_{n-1}}], [e_{-\alpha_{n-3}-2\alpha_{n-2}-\alpha_{n-1}-\alpha_n}, F_\Lambda]]$	$[[E_M, e_{\alpha+\alpha_{n-1}}], [e_{\alpha_{n-2}+\alpha_n}, E_M]]$	
E_6	ω_1	ω_3	$[[F_\Lambda, e_{-\alpha_1}], [e_{-(\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5)}], F_\Lambda]]$	$[[E_M, e_{\alpha+\alpha_1}], [e_{\alpha_3+\alpha_4+\alpha_5}, E_M]]$	
E_6	ω_6	ω_5	$[[F_\Lambda, e_{-\alpha_6}], [e_{-(\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6)}, F_\Lambda]]$	$[[E_M, e_{\alpha+\alpha_6}], [e_{\alpha_3+\alpha_4+\alpha_5}, E_M]]$	
E_6	ω_2	$\omega_1 + \omega_6$	$[[F_\Lambda, e_{-\alpha_2}], [e_{-(\alpha_2+\alpha_3+2\alpha_4+\alpha_5)}, F_\Lambda]]$	$[[E_M, e_{\alpha+\alpha_2}], [e_{\alpha_4+\alpha_5+\alpha_6}, E_M]]$	
E_7	ω_1	ω_6	$[[F_\Lambda, e_{-\alpha_1}], [e_{-(\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5)}, F_\Lambda]]$	$[[E_M, e_{\alpha+\alpha_1}], [e_{\alpha_3+\alpha_4+\alpha_5+\alpha_6}, E_M]]$	
E_7	ω_7	ω_2	$[[F_\Lambda, e_{-\alpha_7}], [e_{-\gamma}, F_\Lambda]]$	$[[E_M, e_{\alpha+\alpha_7}], [e_{\alpha_2+\alpha_4+\alpha_5+\alpha_6}, E_M]]$	$\gamma = \alpha_2 + \alpha_3 +$
E_8	ω_8	ω_1	$[[F_\Lambda, e_{-\alpha_8}], [e_{-\gamma}, F_\Lambda]]$	$[[E_M, e_{\alpha+\alpha_8}], [e_{\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7}, E_M]]$	$\gamma = \alpha_2 + \alpha_3 + 2\alpha$
E_8	ω_7	ω_2	$[[F_\Lambda, e_{-\alpha_7}], [e_{-\gamma}, F_\Lambda]]$	$[[E_M, e_{\alpha+\alpha_7}], [e_{\alpha_2+\alpha_4+\alpha_5+\alpha_6}, E_M]]$	$\gamma = \alpha_2 + \alpha_3 +$
F_4	ω_1	$2\omega_4$	$[[F_\Lambda, e_{-\alpha_1}], [e_{-\alpha_1-2\alpha_2-2\alpha_3}, F_\Lambda]]$	$[[E_M, e_{\alpha+\alpha_1}], [e_{\alpha_2+\alpha_3+\alpha_4}, E_M]]$	
G_2	Ad	$2\omega_1$	$[[F_\Lambda, e_{-\alpha_1-\alpha_2}], [e_{-\alpha_1-\alpha_2}, F_\Lambda]]$	$[[E_M, e_{2\alpha_1+\alpha_2}], [e_{\alpha_1}, E_M]]$	

Table 2

$\mathcal{G}_0$	$\mathcal{G}_{-1}$	$\mathcal{G}_1$	x	y
A_n $(n \geq 4)$	$\omega_s + \omega_{s+2}$ $(1 \leq s \leq n-2)$	$2\omega_{n-s}$	$[[F_\Lambda, e_{-\alpha_s}], [e_{-\alpha_{s+2}}, F_\Lambda]]$	$[[E_M, e_{\alpha+\alpha_s}], [e_{\alpha_{s+1}+\alpha_{s+2}}, E_M]]$
C_n $(n \geq 3)$	$2\omega_{n-1}$	$2\omega_n$	$[[F_\Lambda, e_{-\alpha_{n-1}}], [e_{-\alpha_{n-1}}, F_\Lambda]]$	$[[E_M, e_{\alpha_{n-1}+\alpha_n}], [e_{\alpha_{n-1}+\alpha_n}, E_M]]$
D_n $(n > 4)$	ω_{n-2}	$2\omega_n$	$[[F_\Lambda, e_{-\alpha_{n-3}-\alpha_{n-2}-\alpha_n}], [e_{-\alpha_{n-2}}, F_\Lambda]]$	$[[E_M, e_{\alpha_{n-3}+\alpha_{n-2}+\alpha_{n-1}}], [e_{\alpha_{n-2}+\alpha_{n-1}+\alpha_n}, E$
D_n $(n > 4)$	ω_{n-2}	$2\omega_{n-1}$	$[[F_\Lambda, e_{-\alpha_{n-3}-\alpha_{n-2}-\alpha_{n-1}}], [e_{-\alpha_{n-2}}, F_\Lambda]]$	$[[E_M, e_{\alpha_{n-3}+\alpha_{n-2}+\alpha_n}], [e_{\alpha_{n-2}+\alpha_{n-1}+\alpha_n}, E_M$
E_6	ω_3	$2\omega_6$	$[[F_\Lambda, e_{-\alpha_3}], [e_{-\alpha_2-\alpha_3-2\alpha_4-\alpha_5}, F_\Lambda]]$	$[[E_M, e_{\alpha_1}], [e_{\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5}, E_M]]$
E_6	ω_5	$2\omega_1$	$[[F_\Lambda, e_{-\alpha_5}], [e_{-\alpha_2-\alpha_3-2\alpha_4-\alpha_5}, F_\Lambda]]$	$[[E_M, e_{\alpha_6}], [e_{\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6}, E_M]]$
E_7	ω_6	$2\omega_7$	$[[F_\Lambda, e_{-\alpha_6}], [e_{-\alpha_2-\alpha_3-2\alpha_4-2\alpha_5-\alpha_6}, F_\Lambda]]$	$[[E_M, e_{\alpha_7}], [e_{\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7}, E_M]]$

Table 3

lowest weight vector y of $\mathcal{G}_2$, and the highest weights of the $\mathcal{G}_0$-modules $\mathcal{G}_{-1}$ and $\mathcal{G}_1$.

In the same way the classification of affine Kac-Moody algebras shows that the following cases are allowed:

1) $\mathcal{G}_0$ of type A_3, $\mathcal{G}_{-1} = V(\omega_1+\omega_3)$, $\mathcal{G}_1 = V(2\omega_2)$, $\alpha=\alpha_2$: under these hypotheses $\mathcal{G}_{-2}$ contains the highest weight vector $x = [[F_\Lambda, e_{-\alpha_1}],[e_{-\alpha_3}, F_\Lambda]]$ and $\mathcal{G}_2$ contains the lowest weight vector $y = [[E_M, e_{\alpha_1+\alpha_2}],[e_{\alpha_2+\alpha_3}, E_M]]$. The algebra $\mathcal{A}$ is an affine Kac-Moody algebra of type $A_5^{(2)}$.

2) $\mathcal{G}_0$ of type B_n $(n\geq 3)$, $\mathcal{G}_{-1} = V(\omega_2)$, $\mathcal{G}_1 = V(2\omega_1)$, $\alpha=\alpha_1$: $\mathcal{G}_{-2}$ contains the highest weight vector $x = [[F_\Lambda, e_{-\alpha_2}],[e_{-\alpha_2-2\alpha_3-\ldots-2\alpha_n}, F_\Lambda]]$ and $\mathcal{G}_2$ contains the lowest weight vector $y = [[E_M, e_{\alpha_1+\alpha_2}],[e_{\alpha_1+\alpha_2+2\alpha_3+\ldots+2\alpha_n}, E_M]]$. The algebra $\mathcal{A}$ is an affine Kac-Moody algebra of type $A_{2n}^{(2)}$.

3) $\mathcal{G}_0$ of type B_2, $\mathcal{G}_{-1} = V(2\omega_2)$, $\mathcal{G}_1 = V(2\omega_1)$, $\alpha=\alpha_1$: $\mathcal{G}_{-2}$ contains the highest weight vector $x = [[F_\Lambda, e_{-\alpha_2}],[e_{-\alpha_2}, F_\Lambda]]$ and $\mathcal{G}_2$ contains the lowest weight vector $y = [[E_M, e_{\alpha_1+2\alpha_2}],[e_{\alpha_1}, E_M]]$. The algebra $\mathcal{A}$ is an affine Kac-Moody algebra of type $A_4^{(2)}$.

4) $\mathcal{G}_0$ of type C_n $(n\geq 3)$, $\mathcal{G}_{-1} = V(\omega_2)$, $\mathcal{G}_1 = V(2\omega_1)$, $\alpha=\alpha_1$: $\mathcal{G}_{-2}$ contains the highest weight vector $x = [[F_\Lambda, e_{-\alpha_2-\ldots-\alpha_n}],[e_{-\alpha_1-\ldots-\alpha_{n-1}}, F_\Lambda]]$ and $\mathcal{G}_2$ contains the lowest weight vector $y = [[E_M, e_{\alpha_1}],[e_{2\alpha_1+\ldots+2\alpha_{n-1}+\alpha_n}, E_M]]$. The algebra $\mathcal{A}$ is an affine Kac-Moody algebra of type $A_{2n-1}^{(2)}$.

5) $\mathcal{G}_0$ of type D_n $(n\geq 4)$, $\mathcal{G}_{-1} = V(\omega_2)$, $\mathcal{G}_1 = V(2\omega_1)$, $\alpha=\alpha_1$: in this case $x = [[F_\Lambda, e_{-\alpha_2}],[e_{-\alpha_2-2\alpha_3-\ldots-2\alpha_{n-2}-\alpha_{n-1}-\alpha_n}, F_\Lambda]]$ and $y = [[E_M, e_{\alpha_1+\alpha_2}],[e_{\alpha_1+\alpha_2+2\alpha_3+\ldots+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n}, E_M]]$. The algebra $\mathcal{A}$ is an affine Kac-Moody algebra of type $A_{2n-1}^{(2)}$.

We finally analyze and rule out the remaining cases:

- $\mathcal{G}_0$ of type A_n, $\mathcal{G}_{-1} = V(\omega_s)$:

(i) if $s=1$ (or, equivalently, $s=n$) then $\mathcal{G}_1 = V(\omega_1+\omega_{n-1})$ and $\mathcal{G}_{-2}\subset S^2\mathcal{G}_{-1} = S^2V(\omega_1) = 0$ since $S^2V(\omega_1) = V(2\omega_1)$ and $[F_\Lambda, F_\Lambda]=0$, therefore $\mathcal{G}$ has finite depth;

(ii) if $s=2$, $\mathcal{G}_1 = V(2\omega_n)$, $\alpha=\alpha_1$, (or, equivalently, $s=n-1$, $\mathcal{G}_1 = V(2\omega_1)$, $\alpha=\alpha_n$) then $\mathcal{G}$ is isomorphic to the finite-dimensional Lie superalgebra $p(n)$ (for the definition of $p(n)$ see [K2]);

(iii) if $n=4$ and $s=3$, i.e. $\mathcal{G}_{-1}\cong \Lambda^2 sl_5^*$, $\mathcal{G}_1 = V(\omega_3+\omega_4)$, $\alpha=\alpha_1+\alpha_2$ (or, equivalently, $\mathcal{G}_{-1} = V(\omega_2)$, $\mathcal{G}_1 = V(\omega_1+\omega_2)$, $\alpha=\alpha_3+\alpha_4$), then $\mathcal{G}$ is isomorphic to the infinite-dimensional Lie superalgebra $E(5,10)$ (for the definition of $E(5,10)$ see [K3]).

- $\mathcal{G}_0$ of type B_n $(n\geq 2)$, $\mathcal{G}_{-1} = V(\omega_1)$ and:

(i) $\mathcal{G}_1 = V(\omega_3)$ if $n>3$ $(\alpha=\alpha_2+2\alpha_3+\ldots+2\alpha_n)$,

(ii) $\mathcal{G}_1 = V(2\omega_3)$ if $n=3$ $(\alpha=\alpha_2+2\alpha_3)$,

(iii) $\mathcal{G}_1 = V(\omega_2)$ if $n=2$ $(\alpha=\alpha_2)$.

For all these cases $\mathcal{G}_{-2}\subset S^2\mathcal{G}_{-1} = S^2V(\omega_1) = V(2\omega_1)+1 = 1$ since $[F_\Lambda, F_\Lambda]=0$. Thus $\mathcal{G}$ has finite depth.

- $\mathcal{G}_0$ of type D_n ($n \geq 4$):

(i) $\mathcal{G}_{-1} = V(\omega_1)$, $\mathcal{G}_1 = V(\omega_1 + \omega_3)$, then $\mathcal{G}_{-2} \subset S^2\mathcal{G}_{-1} = S^2V(\omega_1) = V(2\omega_1) + 1 = 1$ hence $\mathcal{G}$ has finite depth.

(ii) $n = 4$, $\mathcal{G}_{-1} = V(\omega_4)$, $\mathcal{G}_1 = V(\omega_1+\omega_4)$ ($\alpha = \alpha_1+\alpha_2+\alpha_3$) (or, equivalently, $\mathcal{G}_{-1} = V(\omega_3)$, $\mathcal{G}_1 = V(\omega_1 + \omega_3)$), then we can use the same argument as in (i) and conclude. □

2.2 Case $(\Lambda, \alpha) \neq 0$

In the following we assume $(\Lambda, \alpha) \neq 0$.

Remark 2.11 Under the hypothesis $(\Lambda, \alpha) \neq 0$ the vector $[F_\Lambda, F_\Lambda]$ is different from 0: $[E_M, [F_\Lambda, F_\Lambda]] = 2[e_{-\alpha}, F_\Lambda] \neq 0$.
Nevertheless, $[E_M, E_M] = 0$ and therefore $[[E_M, e_\beta], E_M] = 0$ for every positive root β (see Lemma 2.2).

Corollary 2.12 *If $(\Lambda, \alpha) \neq 0$ then either $(\Lambda, \alpha) = (\alpha, \alpha)$ or $(\Lambda, \alpha) = (\alpha, \alpha)/2$.*

Proof. It is enough to apply Lemma 1.14 to the following vectors:

$$x_\lambda = F_\Lambda, \;\; x_\mu = E_M.$$

□

Lemma 2.13 *Suppose that α is not simple. Then there exists j such that: $\alpha - \alpha_j$ is a root and $\alpha + \alpha_j$, $2\alpha - \alpha_j$, $\alpha - 2\alpha_j$ are not roots, in all cases except those in the following list:*
- $\mathcal{G}_0$ of type B_n and $\alpha = \alpha_i + \alpha_{i+1} + \ldots + \alpha_n$, $\alpha = \alpha_{n-1} + 2\alpha_n$;
- $\mathcal{G}_0$ of type C_n and $\alpha = 2\alpha_i + \ldots + 2\alpha_{n-1} + \alpha_n$, $\alpha = \alpha_{n-1} + \alpha_n$;
- $\mathcal{G}_0$ of type F_4 and $\alpha = \alpha_1 + \alpha_2 + \alpha_3, \alpha = \alpha_2 + \alpha_3, \alpha = \alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4$, $\alpha = \alpha_2 + 2\alpha_3, \alpha = \alpha_2 + 2\alpha_3 + 2\alpha_4, \alpha = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4$;
- $\mathcal{G}_0$ of type G_2 and $\alpha = 2\alpha_1 + \alpha_2$, $\alpha = \alpha_1 + \alpha_2$, $\alpha = 3\alpha_1 + \alpha_2$.

Proof. Case by case check. □

Lemma 2.14 *Let α be a positive root of $\mathcal{G}_0$ and suppose that it is not simple. If α_j is a simple root of $\mathcal{G}_0$ such that $\alpha - \alpha_j$ is a root and $\alpha + \alpha_j$, $2\alpha - \alpha_j$, $\alpha - 2\alpha_j$ are not roots, then either*

$$\tilde{x} := [[[E_M, e_{\alpha_j}], e_{\alpha - \alpha_j}], E_M]$$

is a lowest weight vector in $\mathcal{G}_{-2}$ or $\tilde{x} = 0$ and

$$x := [[[E_M, e_{\alpha_j}], e_\alpha], E_M]$$

is a lowest weight vector in $\mathcal{G}_{-2}$.

Proof. If $\tilde{x} \neq 0$ then, using the transitivity of $\mathcal{G}$, one can show that it is a lowest weight vector in $\mathcal{G}_{-2}$. If $\tilde{x} = 0$ then $[x, e_{-k}] = 0$ for every $k = 1, \ldots, n$.
□

Proposition 2.15 *If α is not a simple root and the growth of $\mathcal{G}$ is finite then either $(\mathcal{G}_0,\alpha)$ belongs to the list in Lemma 2.13 or $(\mathcal{G}_0,\alpha)=(A_n, longest\,root)$ and $\tilde{x}:=[[[E_M,e_{\alpha_j}],e_{\alpha-\alpha_j}],E_M]\neq 0$.*

Proof. Suppose that $(\mathcal{G}_0,\alpha)$ is not in the list in Lemma 2.13. Since α is not a simple root we can apply Lemma 2.14: in the case $\tilde{x}=0$ we take $y=[F_\Lambda,F_\Lambda]$. Then $[x,y]=2\Lambda(h_\alpha)e_{-\alpha+\alpha_j}\neq 0$, and, by Theorem 1.16, we get infinite growth.

If $\tilde{x}:=[[[E_M,e_{\alpha_j}],e_{\alpha-\alpha_j}],E_M]\neq 0$ then, by bitransitivity, $[\tilde{x},F_\Lambda]=(\Lambda(h_\alpha)-2\Lambda(h_j))E_M\neq 0$, thus $[\tilde{x},y]=(\Lambda(h_\alpha)-2\Lambda(h_j))e_{-\alpha}$ is different from zero. Then the thesis follows from Theorem 1.16. (Notice that the case $\mathcal{G}_0$ of type C_n, α its longest root, is in the list of Lemma 2.13 and is therefore excluded by the hypotheses.) $\square$

Lemma 2.16 *If the growth of $\mathcal{G}$ is finite and β is a positive root such that $\alpha+\beta$ and $\alpha-\beta$ are not roots, then $(\Lambda,\beta)=0$.*

Proof. Suppose $(\Lambda,\beta)\neq 0$. We define:

$$
(2)\qquad \begin{array}{ll} E_1=[e_\alpha,E_M], & E_2=[[E_M,e_\alpha],e_\beta],\\ F_1=F_\Lambda, & F_2=\Lambda(h_\beta)^{-1}[F_\Lambda,e_{-\beta}],\\ H=h_\alpha. & \end{array}
$$

It is easy to verify that the conditions of Lemma 1.13 are satisfied with $a_1=a_2=-\Lambda(h_\alpha)$, thus $r(\mathcal{G})=\infty$. $\square$

Theorem 2.17 *Let $\mathcal{G}=\oplus_{i\in\mathbb{Z}}\mathcal{G}_i$ be a $\mathbb{Z}$-graded, consistent, simple, irreducible Lie superalgebra of finite growth. Assume that $\mathcal{G}_0$ is a simple Lie algebra, that $\mathcal{G}_1$ is an irreducible $\mathcal{G}_0$-module which is not contragredient to $\mathcal{G}_{-1}$ and that the local part generates $\mathcal{G}$. Let F_Λ be a highest weight vector in $\mathcal{G}_{-1}$ and E_M a lowest weight vector in $\mathcal{G}_1$ so that $\Lambda+M=-\alpha$ for a positive root α. If $(\Lambda,\alpha)\neq 0$ then $\mathcal{G}_0$ has rank 1.*

Proof. By Proposition 2.15 and its proof only the following cases may occur:
- α is a simple root;
- $(\mathcal{G}_0,\alpha)=(A_n,\text{longest root})$;
- $(\mathcal{G}_0,\alpha)$ is in the list of Lemma 2.13.

Let us analyze these possibilities case by case:
1) $\mathcal{G}_0$ of type A_n, $\alpha=\alpha_1+\ldots+\alpha_n$. If $n=1$ we get the thesis. Now suppose $n\geq 2$. The proof of Proposition 2.15 shows that this possibility holds if

$$\tilde{x}=[[[E_M,e_j],e_{\alpha-\alpha_j}],E_M]$$

is a nonzero vector, thus either $j=1$ or $j=n$. If we apply Lemma 2.16 to $\alpha=\alpha_1+\ldots+\alpha_n$ and $\beta=\alpha_2\ldots+\alpha_{n-1}$ we deduce that $(\Lambda,\alpha_i)=0$ for every $i=2,\ldots,n-1$, therefore $(\Lambda,\alpha)=(\Lambda,\alpha_1)+(\Lambda,\alpha_n)$.

As we already noticed in the proof of Proposition 2.15, for every $k=1,\ldots,n$, $[\tilde{x},e_{-k}]=0$ thus, since we assume $\tilde{x}\neq 0$, transitivity implies $[\tilde{x},F_\Lambda]\neq 0$. Since $[\tilde{x},F_\Lambda]=(\Lambda(h_\alpha)-2\Lambda(h_j))E_M$, it turns out that $\Lambda(h_1)\neq\Lambda(h_n)$. Corollary

2.12 now implies that either $(\Lambda,\alpha_1)=0$ or $(\Lambda,\alpha_n)=0$. But this hypothesis contradicts Theorem 1.16, since if we take the highest weight vector $y=[F_\Lambda,F_\Lambda]$ in $\mathcal{G}_{-2}$, then $[\tilde{x},y]\neq 0$ but the irreducible submodule of $\mathcal{G}_{-2}$ generated by $[F_\Lambda,F_\Lambda]$ is not the standard A_n-module.

2) $\mathcal{G}_0$ of type A_n, α simple, $n\geq 2$.

2a) $n\geq 3$, $\alpha=\alpha_j$ with $j\neq 1,n$

If we apply Lemma 2.16 with $\alpha=\alpha_j$ and $\beta=\alpha_{j-1}+\alpha_j+\alpha_{j+1}$ we find a contradiction.

2b) $\alpha=\alpha_1$ (or, equivalently, $\alpha=\alpha_n$).

Again, by applying Lemma 2.16 with $\beta=\alpha_3+\ldots+\alpha_n$, we find $(\Lambda,\alpha_i)=0$ for every $i\geq 3$. On the other hand, $(\Lambda,\alpha_2)\neq 0$ since $[E_M,[F_\Lambda,e_{-\alpha_2}]]=e_{-\alpha_1-\alpha_2}\neq 0$. We distinguish two cases:

CASE 1: $(\Lambda,\alpha_2)\neq 1$

Under this hypothesis let us consider the following vectors:

$$x_\mu=[[[E_M,e_1],[E_M,e_2]],[E_M,e_1]],$$

$$x_\lambda=\Lambda(h_1)^{-1}(1-\Lambda(h_2))^{-1}(3+\Lambda(h_1))^{-1}[F_\Lambda,[F_\Lambda,[F_\Lambda,e_{-2}]]].$$

Then x_λ and x_μ satisfy the hypotheses of Lemma 1.14 with $\delta=\alpha_1$. Since $(3\Lambda-\alpha_2,\alpha_1)=3(\Lambda,\alpha_1)+1\geq 4$ we find a contradiction.

CASE 2: $(\Lambda,\alpha_2)=1$

By Corollary 2.12, either $\Lambda(h_1)=1$ or $\Lambda(h_1)=2$. Notice that $x:=[F_\Lambda,F_\Lambda]$ is a highest weight vector in $\mathcal{G}_{-2}$ and $y:=[[E_M,e_1],[E_M,e_1]]$ is a lowest weight vector in $\mathcal{G}_2$. Since $[x,y]=-4\Lambda(h_1)h_1$, $\mathcal{G}_{\bar{0}}$ contains a $\mathbb{Z}$-graded Lie subalgebra with local part $s_{-2}\oplus\mathcal{G}_0\oplus s_2$, where s_{-2} is the irreducible submodule of $\mathcal{G}_{-2}$ generated by x and s_2 is the irreducible submodule of $\mathcal{G}_2$ generated by y. The classification of Kac-Moody Lie algebras immediately allows us to rule out the case $\Lambda(h_1)=2$ and the case $\Lambda(h_1)=1$, $n>2$.

Now suppose $n=2$, $\Lambda(h_1)=1=\Lambda(h_2)$. Under these hypotheses $\mathcal{G}_{-2}$ contains the highest weight vector

$$z:=-4[[F_\Lambda,e_{-\alpha_1-\alpha_2}],F_\Lambda]+5[[[F_\Lambda,e_{-\alpha_1}],e_{-\alpha_2}],F_\Lambda]-3[[[F_\Lambda,e_{-\alpha_2}],F_\Lambda],e_{-\alpha_1}]$$

of weight Λ. Besides, $[z,y]=-24e_{-\alpha_1-\alpha_2}$ and this contradicts Theorem 1.16 since the irreducible $\mathcal{G}_0$-submodule of $\mathcal{G}_{-2}$ containing z is the adjoint module and not the standard one.

3) $\mathcal{G}_0$ of type B_n $(n\geq 2)$, $\alpha=\alpha_i+\ldots+\alpha_n$ $(1\leq i\leq n-1)$.

3a) If $i>1$ take $\beta=\alpha_{i-1}+\alpha_i+2\alpha_{i+1}+\ldots+2\alpha_n$, then $\alpha+\beta$ and $\alpha-\beta$ are not roots and, by Lemma 2.16, $(\Lambda,\beta)=0$, i.e. $(\Lambda,\alpha_j)=0$ for every $j\geq i-1$ which contradicts the hypothesis $(\Lambda,\alpha)\neq 0$.

3b) If $i=1$ and $n\geq 3$ take $\beta=\alpha_2+\ldots+2\alpha_n$. Then, by Lemma 2.16, $(\Lambda,\alpha_i)=0$ for every $i\neq 1$. This implies the following contradiction:

$$0=[E_M,[F_\Lambda,e_{-\alpha_n}]]=[e_{-\alpha},e_{-\alpha_n}]\neq 0.$$

3c) Let $i=1$ and $n=2$, i.e. $\alpha=\alpha_1+\alpha_2$.

If $(\Lambda, \alpha_2) = 0$, as above we have:

$$0 = [E_M, [F_\Lambda, e_{-\alpha_2}]] = [e_{-\alpha}, e_{-\alpha_2}] \neq 0.$$

Thus suppose $(\Lambda, \alpha_2) \neq 0$. Since α and α_2 have both length 1, Corollary 2.12 implies $(\Lambda, \alpha_1) = 0$ and either $\Lambda(h_2) = 1$ or $\Lambda(h_2) = 2$.

Notice that $\mathcal{G}_{-2}$ contains the highest weight vector $x := [F_\Lambda, F_\Lambda]$. Now, if $\Lambda(h_2) = 1$ then $\mathcal{G}_2$ contains the lowest weight vector $y := [[E_M, e_{\alpha_1}], [E_M, e_{\alpha_2}]]$ and $[x, y] = 2e_{-\alpha}$ thus $\mathcal{G}_{\bar{0}}$ has infinite growth according to Theorem 1.16.

If $\Lambda(h_2) = 2$, by bitransitivity, then $y = 0$ and the vector

$$z := [[E_M, e_{\alpha_1+\alpha_2}], [E_M, e_{\alpha_2}]]$$

is a lowest weight vector in $\mathcal{G}_2$. Again, since $[x, z] = -8e_{-\alpha_1}$, this contradicts Theorem 1.16.

4) $\mathcal{G}_0$ of type B_n, α simple.

4a) If $\alpha = \alpha_i$ with $i \neq 1, n$, we proceed as for A_n.

4b) If $\alpha = \alpha_1$ we take $\beta = \alpha_1 + 2\alpha_2 + \ldots + 2\alpha_n$ and apply Lemma 2.16.

4c) If $\alpha = \alpha_n$ and $n \geq 3$ we take $\beta = \alpha_{n-2} + 2\alpha_{n-1} + 2\alpha_n$. Then Lemma 2.16 holds and we get a contradiction.

4d) $n = 2$, $\alpha = \alpha_2$. In this case relation $[E_M, [F_\Lambda, e_{-\alpha_1}]] = e_{-\alpha_2-\alpha_1}$ implies $(\Lambda, \alpha_1) \neq 0$. This possibility is therefore ruled out by the classification of Kac-Moody Lie algebras once we have noticed that since $\mathcal{G}_{-2}$ contains the highest weight vector $x := [F_\Lambda, F_\Lambda]$ and $\mathcal{G}_2$ contains the lowest weight vector $y := [[E_M, e_{\alpha_2}], [E_M, e_{\alpha_2}]]$, with $[x, y] \neq 0$, $\mathcal{G}_{\bar{0}}$ contains an affine Kac-Moody, $\mathbb{Z}$-graded Lie subalgebra with local part $s_{-2} \oplus \mathcal{G}_0 \oplus s_2$, where s_{-2} is the $\mathcal{G}_0$-irreducible module with highest weight 2Λ and s_2 is the $\mathcal{G}_0$-module contragredient to s_{-2}.

5) $\mathcal{G}_0$ of type B_n, $\alpha = \alpha_{n-1} + 2\alpha_n$.

5a) If $n \geq 3$ take $\beta = \alpha_{n-2} + \alpha_{n-1} + \alpha_n$ and use Lemma 2.16.

5b) Let $n = 2$, $\alpha = \alpha_1 + 2\alpha_2$. If we take $\beta = \alpha_1$ then Lemma 2.16 implies $(\Lambda, \alpha_1) = 0$ thus $\Lambda(h_\alpha) = \Lambda(h_2)$ is either 1 or 2. One can easily verify, using the bitransitivity of $\mathcal{G}$, that the vector $z := [[E_M, e_{\alpha_1+\alpha_2}], [E_M, e_{\alpha_2}]]$ is equal to 0, the vector $y := [[E_M, e_{\alpha_1+2\alpha_2}], [E_M, e_{\alpha_2}]]$ is a lowest weight vector in $\mathcal{G}_2$ and, as in the previous cases, $x := [F_\Lambda, F_\Lambda]$ is a highest weight vector in $\mathcal{G}_{-2}$. Since $[x, y] = 24(\Lambda(h_2) + 1)e_{-\alpha_1-\alpha_2}$ this contradicts Theorem 1.16.

6) $\mathcal{G}_0$ of type C_n $(n \geq 3)$, $\alpha = 2\alpha_i + \ldots + 2\alpha_{n-1} + \alpha_n$ $(1 \leq i \leq n-1)$.

If $i \neq 1$ we apply Lemma 2.16 to $\beta = \alpha_{i-1} + \alpha_i + 2\alpha_{i+1} + \ldots + 2\alpha_{n-1} + \alpha_n$ and get a contradiction.

If $i = 1$ take $\beta = 2\alpha_2 + \ldots + 2\alpha_{n-1} + \alpha_n$. Then Lemma 2.16 implies $(\Lambda, \alpha_i) = 0$ for every $i \geq 2$. Thus $(\Lambda, \alpha) = 2(\Lambda, \alpha_1)$.

Consider the following vectors:

$$x = [F_\Lambda, F_\Lambda]$$

$$y = [[[E_M, e_1], e_\alpha], E_M].$$

Then x is a highest weight vector in $\mathcal{G}_{-2}$ and y is a lowest weight vector in $\mathcal{G}_2$. Besides, $[x,y] = 2\Lambda(h_\alpha)e_{\alpha_1-\alpha}$. This contradicts Theorem 1.16 since $\alpha - \alpha_1$ is not the highest root of $\mathcal{G}_0$.

7) $\mathcal{G}_0$ of type C_n ($n \geq 3$), $\alpha = \alpha_{n-1} + \alpha_n$.

If we take $\beta = 2\alpha_{n-2} + 2\alpha_{n-1} + \alpha_n$, by Lemma 2.16, we get a contradiction.

8) $\mathcal{G}_0$ of type C_n ($n \geq 3$), α simple.

8a) If $\alpha = \alpha_i$ with $i \neq 1, n-1, n$ then we proceed as for A_n, case 2a).

8b) If $\alpha = \alpha_{n-1}$, take $\beta = 2\alpha_{n-2} + 2\alpha_{n-1} + \alpha_n$ and apply Lemma 2.16.

8c) If $\alpha = \alpha_1$, $[E_M, [F_\Lambda, e_{-\alpha_2}]] = e_{-\alpha_1-\alpha_2}$ implies $(\Lambda, \alpha_2) \neq 0$. Thus we apply the same argument as in case 4d) with $x = [F_\Lambda, F_\Lambda]$ and $y = [[E_M, e_{\alpha_1}], [E_M, e_{\alpha_1}]]$.

8d) If $\alpha = \alpha_n$ we take $\beta = 2\alpha_{n-1} + \alpha_n$. By Lemma 2.16 we find a contradiction.

9) $\mathcal{G}_0$ of type D_n ($n \geq 4$), α simple.

9a) If $\alpha = \alpha_i$, $i \neq 1, n-1, n$ we proceed as for A_n, case 2a).

9b) If $\alpha = \alpha_1$ we apply Lemma 2.16 to $\beta = \alpha_1 + 2\alpha_2 + \ldots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$ and find a contradiction.

9c) If $\alpha = \alpha_n$ (or, equivalently, $\alpha = \alpha_{n-1}$) we apply Lemma 2.16 to $\beta = \alpha_{n-3} + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$.

10) $\mathcal{G}_0$ of type E_6, α simple, $\alpha = \alpha_i$.

If $i \neq 1$, 2, 6 we proceed as for A_n, case 2a). Otherwise we apply Lemma 2.16 as follows:

if $i = 1$ we take $\beta = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5$;

if $i = 6$ we take $\beta = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$;

if $i = 2$ we take $\beta = \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5$.

11) $\mathcal{G}_0$ of type E_7 or E_8.

The situation is analogous to case 10).

12) $\mathcal{G}_0$ of type F_4 and α in the list.

We apply Lemma 2.16 with the following roots α and β:

- $\alpha = \alpha_1 + \alpha_2 + \alpha_3$, $\beta = \alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$;
- $\alpha = \alpha_2 + \alpha_3$, $\beta = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$;
- $\alpha = \alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4$, $\beta = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4$;
- $\alpha = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4$, $\beta = \alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4$;
- $\alpha = \alpha_2 + 2\alpha_3$, $\beta = \alpha_2 + \alpha_3 + \alpha_4$;
- $\alpha = \alpha_2 + 2\alpha_3 + 2\alpha_4$, $\beta = \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4$.

13) $\mathcal{G}_0$ of type F_4, α simple.

We apply Lemma 2.16 with the following roots α and β:

- $\alpha = \alpha_1$, $\beta = \alpha_1 + 2\alpha_2 + 2\alpha_3$;
- $\alpha = \alpha_2$, $\beta = \alpha_1 + \alpha_2 + \alpha_3$;
- $\alpha = \alpha_3$, $\beta = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$;
- $\alpha = \alpha_4$, $\beta = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$.

14) $\mathcal{G}_0$ of type G_2, α in the list.

14a) $\alpha = 2\alpha_1 + \alpha_2$

If we apply Lemma 2.16 with $\beta = \alpha_2$ we find $(\Lambda, \alpha_2) = 0$ thus $(\Lambda, \alpha) = 2(\Lambda, \alpha_1)$. Besides, Corollary 2.12 implies $\Lambda(h_\alpha) = 2$, i.e. $\Lambda(h_1) = 1$.

Consider the vector $x := [[E_M, e_\alpha], [E_M, e_{\alpha_1}]]$. Then one can verify that x is a lowest weight vector. Now, if we take $y := [F_\Lambda, F_\Lambda]$ in $\mathcal{G}_{-2}$, then $[x, y] \neq 0$ and this contradicts Theorem 1.16.

14b) $\alpha = \alpha_1 + \alpha_2$

In this case we apply Lemma 2.16 with $\beta = 3\alpha_1 + \alpha_2$ and find a contradiction.

14c) $\alpha = 3\alpha_1 + \alpha_2$

We proceed as in 14b) with $\beta = \alpha_1 + \alpha_2$.

15) $\mathcal{G}_0$ of type G_2, α simple.

If $\alpha = \alpha_1$ apply Lemma 2.16 with $\beta = 3\alpha_1 + 2\alpha_2$.

If $\alpha = \alpha_2$ apply Lemma 2.16 with $\beta = 2\alpha_1 + \alpha_2$. $\square$

3 The classification theorem

Let L be a finite-dimensional Lie superalgebra and let σ be an automorphism of L of finite order k. Then

(3) $$L = \oplus_{i=0}^{k-1} L_i$$

where $L_i = \{x \in L | \sigma(x) = \epsilon^i x\}$, $\epsilon = e^{2\pi i/k}$. Notice that (3) is a mod-k gradation of L.

Consider the Lie superalgebra $\mathbf{C}[x, x^{-1}] \otimes L = \oplus_{i=-\infty}^{+\infty} x^i \otimes L$ and its subalgebra

$$G^k(L, \sigma) := \oplus_{i=-\infty}^{+\infty} x^i \otimes L_{i\,(mod k)}$$

called the *covering superalgebra* of L. Then $G^k(L, \sigma)$ is a $\mathbb{Z}$-graded Lie superalgebra of infinite depth and growth 1.

Example 1 (The Lie superalgebra $\mathbf{S_1(n)}$) We recall that $sl(m, n)$ is the Lie superalgebra of $(m+n) \times (m+n)$ matrices with supertrace equal to 0, i.e., in suitable coordinates, the set of matrices $\{\left(\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right) | tr(a) = tr(d)\}$.

Let $\tilde{Q}(n)$ $(n \geq 2)$ be the subalgebra of $sl(n+1, n+1)$ consisting of the matrices of the form $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$, where $tr(b) = 0$. Then $\tilde{Q}(n)$ has a one-dimensional centre $C = \langle I_{2n+2} \rangle$ and we define $Q(n) = \tilde{Q}(n)/C$. Notice that $Q(n)$ has even part isomorphic to the Lie algebra of type A_n and odd part isomorphic to $ad\, sl_{n+1}$ and has therefore dimension $2(n^2 + 2n)$. We consider the following automorphism σ of $Q(n)$:

$$\sigma \begin{pmatrix} a & b \\ b & a \end{pmatrix} = \begin{pmatrix} -a^t & ib^t \\ ib^t & -a^t \end{pmatrix}.$$

Then σ has order 4 and $Q(n) = \oplus_{i=0}^{3} Q(n)_i$ where

$Q(n)_0 \cong so_{n+1}$,

$Q(n)_1 = \{b \in sl_{n+1} | b = b^t\}$,

$Q(n)_2 = \{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} | a = a^t\}/C$,

$Q(n)_3 = \{b \in sl_{n+1} | b = -b^t\}$.

Let us suppose $n \neq 3$ and denote by $S_1(n)$ the covering superalgebra $G^4(Q(n), \sigma)$. Notice that $Q(n)_3$ is isomorphic to the adjoint module of so_{n+1} and if $n > 2$ then $Q(n)_1$ and $Q(n)_2$ are isomorphic, as so_{n+1}-modules, to the highest weight module $V(2\omega_1)$, while if $n = 2$ $Q(n)_1$ and $Q(n)_2$ are $sl(2)$-irreducible modules of dimension 5.

Example 2 (The Lie superalgebra $\mathbf{S_2(m)}$) Suppose $m = 2n-1$ and consider the following automorphism τ of $Q(m)$:

$$\tau\left(\begin{array}{cc|cc} a & b & r & s \\ c & d & v & w \\ \hline r & s & a & b \\ v & w & c & d \end{array}\right) = \left(\begin{array}{cc|cc} -d^t & b^t & -iw^t & is^t \\ c^t & -a^t & iv^t & -ir^t \\ \hline -iw^t & is^t & -d^t & b^t \\ iv^t & -ir^t & c^t & -a^t \end{array}\right)$$

where a, b, c, d, r, s, v, w are $n \times n$-blocks and $tr(r) + tr(w) = 0$.
Then $\tau^4 = 1$ and $Q(m) = \oplus_{i=0}^{3} Q(m)_i$ where
$Q(m)_0 \cong sp(2n)$,
$Q(m)_1 = \{\begin{pmatrix} r & s \\ v & w \end{pmatrix} | r = -w^t, s = s^t, v = v^t\}$,
$Q(m)_2 = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} | b^t = -b, c^t = -c, a^t = d\}/C$,
$Q(m)_3 = \{\begin{pmatrix} r & s \\ v & w \end{pmatrix} | w^t = r, s^t = -s, v^t = -v, tr(r) = 0\}$.

Let us denote by $S_2(m)$ the covering superalgebra $G^4(Q(m), \tau)$. Notice that $Q(m)_1$ is isomorphic to the adjoint module of the Lie algebra $sp(2n)$ and $Q(m)_2$, $Q(m)_3$ are isomorphic to the $sp(2n)$-module $\Lambda_0^2 sp_{2n}$.

Example 3 (The Lie superalgebra $\mathbf{S_3}$) Let $D(2, 1; \alpha)$ be the one-parameter family of 17-dimensional Lie superalgebras with even part isomorphic to $A_1 \oplus A_1 \oplus A_1$ and odd part isomorphic to $sl_2 \otimes sl_2 \otimes sl_2$. We recall that two members $D(2, 1; \alpha)$ and $D(2, 1; \beta)$ of this family are isomorphic if and only if α and β lie in the same orbit of the group V of order 6 generated by $\alpha \mapsto -1-\alpha$, $\alpha \mapsto 1/\alpha$.

$D(2, 1; \alpha)$ is the contragredient Lie superalgebra associated to the matrix

$$\begin{pmatrix} 0 & 1 & -1-\alpha \\ 1/\alpha & 0 & 1 \\ 1 & -\alpha/(1+\alpha) & 0 \end{pmatrix}.$$

Suppose that $\alpha^2 + \alpha + 1 = 0$ and consider the following automorphism φ of $D(2, 1; \alpha)$:

$$\begin{array}{lll} \varphi(e_1) = -e_2 & \varphi(f_1) = -f_2 & \varphi(h_1) = h_2 \\ \varphi(e_2) = -e_3 & \varphi(f_2) = -f_3 & \varphi(h_2) = h_3 \\ \varphi(e_3) = -e_1 & \varphi(f_3) = -f_1 & \varphi(h_3) = h_1. \end{array}$$

Then φ has order 6 and $D(2,1;\alpha) = \oplus_{i=0}^{5} V_i$ where
– V_0 is isomorphic to the Lie algebra of type A_1;
– V_1 is isomorphic, as a V_0-module, to the $sl(2)$-irreducible module of dimension 4;
– V_2 is isomorphic, as a V_0-module, to the adjoint module of $sl(2)$;
– V_3 is isomorphic to the $sl(2)$-irreducible module of dimension 2;
– V_4 is isomorphic to the adjoint module of $sl(2)$;
– V_5 is isomorphic to the $sl(2)$-irreducible module of dimension 2.
We denote by S_3 the covering superalgebra $G^6(D(2,1;\alpha), \varphi)$.

Theorem 3.1 *Let $\mathcal{G} = \oplus_{i\in\mathbb{Z}}\mathcal{G}_i$ be an infinite-dimensional $\mathbb{Z}$-graded Lie superalgebra. Suppose that:*
- $\mathcal{G}$ is simple and generated by its local part,
- the $\mathbb{Z}$-gradation is consistent and has infinite depth,
- $\mathcal{G}_0$ is simple,
- $\mathcal{G}_{-1}$ and $\mathcal{G}_1$ are irreducible $\mathcal{G}_0$-modules which are not contragredient.
Then $\mathcal{G}$ has finite growth if and only if it is isomorphic to one of the Lie superalgebras S_i for some $1 \leq i \leq 3$.

Proof. Theorems 2.10 and 2.17 show that under our hypotheses either $\mathcal{G}_0$ has rank 1 or one of the following possibilities occur:
a) $\mathcal{G}_0$ is of type A_3, $\mathcal{G}_{-1}$ is its adjoint module and $\mathcal{G}_1 = V(2\omega_2)$;
b) $\mathcal{G}_0$ is of type B_n, $\mathcal{G}_{-1}$ is its adjoint module and $\mathcal{G}_1 = V(2\omega_1)$;
c) $\mathcal{G}_0$ is of type C_n $(n \geq 3)$, $\mathcal{G}_1$ is its adjoint module and $\mathcal{G}_{-1} \cong \Lambda_0^2 sp_{2n}$;
d) $\mathcal{G}_0$ is of type D_n $(n \geq 4)$, $\mathcal{G}_{-1}$ is its adjoint module and $\mathcal{G}_1 = V(2\omega_1)$.
Besides, if $\mathcal{G}_0$ has rank 1, by Corollary 2.12, either
e) $\mathcal{G}_{-1} \cong V(\omega)$ and $\mathcal{G}_1 \cong V(3\omega)$ or
f) $\mathcal{G}_{-1}$ is isomorphic to the adjoint module of A_1 and $\mathcal{G}_1 \cong V(4\omega)$.
By Propositions 1.7 and 1.9 we conclude that $\mathcal{G}$ is isomorphic to the Lie superalgebra $S_1(m) = G^4(Q(m), \sigma)$ with $m = 5$ in case a), $m = 2n$ in case b), $m = 2$ in case f) and $m = 2n - 1$ in case d); in case c) $\mathcal{G}$ is isomorphic to the Lie superalgebra $S_2(m) = G^4(Q(m), \tau)$ with $m = 2n - 1$. Finally, in case e) $\mathcal{G}$ is isomorphic to the Lie superalgebra S_3. □

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