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The Quantized Enveloping Algebra $\mathcal{U}_q(sl(n))$ at the Roots of Unity

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Abstract: In this paper we classify the irreducible, subregular representations of the quantum group $\mathcal{U}_\varepsilon(sl(n))$ at a primitive, ℓ^{th}-root of unity ε, for $\ell = p^k$ with p prime and $k \in \mathbf{N}$. We show that every such a representation is induced from an irreducible $\mathcal{U}_\varepsilon(sl(2))$-module and prove the De Concini, Kac, Procesi conjecture about the dimension of the $\mathcal{U}_\varepsilon(\mathcal{G})$-modules.

0. Introduction

In this paper we consider the quantized enveloping algebra $\mathcal{U}_q(\mathcal{G})$ with $\mathcal{G} = sl(n+1)$ and its specialization at a primitive ℓ^{th}-root of unity ϵ. We suppose that ℓ is an odd integer strictly greater than 1, $\ell = p^k$ for a prime $p > n+1$ and k in $\mathbf{N}$.

We start from the parametrization of the irreducible $\mathcal{U}_\varepsilon(sl(n+1))$-modules through the conjugacy classes of the group $\mathrm{SL}(n+1)$ ([DC-K 1]) and make use of the De Concini–Kac theorem of reduction to the unipotent case ([DC-K 2]). We aim to classify the subregular modules (see Sect. 3) *i.e.* the modules parametrized by the conjugacy classes of dimension $2N-2$ (N is the number of positive roots).

Let us briefly describe the content of this paper: let us embed $\mathcal{U}_\varepsilon(sl(2))$ into $\mathcal{U}_\varepsilon(sl(n+1))$ through the map

$$j : \mathcal{U}_\varepsilon(sl(2)) \hookrightarrow \mathcal{U}_\varepsilon(sl(n+1))$$

$j(E) = E_n,\ j(F) = F_n,\ j(K^{\pm1}) = K_n^{\pm1}$ and let V be an irreducible $\mathcal{U}_\varepsilon(sl(2))$-module of dimension $\lambda_n + 1$ for some integer λ_n, $0 \le \lambda_n \le \ell - 1$, such that $E_n^\ell = 0 = F_n^\ell$. Then it is known that there exists a vector $v \in V \setminus \{0\}$ such that $K_n v = \varepsilon^{\lambda_n} v$, $E_n v = 0$, $F_n^{\lambda_n+1} v = 0$ and the set $\{v, F_n v, \ldots, F_n^{\lambda_n} v\}$ is a basis of V. We consider the subalgebra $\tilde{\mathcal{U}}$ of $\mathcal{U}_\varepsilon(sl(n+1))$ with generators $E_i,\ K_i$ for $i = 1, \ldots, n$, $F_n,\ F_\alpha^\ell$ for every positive root α, and extend the action of $\mathcal{U}_\varepsilon(sl(2))$ on V to an action of $\tilde{\mathcal{U}}$ as follows (Sect. 3.1):

– $E_i = 0$ for every $i = 1, \ldots, n-1$;
– $K_i F_n^r v = \varepsilon^{\lambda_i} F_n^r v$ for every $0 \le r \le \lambda_n$, $i \le n-2$, for some $\lambda_i \in \mathbf{Z}$, $0 \le \lambda_i \le \ell - 1$;
– $K_{n-1} F_n^r v = \varepsilon^{\lambda_{n-1}+r} F_n^r v$ $(0 \le r \le \lambda_n)$ for some $\lambda_{n-1} \in \mathbf{Z}$, $0 \le \lambda_{n-1} \le \ell - 1$;
– $F_\alpha^\ell = 1$ for every $\alpha \neq \alpha_n$, $F_n^\ell = 0$.

We then consider the induced subregular module

$$S_{\underline{\lambda}} = \mathcal{U}_\varepsilon(sl(n+1)) \otimes_{\tilde{\mathcal{U}}} V,$$

where $\underline{\lambda} = (\lambda_1, \ldots, \lambda_n)$.

The main idea of the paper consists in introducing the notion of nice weight (Definition 3.1) and proving that if $\underline{\lambda}$ is nice then $S_{\underline{\lambda}}$ is an irreducible $\mathcal{U}_\varepsilon(sl(n+1))$-module. Furthermore we will show that any finite-dimensional, irreducible, subregular $\mathcal{U}_\varepsilon(sl(n+1))$-module is isomorphic to $S_{\underline{\lambda}}$ for some nice $\underline{\lambda}$ and that $S_{\underline{\lambda}}$ is isomorphic to $S_{\underline{\mu}}$ if and only if there exists an element w, lying in the subgroup of the Weyl group $\mathcal{W}$ generated by the reflections $s_1, s_2, \ldots, s_{n-2}$, such that $\underline{\lambda} = w \cdot \underline{\mu}$, where "$\cdot$" is the "dot" action of $\mathcal{W}$ on the set of weights.

In Sect. 1 we recall the basic definitions and the main results due to De Concini, Kac and Procesi. We describe the center of $\mathcal{U}_\varepsilon$ and recall its main properties.

Section 2 is dedicated to the regular representations *i.e.* those parametrized by the conjugacy classes of maximal dimension $(2N)$. In [DC-K-P 1] it is shown that the dimension of a regular representation is ℓ^N. These modules play a fundamental role in our classification, thanks to Lemma 2.10. We want to stress the fact that this is the only point in which the hypothesis $\ell = p^k$ is used. If this lemma were generalized to any odd integer ℓ, the whole classification would work in the same way.

In Sect. 3 we finally introduce and study the subregular representations. We analyze the main properties of the nice weights and prove the main theorems. We point out that our results demonstrate the De Concini–Kac-Procesi conjecture about the dimension of the $\mathcal{U}_\varepsilon(\mathcal{G})$-modules ([DC-K-P 2]) in the case of the $\mathcal{U}_\varepsilon(sl(n))$-subregular modules.

1. The General Setting

1.1. Let $(a_{ij})_{i,j=1,..,n}$ be the Cartan matrix associated to the Lie algebra $sl(n+1)$. Fix a system of simple roots $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and the corresponding sets of roots and positive roots R and R^+, respectively, with $N = |R^+|$. By Q and P we shall denote the root and weight lattice respectively and we shall consider the basis ω_i of P such that

$$\alpha_j = \sum_{i=1}^{n} a_{ij}\omega_i \quad (j = 1, \ldots, n).$$

Define a bilinear pairing: $P \times Q \longrightarrow \mathbf{Z}$ by:

$$(\omega_i, \alpha_j) = \delta_{ij}.$$

We introduce the following element:

$$\rho = \sum_{i=1}^{n} \omega_i.$$

Let $\mathcal{W} = S_{n+1}$ be the Weyl group associated to $sl(n+1)$ with generating reflections s_i for $i = 1, \ldots, n$ and let $w_0 \in \mathcal{W}$ be the longest element in $\mathcal{W}$ so that the length of w_0 is N.

By $\mathcal{W}_{k,\ldots,t}$, $1 \le k \le t \le n$, we shall denote the subgroup of $\mathcal{W}$ generated by the reflections $s_k, \ldots, s_t$. In the same way by $R_{k,\ldots,t}$ and $R^+_{k,\ldots,t}$ we will denote the sets of roots and positive roots associated to the copy of $sl(t-K+2)$ whose Chevalley generators are those of $sl(n+1)$ corresponding to the simple roots $\alpha_k, \ldots, \alpha_t$.

Lemma 1.1. *Given any element w in $\mathcal{W}$ either the reflection s_1 does not appear in any reduced expression of w or we can write*

$$w = s_{i_1} \ldots s_{i_r} s_1 s_2 \ldots s_t$$

with $2 \le i_h \le n$ for every $h = 1, \ldots, r$ and $1 \le t \le n$.

Proof. Take an element $w \in \mathcal{W}$ and any of its reduced expressions : $w = s_{i_1} \ldots s_{i_k}$. What we need to prove is that we can find a reduced expression of w in which s_1 appears at most once.

Let us proceed by induction on n, the case $n = 1$ being trivial.

Now, if $i_h \neq 1$ for every $h = 1, \ldots, k$ then we succeeded. Otherwise suppose that s_1 appears more than once, then, by the inductive hypotheses, we may assume that only one s_2 appears in the expression of w between one s_1 and the following. Thus, since $s_1 s_j = s_j s_1$ for every $j > 2$, we may isolate the product $s_1 s_2 s_1$ and replace it with $s_2 s_1 s_2$. We obtain the result by repeating this procedure till we are left with only one s_1. □

We fix an odd integer $\ell > 1$ and a primitive ℓ^{th} root of unity ε.

Definition 1.2. *The quantized enveloping algebra $\mathcal{U}_\varepsilon \equiv \mathcal{U}_\varepsilon(sl(n+1))$ associated to (a_{ij}) at the root ε of unity is the associative algebra over* $\mathbf{C}$ *with generators E_i, F_i, K_i, K_i^{-1}, $i = 1, .., n$, and relations:*

$$\begin{gathered}
K_i K_j = K_j K_i = K_{i+j}, \quad K_i K_i^{-1} = 1, \\
K_i E_j = \varepsilon^{a_{ij}} E_j K_i, \quad K_i F_j = \varepsilon^{-a_{ij}} F_j K_i, \\
E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{\varepsilon - \varepsilon^{-1}}, \\
E_i^2 E_j - (\varepsilon + \varepsilon^{-1}) E_i E_j E_i + E_j E_i^2 = 0 \quad \textit{for } j = i \pm 1, \\
F_i^2 F_j - (\varepsilon + \varepsilon^{-1}) F_i F_j F_i + F_j F_i^2 = 0 \quad \textit{for } j = i \pm 1, \\
E_i E_j = E_j E_i, \quad F_i F_j = F_j F_i \quad \textit{for } j \neq i \pm 1.
\end{gathered}$$

By $T_{\mathcal{W}}$ we denote the Braid group associated to $\mathcal{W}$, with generators T_i for $i = 1, \ldots, n$. We recall that $T_{\mathcal{W}}$ acts on $\mathcal{U}_\varepsilon$ as follows ([L2]):

$$\begin{gathered}
T_i K_j = K_{s_i(\alpha_j)}, \\
T_i E_i = -F_i K_i \quad T_i E_j = -E_i E_j + \varepsilon^{-1} E_j E_i, \quad \text{if } j = i \pm 1, \\
T_i F_i = -K_i^{-1} E_i, \quad T_i F_j = -F_j F_i + \varepsilon F_i F_j \quad \text{if } j = i \pm 1, \\
T_i E_j = E_j, \quad T_i F_j = F_j \quad \text{if } j \neq i \pm 1.
\end{gathered}$$

For every fixed reduced expression of w_0, $w_0 = s_{i_1} \dots s_{i_N}$, a total convex ordering in R^+ can be defined by putting

$$\beta_j = s_{i_1} \dots s_{i_{j-1}}(\alpha_{i_j}) \quad j = 1, .., N.$$

Introduce the following vectors:

$$E_{\beta_j} = T_{i_1} \dots T_{i_{j-1}}(E_{i_j}), \quad F_{\beta_j} = T_{i_1} \dots T_{i_{j-1}}(F_{i_j})$$

and, for every $k = (k_1, .., k_N) \in \mathbf{Z}_+^N$, put

$$E^k = E_{\beta_1}^{k_1} \dots E_{\beta_N}^{k_N}, \quad F^k = F_{\beta_N}^{k_N} \dots F_{\beta_1}^{k_1}.$$

1.2. Let Z_ε be the center of the algebra $\mathcal{U}_\varepsilon$. Then the following lemma holds:

Lemma 1.3 ([DC-K 1]**).** *The elements E_α^ℓ, F_α^ℓ $(\alpha \in R^+)$ and K_j^ℓ $(j = 1, .., n)$ lie in Z_ε.*

By Z_0 we shall denote the subalgebra of Z_ε generated by the elements E_α^ℓ, F_α^ℓ and K_j^ℓ with $\alpha \in R^+$ and $j = 1, .., n$. Then $Z_0 = Z_0^- \otimes Z_0^0 \otimes Z_0^+$, where Z_0^-, Z_0^0 and Z_0^+ are the subalgebras of Z_0 generated by F_α^ℓ, K_j^ℓ, E_α^ℓ, respectively, with $\alpha \in R^+$, $j = 1, .., n$.

Let us introduce the following sets:

$$\mathrm{Par}_\ell = \{k \in \mathbf{Z}_+^N \ : \ 0 \le k_i < \ell, \ i = 1, \dots, N\};$$
$$\mathrm{Par}_\ell(\eta) = \{k \in \mathrm{Par}_\ell \ : \ \sum_i k_i \beta_i = \eta\} \quad \text{for } \eta \in Q.$$

Theorem 1.4 ([DC-K 1]**).** *$\mathcal{U}_\varepsilon$ is a free Z_0-module with a basis consisting of the vectors $\{F^k K_1^{m_1} \dots K_n^{m_n} E^r \,|\, k, r \in \mathrm{Par}_\ell, \ 0 \le m_j < \ell\}$.*

If z is a central element in $\mathcal{U}_\varepsilon$ and $\mathcal{U}^0$ is the subalgebra of $\mathcal{U}_\varepsilon$ generated by the K_i's, then

$$z = \sum_{\eta \in Q^+} \sum_{k, r \in \mathrm{Par}_\ell(\eta)} F^k \varphi_{k,r} E^r,$$

where $\varphi_{k,r} \in \mathcal{U}^0$.

Definition 1.5. *We call Harish–Chandra homomorphism the homomorphism*

$$h \ : \ Z_\varepsilon \longrightarrow \mathcal{U}^0,$$
$$z \mapsto \varphi_{0,0}.$$

Any element $\varphi \in \mathcal{U}^0$ may be regarded as a $\mathbf{C}$-valued function on the weight lattice P in an obvious way: if $\varphi = \prod_{i=1}^n K_i^{t_i}$, $t_i \in \mathbf{Z}$, set

$$\varphi(\lambda) = \varepsilon^{\sum_i t_i (\alpha_i, \lambda)}$$

and extend it to $\mathcal{U}^0$ by linearity.

Using the common notation we shall say that a vector x in a $\mathcal{U}_\varepsilon(sl(n+1))$-module M is a *weight vector* if it is an eigenvector for the K_i's. If furthermore $E_i(x) = 0$ for every $i = 1, \dots, n$ then we shall call x a *highest weight vector*.

Remark. Let V be an $\mathcal{U}_\varepsilon(sl(n+1))$-Verma module. If v is a highest weight vector in V with weight $\underline{\lambda}$ and $z \in Z_\varepsilon$ then

$$z.v = \varphi_{0,0}(\underline{\lambda})v$$

and z acts on V as the scalar $\varphi_{0,0}(\underline{\lambda})$ since it commutes with F_α for every α.

We point out that if ϕ_η is the matrix with entries $(\varphi_{k,r})_{k,r\in \mathrm{Par}_\ell(\eta)}$, one can compute ϕ_η by induction on $\eta \in Q_+$, essentially by using the conditions $[z, \mathcal{U}_\varepsilon^+] = 0$. In order to explain this idea let us introduce a bit of terminology: let V_η be the linear span of the $F^k v$ with $k \in \mathrm{Par}_\ell(\eta)$, H the Hermitian form on the Verma module V (see [DC-K 1]), H_η its restriction to V_η and G_γ the matrix of the operator $\sum_{k,r\in\mathrm{Par}_\ell(\gamma)} F^k \varphi_{k,r} E^r$ on V_η with respect to the basis $F^s v$ with $s \in \mathrm{Par}_\ell(\eta)$. By the inductive hypothesis one knows G_γ for $\gamma < \eta$. Then

$$G_\eta = \phi_\eta H_\eta.$$

Since z acts on V_η as the scalar $\varphi_{0,0}(\underline{\lambda})$,

$$\phi_\eta H_\eta + \sum_{\gamma<\eta} G_\gamma = \varphi_{0,0} I. \tag{1.1}$$

Therefore, once we know $\varphi_{0,0}$, we can compute all the $\varphi_{k,r}$'s using (1.1).

Definition 1.6. *We define the following endomorphism of the algebra $\mathcal{U}^0$:*

$$\gamma : \mathcal{U}^0 \longrightarrow \mathcal{U}^0,$$
$$K_i \mapsto \varepsilon K_i.$$

Remark. For every $\lambda \in P$: $(\gamma\varphi)(\lambda) = \varphi(\lambda + \rho)$.

Given $\varphi \in \mathcal{U}^0$, by z_φ we shall denote the series $\sum_{\eta\in Q^+} \sum_{k,r\in\mathrm{Par}_\ell(\eta)} F^k \varphi_{k,r} E^r$ with $\varphi_{0,0} = \gamma(\varphi)$ constructed as above. Let Q_2^* be the group of homomorphisms of Q into the group $\{\pm 1\}$. Then Q_2^* acts on $\mathcal{U}^0$ as follows:

$$\delta K_\beta = \delta(\beta) K_\beta$$

($\delta \in Q_2^*$, $\beta \in Q$). Since $\mathcal{W}$ acts on P by: $s_i(\omega_j) = \omega_j - \delta_{ij}\alpha_i$, then $\mathcal{W} \ltimes Q_2^*$ acts on $\mathcal{U}^0$. Let $\tilde{\mathcal{W}}$ be the subgroup of $\mathcal{W} \ltimes Q_2^*$ generated by $\sigma \mathcal{W} \sigma^{-1}$ with $\sigma \in Q_2^*$.

Proposition 1.7. (a) *Given $\varphi_{0,0} \in \mathcal{U}^0$, all the coefficients $\varphi_{k,r}$ of the corresponding series z lie in $\mathcal{U}^0$ if and only if*

$$\gamma^{-1}(\varphi_{0,0}) \in (\mathcal{U}^0)^{\tilde{\mathcal{W}}} = \{\psi \in \mathcal{U}_\varepsilon^0 \mid w\psi = \psi,\ \forall w \in \tilde{\mathcal{W}}\}.$$

Furthermore we have:

$$\deg \varphi_{k,r} + \min(|k|, |m|) \le \deg \varphi_{0,0}.$$

(b) *The map*

$$\varphi \mapsto z_\varphi$$

is an injective homomorphism: $(\mathcal{U}^0)^{\tilde{\mathcal{W}}} \longrightarrow Z_\varepsilon$.

Proof. See [DC-K 1]. □

1.3. For every associative algebra A by Rep(A) we shall denote the set of the equivalence classes of the irreducible, finite dimensional representations of A.

Given a homomorphism $\chi : Z_0 \longrightarrow \mathbf{C}$ we put $\mathcal{U}_\chi := \mathcal{U}_\varepsilon/(z - \chi(z),\ z \in Z_0)$.

Notice that if χ is the central character of a representation $\sigma \in \mathrm{Rep}(\mathcal{U}_\varepsilon)$ then σ is in fact a representation of the algebra $\mathcal{U}_\chi$.

Theorem 1.8 ([DC-K-P 1]**).** *There exists a map*

$$\varphi : \mathrm{Rep}(\mathcal{U}_\varepsilon(sl(n+1))) \longrightarrow \mathrm{SL}(n+1)$$

such that:

a) $\mathrm{Im}\,\varphi$ *is the big cell of* $\mathrm{SL}(n+1)$*;*
b) *if* $x, y \in \mathrm{SL}(n+1)$ *are conjugated elements in* $\mathrm{SL}(n+1)$ *then the representations in* $\varphi^{-1}(x)$ *are isomorphic (up to a twist) to those in* $\varphi^{-1}(y)$ *through the elements of a group* $\tilde{G}$ *of automorphisms of* $\mathcal{U}_\varepsilon$*.*

Definition 1.9. *We say that* $\sigma \in \mathrm{Rep}(\mathcal{U}_\varepsilon)$ *is unipotent if* $\varphi(\sigma)$ *is a unipotent element of* $\mathrm{SL}(n)$.

Now take a non-unipotent element σ in $\mathrm{Rep}(\mathcal{U}_\varepsilon)$ with central character χ, such that $g = \varphi(\sigma)$. Then we may suppose, up to the action of $\tilde{G}$, that if $g = g_s \cdot g_u$ is the Jordan decomposition of $g \in \mathrm{SL}(n+1)$ then $g_s \neq 1$, g_u is a lower triangular matrix and if we define T' as the center of the centralizer of g_s in $\mathrm{SL}(n+1)$ and h' as the Lie algebra of T' then h' is a proper subalgebra of the Cartan subalgebra h of $sl(n+1)$.

Let $R' := \{\alpha \in R \mid \alpha \text{ vanishes on } h'\}$. Then

$$R' = Q' \cap R,$$

where $Q' = \mathbf{Z}\Delta'$ is a sublattice of Q spanned by a proper subset Δ' of Δ.

Let $\mathcal{U}'$ be the subalgebra of $\mathcal{U}_\varepsilon$ generated by the K_j's with $j = 1, \ldots, n$ and by the elements E_i, F_i such that $\alpha_i \in \Delta'$ and let $\mathcal{U}^+$ be the subalgebra of $\mathcal{U}_\varepsilon$ generated by the E_i's and K_i's with $i = 1, \ldots, n$. We put $\tilde{\mathcal{U}} = \mathcal{U}'\mathcal{U}^+$, $\mathcal{U}'_\chi := \mathcal{U}'/(z - \chi(z),\ z \in Z_0 \cap \mathcal{U}')$ and $\tilde{\mathcal{U}}_\chi := \tilde{\mathcal{U}}/(z - \chi(z),\ z \in Z_0 \cap \tilde{\mathcal{U}})$.

Theorem 1.10 ([DC-K 2]**).** *If* $\sigma \in \mathrm{Rep}(\mathcal{U}_\varepsilon)$ *is a non unipotent representation of* $\mathcal{U}_\varepsilon(sl(n+1))$ *on a vector space V, with central character* χ*, then the* $\mathcal{U}_\chi$*-module V contains a unique irreducible* $\tilde{\mathcal{U}}_\chi$*-submodule V′ such that:*

1) V' *is a* $\mathcal{U}'_\chi$*-module;*
2) $V = \mathcal{U}_\chi \otimes_{\tilde{\mathcal{U}}_\chi} V'$*;*
3) *the map* $V \longrightarrow V'$ *establishes a bijection between* Rep $(\mathcal{U}_\chi)$ *and* (Rep $\mathcal{U}'_\chi$)*.*

Remark. We point out that if m is the cardinality of Δ', then Δ' is the set of simple roots of the Lie algebra $sl(m+1)$ having as Chevalley generators those of our copy of $sl(n+1)$ which correspond to $\alpha_i \in \Delta'$. If we restrict the representation of $\mathcal{U}'_\chi$ in V′ to the subalgebra of $\mathcal{U}'_\chi$ generated by the elements E_i, F_i, K_i such that $\alpha_i \in \Delta'$, we in fact find a unipotent representation of $\mathcal{U}_\varepsilon(sl(m+1))$.

Therefore Theorem 1.10 reduces the classification of the $\mathcal{U}_\varepsilon(sl(n))$-modules to the analysis of the unipotent ones.

2. Regular Representations

2.1. According to the accepted terminology we shall say that an element τ in Rep($\mathcal{U}_\varepsilon$) is a *regular representation* if $\varphi(\tau)$ belongs to a conjugacy class of maximal dimension (2N) in SL($n+1$). Using the De Concini–Kac theorem of reduction to the unipotent case we can focus our attention on the unipotent modules and therefore consider the representations lying over the unipotent conjugacy class of $SL(n+1)$ of dimension $2N$. Notice that the $n \times n$ matrix

$$u = \begin{pmatrix} 1 & & & 0 \\ 1 & 1 & & \\ \vdots & \vdots & \ddots & \\ 1 & 1 & \dots & 1 \end{pmatrix}$$

belongs to this conjugacy class, therefore, according to Theorem 1.8, we take a representation $\tau \in$ Rep($\mathcal{U}_\varepsilon$) such that $\varphi(\tau) = u$, satisfying the following conditions:

$$\tau(K_j^\ell) = 1 \quad \text{for every } j = 1, \dots, n;$$
$$\tau(F_\alpha^\ell) = 1, \quad \tau(E_\alpha^\ell) = 0 \quad \text{for every } \alpha \in R^+.$$

In what follows we shall often identify the elements x in $\mathcal{U}_\varepsilon(sl(n+1))$ with their images $\tau(x)$.

Theorem 2.1. *The irreducible $\mathcal{U}_\varepsilon$-modules parametrized by the conjugacy class of maximal dimension (2N) in SL(n+1) have dimension equal to ℓ^N.*

Proof. See ([DC-K-P 2]). □

Using Theorem 2.1 we can give an explicit construction of the irreducible regular modules: fix a reduced expression of w_0, $w_0 = s_{i_1} \dots s_{i_N}$, and the corresponding ordering of the positive roots, $\beta_1, \dots, \beta_N$. Then the following proposition holds:

Proposition 2.2. *Let W be an irreducible regular $\mathcal{U}_\varepsilon(sl(n+1))$-module. Then the set*

$$\mathcal{B} = \{F_{\beta_N}^{h_N} \dots F_{\beta_1}^{h_1} v \mid 0 \le h_i \le \ell - 1\},$$

where v is a highest weight vector, is a basis of W.

Proof. A highest weight vector v in W exists since the subalgebra of $\mathcal{U}_\varepsilon(sl(n+1))$ generated by the E_i's acts nilpotently on W. Therefore

$$\mathcal{B} = \{F_{\beta_N}^{h_N} \dots F_{\beta_1}^{h_1} v \mid 0 \le h_i \le \ell - 1\} \tag{2.2}$$

is a set of generators of W since W is irreducible and therefore generated by v as a $\mathcal{U}_\varepsilon(sl(n+1))$-module. Since the cardinality of $\mathcal{B}$ is ℓ^N, Theorem 2.1 implies that $\mathcal{B}$ is a basis of W. □

Taking into account Proposition 2.2, if λ is the weight of v we shall denote the module W by W_λ.

Now we want to establish when two regular modules W_λ and W_μ are isomorphic.

As we recalled in 1.2, the Weyl group $\mathcal{W}$ acts on the set of weights as follows: $s_i(w_j) = w_j - \delta_{ij}\alpha_i$.

Definition 2.3. *The action of $\mathcal{W}$ on the set of weights defined by*

$$w \cdot \lambda = w(\lambda + \rho) - \rho$$

with $w \in \mathcal{W}$, $\lambda \in P$, is called the "dot" action.

Definition 2.4. *Two weights λ and μ are said to be linked if there exists an element $w \in \mathcal{W}$ such that $w \cdot \lambda = \mu$.*

Proposition 2.5. *Let M be a $\mathcal{U}_\varepsilon(sl(n+1))$-Verma module. If v and w are highest weight vectors in M then the weight $\underline{\lambda}$ of v is linked to the weight $\underline{\mu}$ of w (linkage principle).*

Proof. Let $z = \sum_{\eta \in Q^+} \sum_{k,r \in \mathrm{Par}_\ell(\eta)} F^k \varphi_{k,r} E^r$ be a central element in $\mathcal{U}_\varepsilon(sl(n+1))$. Then $\varphi_{0,0} v = \alpha v$ for some $\alpha \in \mathbf{C}$ $(\alpha = \varphi_{0,0}(\underline{\lambda}))$ and for every x in M $z.x = \alpha x$ since M is a Verma module. In particular $\varphi_{0,0}(\underline{\lambda}) w = zw = \varphi_{0,0}(\underline{\mu}) w$. We conclude by using the injective homomorphism $(\mathcal{U}^0)^{\tilde{\mathcal{W}}} \longrightarrow Z_\varepsilon$ introduced in 1.2. □

Lemma 2.6. *Let $W_{\underline{\lambda}}$ be a regular $\mathcal{U}_\varepsilon(sl(n+1))$-module. For every highest weight $\underline{\mu}$ the highest weight vector of weight $\underline{\mu}$ is unique up to scalar multiples.*

Proof. Let v, w be highest weight vectors in $W_{\underline{\lambda}}$ of the same weight and consider the bases:

$$\mathcal{B}_1 = \{F_{\beta_N}^{h_N} \dots F_{\beta_1}^{h_1} v \mid 0 \le h_i \le \ell - 1\}$$

and

$$\mathcal{B}_2 = \{F_{\beta_N}^{h_N} \dots F_{\beta_1}^{h_1} w \mid 0 \le h_i \le \ell - 1\}$$

of $W_{\underline{\lambda}}$. We define an endomorphism ψ of $W_{\underline{\lambda}}$ setting

$$\psi(F_{\beta_N}^{h_N} \dots F_{\beta_1}^{h_1} v) = F_{\beta_N}^{h_N} \dots F_{\beta_1}^{h_1} w.$$

Then, by Schur's lemma, $\psi = aI$ for some $a \in \mathbf{C}^\times$. In particular $w = av$. □

We shall now exhibit the unique, up to scalars, highest weight vector in $W_{\underline{\lambda}}$ of fixed weight $w \cdot \underline{\lambda}$ $(w \in \mathcal{W})$. Let $W_{\underline{\lambda}}$ be as above, $\underline{\lambda} = (\lambda_1, \dots, \lambda_n)$, and let w be an element in the Weyl group $\mathcal{W} = S_{n+1}$. Take a reduced expression of w, $w = s_{i_k} \dots s_{i_1}$, and associate to it the following vector in $W_{\underline{\lambda}}$:

$$y_w := F_{i_k}^{A_{i_k}} \dots F_{i_1}^{A_{i_1}} v,$$

where $0 \le A_{i_h} \le \ell - 1$ for every $h = 1, \dots, k$,

$$A_{i_1} \equiv \lambda_{i_1} + 1 \pmod{\ell}$$

and, recursively,

$$A_{i_h} \equiv -\sum_{\nu=1}^{h-1} a_{i_h i_\nu} A_{i_\nu} + \lambda_{i_h} + 1 \pmod{\ell}$$

$(2 \le h \le k)$.

Proposition 2.7. *y_w is the unique (up to scalars) highest weight vector in $W_{\underline{\lambda}}$ of weight $\underline{\mu} = w \cdot \underline{\lambda}$.*

Proof. First we notice that if v is a highest weight vector of weight $\underline{\lambda} = (\lambda_1, \dots, \lambda_n)$ then $F_i^{\lambda_i+1} v$ is a highest weight vector, for every $i = 1, \dots, n$. Therefore, since by definition $y_w = F_{i_k}^{1+\lambda_{i_k}^z} z$ where z is a highest weight vector and $\underline{\lambda}^z = (\lambda_1^z, \dots, \lambda_n^z)$ is the weight of z, $E_i(y_w) = 0$ for every i.

Now let us compute the weight $\underline{\mu}$ of y_w: if we proceed by induction on the length of w then it is sufficient to consider the case of a simple reflection $w = s_i$.

The vector associated to w is then $y_w = F_i^{\lambda_i+1} v$ with weight $\underline{\mu} = (\lambda_1, \dots, \lambda_{i-2}, \lambda_{i-1} + \lambda_i + 1, -\lambda_i - 2, 1 + \lambda_i + \lambda_{i+1}, \lambda_{i+2}, \dots, \lambda_n)$.

Let us now calculate $s_i(\underline{\lambda} + \rho) - \rho$: we write s_i with respect to the basis ω_i introduced in Sect. 1:

$$s_i = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & 0 & \\ & & 1 & 1 & & & \\ & & & -1 & & & \\ & & & 1 & 1 & & \\ & 0 & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}$$

therefore

$$s_i(\underline{\lambda} + \rho) - \rho = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_{i-2} \\ \lambda_{i-1} + \lambda_i + 1 \\ -\lambda_i - 2 \\ \lambda_{i+1} + \lambda_i + 1 \\ \lambda_{i+2} \\ \vdots \\ \lambda_n \end{pmatrix}.$$

This concludes the proof. □

We are now able to classify the regular modules:

Proposition 2.8. *The regular irreducible $\mathcal{U}_\varepsilon(sl(n+1))$-modules are parametrized, up to isomorphisms, by the $\mathcal{W}$-orbits of weights with respect to the 'dot' action.*

Proof. Let us consider two regular modules W_λ and W_μ ($\lambda, \mu \in \mathcal{W}$) and their respective bases

$$\mathcal{B}_\lambda = \{F_{\beta_N}^{h_N} \dots F_{\beta_1}^{h_1} v \mid 0 \le h_1 \le \dots \le h_N \le \ell - 1\}$$

and

$$\mathcal{B}_\mu = \{F_{\beta_N}^{h_N} \dots F_{\beta_1}^{h_1} z \mid 0 \le h_1 \le \dots \le h_N \le \ell - 1\}.$$

If $W_\lambda \cong W_\mu$ then z is a highest weight vector in W_λ of weight μ, therefore μ is linked to λ (Proposition 2.5).

Conversely, if $\mu = w \cdot \lambda$ then W_λ contains the highest weight vector y_w of weight μ so that the set

$$\mathcal{B}_{w \cdot \lambda} = \{F_{\beta_N}^{h_N} \dots F_{\beta_1}^{h_1} y_w \mid 0 \le h_1 \le \dots \le h_N \le \ell - 1\}$$

is a basis of W_λ. Therefore the map

$$F_{\beta_N}^{h_N}\dots F_{\beta_1}^{h_1}y_w \longmapsto F_{\beta_N}^{h_N}\dots F_{\beta_1}^{h_1}z$$

is an isomorphism between W_λ and W_μ. □

From now on we shall fix the reduced expression of w_0:

$$w_0 = s_n\dots s_1 s_n\dots s_2 s_n\dots s_n,$$

and consider the corresponding basis (2.2).

Lemma 2.9. *Suppose* $(\ell, n+1) = 1$. *If the vectors* $x = F_1^{p_1^{(1)}}\dots F_{12\dots n}^{p_1^{(n)}}F_2^{p_2^{(2)}}\dots F_n^{p_n^{(n)}}v$ *and* $y = F_1^{P_1^{(1)}}\dots F_{12\dots n}^{P_1^{(n)}}F_2^{P_2^{(2)}}\dots F_n^{P_n^{(n)}}v$ *have the same weight* $\underline{\mu}$ *then, for every* $t = 1,\dots,n$,

$$\begin{aligned}\sum_{i\geq t}p_1^{(i)}+\sum_{i\geq t}p_2^{(i)}+\dots+\sum_{i\geq t}p_t^{(i)}&\\ \equiv \sum_{i\geq t}P_1^{(i)}+\sum_{i\geq t}P_2^{(i)}+\dots+\sum_{i\geq t}P_t^{(i)}&\quad (\mathrm{mod}\,\ell).\end{aligned}$$

Proof. We shall omit the proof of this lemma since it consists of a simple direct calculation. It is indeed sufficient to equal the weight of x and the weight of y and use condition $(\ell, n+1) = 1$. □

2.2. Let us take a central element $z \in Z_\varepsilon$:

$$z = \sum_{\eta\in Q^+}\sum_{k,r\in \mathrm{Par}_\ell(\eta)} F^k\varphi_{k,r}E^r$$

with $\varphi_{k,r} \in \mathcal{U}^0$.

In 1.2 we showed how to compute the elements $\varphi_{k,r}$ by using the recursive formula

$$\phi_\eta H_\eta + \sum_{\gamma<\eta} G_\gamma = \varphi_{0,0} I$$

($\eta \in Q^+ := \sum_i \mathbf{Z}_+\alpha_i$).

Now we are interested in the elements $\varphi_{k,r}$ corresponding to $F^k = F_n^{k_n}$, $E^r = E_n^{k_n}$. Put

$$\bar{z} = \varphi_{0,0} + F_n\alpha_1 E_n + F_n^2\alpha_2 E_n^2 + F_n^3\alpha_3 E_n^3 + \dots,$$

and let us calculate $\alpha_1,\ \alpha_2,\ \alpha_3,\dots$ for

$$\begin{aligned}\varphi_{0,0} &= 1/(\varepsilon-\varepsilon^{-1})^4\Big(\sum_{\alpha\in R^+}K_\alpha^2\varepsilon^{2(\alpha,\rho)}+\sum_{\alpha\in R^+}K_\alpha^{-2}\varepsilon^{-2(\alpha,\rho)}\Big)\\ &= 1/(\varepsilon-\varepsilon^{-1})^4\Big(\sum_{t=1}^{n}\sum_{j=1}^{t}\varepsilon^{2(t-j+1)}K_j^2\dots K_t^2\\ &\qquad +\sum_{t=1}^{n}\sum_{j=1}^{t}\varepsilon^{-2(t-j+1)}K_j^{-2}\dots K_t^{-2}\Big).\end{aligned}$$

Without going into the details of these computations we point out what follows:

$$\alpha_1 = 1/(\varepsilon-\varepsilon^{-1})^2\Bigg(\sum_{j=1}^{n-1}\varepsilon^{2n-2j+1}K_j^2\dots K_{n-1}^2K_n \\ +\sum_{j=1}^{n-1}\varepsilon^{-2n+2j-1}K_j^{-2}\dots K_{n-1}^{-2}K_n^{-1}+(\varepsilon+\varepsilon^{-1})(K_n+K_n^{-1})\Bigg)$$

and

$$\alpha_2 = 1.$$

Therefore $\alpha_t = 0$ for every $t \geq 3$.

Before stating the following lemma let us introduce a bit of terminology: given a regular $\mathcal{U}_\varepsilon(sl(n+1))$-module V with the basis $\mathcal{B}$ associated to the reduced expression of w_0: $w_0 = s_n \dots s_1 s_n \dots s_2 s_n \dots s_n$, by $V_{k\dots t}$ $(1 \leq k \leq t \leq n)$ we shall denote the subspace of V generated by the vectors

$$\left\{F_k^{p_k^{(k)}}F_{kk+1}^{p_k^{(k+1)}}\dots F_{k\dots t}^{p_k^{(t)}}F_{k+1}^{p_{k+1}^{(k+1)}}\dots F_t^{p_t^{(t)}}v \mid 0 \leq p_i^{(j)} \leq \ell-1,\ k \leq i \leq j \leq t\right\}.$$

Notice that for every k, t, $V_{k\dots t}$ is a regular $\mathcal{U}_\varepsilon(sl(t-k+2))$-module; in particular $V_{1\dots n} = V$.

Lemma 2.10. *Let ε be a ℓ^{th}-root of unity with $\ell = p^k$ for some prime $p > n+1$ and $k \in \mathbf{N}$. Let V be a regular $\mathcal{U}_\varepsilon(sl(n+1))$-module, v a highest weight vector of weight $\underline{\lambda} = (\lambda_1, \dots, \lambda_n)$ and x a weight vector such that:*

$$E_i(x) = 0 \quad \textit{for every } i \neq n$$

and

$$E_n(x) = F_n^t v \quad \textit{for some } t,\ 0 \leq t \leq \ell - 1.$$

Then $t \neq \ell-1, \lambda_n$ and $x = \sigma F_n^{t+1}v$ with $\sigma = (1-\varepsilon^2)(\varepsilon-\varepsilon^{-1})/(1-\varepsilon^{2(t+1)})(\varepsilon^{-2t+\lambda_n} - \varepsilon^{-\lambda_n})$.

Proof. If $x = \sigma F_n^{t+1}v$, with $t \neq \lambda_n,\ \ell-1$ and σ is as in the statement, then a direct computation shows that $E_n(x) = F_n^t v$.

Suppose that $t \neq \lambda_n,\ \ell-1$ and that two vectors x and x' satisfy our hypotheses. Put $y := x - x'$. Then $E_i(y) = 0$ for every $i = 1, \dots, n$; therefore if $y \neq 0$, it is a highest weight vector and its weight is equal to $w \cdot \underline{\lambda}$ for some $w \in \mathcal{W}$. Hence we can choose a reduced expression of w in which s_1 appears at most once and, by Proposition 2.7, $y = ay_w$ for some $a \in \mathbf{C}^\times$. Then, since the weight of x is equal to the weight of $F_n^{t+1}v$, Lemma 2.9 implies that F_1 does not appear in the expression of y_w thus $y_w \in V_{2\dots n}$ and $w \in \mathcal{W}_{2,\dots,n}$. With the same argument we can conclude that $y \in V_n$ and is therefore a scalar multiple of $F_n^{t+1}v$ since they have the same weight. But this is a contradiction because $F_n^{t+1}v$ is not a highest weight vector when $t \neq \lambda_n,\ \ell-1$. Therefore $y = 0$.

Now suppose $t = \lambda_n$ and x as in the statement. Take the central element $\bar{z}$ explicitly constructed at the beginning of this section and let it act on x:

$$\bar{z}.x = \sum_{\eta\in Q^+}\sum_{k,r}F^k\varphi_{k,r}E^r x.$$

We recall that, if $\varphi_{0,0}.F_n^{\lambda_n+1}v = \lambda F_n^{\lambda_n+1}v$ for some $\lambda \in \mathbf{C}$ (or equivalently $\varphi_{0,0}.v = \lambda v$) then

$$\bar{z}.x = \lambda x$$

since V is a $\mathcal{U}_\varepsilon$-Verma module. In particular $\bar{z}.x = \varphi_{0,0}.x$ since the weight of x is $s_n \cdot \underline{\lambda}$.

We first want to evaluate $E^r x$: in our hypotheses

$$E_{i\ldots k}(x) = 0 \quad \forall k \neq n,$$
$$E_{i\ldots n}(x) = -E_{i\ldots n-1}E_n x + \varepsilon E_n E_{i\ldots n-1}x = 0.$$

Since $E_n(x) \neq 0$ we have:

$$\bar{z}.x = \varphi_{0,0}x + F_n\alpha_1 E_n x + F_n^2\alpha_2 E_n^2 x,$$

i.e.

$$0 = F_n\alpha_1 F_n^{\lambda_n} v + F_n^2\alpha_2(1-\varepsilon^{2\lambda_n})(\varepsilon^{2-\lambda_n} - \varepsilon^{-\lambda_n})/(1-\varepsilon^2)(\varepsilon-\varepsilon^{-1})F_n^{\lambda_n-1}v.$$

By direct computation we find:

$$\sum_{i=1}^{n-1}\varepsilon^{2n-2i+1+2\lambda_i+\cdots+2\lambda_{n-1}+\lambda_n} + \sum_{i=1}^{n-1}\varepsilon^{-(2n-2i+1+2\lambda_i+\cdots+2\lambda_{n-1}+\lambda_n)} + 2(\varepsilon^{1+\lambda_n} + \varepsilon^{-(1+\lambda_n)}) = 0.$$

Suppose that this relation holds and define the polynomial

$$P(x) = \sum_{i=1}^{n-1} x^{2n-2i+1+2\lambda_i+\cdots+2\lambda_{n-1}+\lambda_n} + \sum_{i=1}^{n-1} x^{[-(2n-2i+1+2\lambda_i+\cdots+2\lambda_{n-1}+\lambda_n)]} + 2(x^{1+\lambda_n} + x^{\ell-1-\lambda_n})$$

(by $[k]$, $k \in \mathbf{Z}$, we denoted the representative of the congruence class of k modulo ℓ, such that $0 \leq [k] \leq \ell - 1$).

Since ε is a root of $P(x)$ then

$$P(x) = \phi_\ell(x)R(x),$$

where $\phi_\ell(x)$ is the ℓ^{th}-cyclotomic polynomial in the variable x and $R(x) \in \mathbf{Z}[x]$. Now, since we assumed $\ell = p^k$ for a prime p such that $p > n+1$, if we evaluate $P(x)$ in $x = 1$ we have a contradiction since $2(n+1) = P(1) = pR(1)$ necessarily implies that $p|n+1$. If $t = \ell - 1$ the computations are exactly the same.

We conclude that V does not contain either any weight vector x such that $E_i(x) = 0\ \forall i \neq n$ and $E_n(x) = F_n^{\lambda_n}v$ or any weight vector such that $E_i(x) = 0\ \forall i \neq n$ and $E_n(x) = F_n^{\ell-1}v$. □

From now on, in order to apply Lemma 2.10, we shall suppose that ε is a primitive ℓ^{th} root of unity with ℓ odd, $\ell = p^k > 1$, for some prime $p > n+1$ and $k \in \mathbf{N}$.

3. Subregular Representations

We shall call subregular a representation τ of $\mathcal{U}_\varepsilon(sl(n+1))$ such that $\varphi(\tau)$ belongs to the unipotent conjugacy class $\mathcal{O}$ of dimension $2N-2$ in $\mathrm{SL}(n+1)$.

Let us choose the representative element m of the class $\mathcal{O}$:

$$m = \begin{pmatrix} 1 & & & & & & 0 \\ 1 & 1 & & & & & \\ \vdots & \vdots & \ddots & & & & \\ & & & & 1 & & \\ \vdots & \vdots & \ddots & & 1 & 1 & \\ 1 & \cdots & \cdots & & 1 & 0 & 1 \end{pmatrix},$$

and a representation $\sigma \in \mathrm{Rep}(\mathcal{U}_\varepsilon(sl(n+1)))$, such that $\varphi(\sigma) = m$, satisfying the following conditions:

$$\sigma(K_j^\ell) = 1 \quad \text{for every } j = 1, \ldots, n\,; \quad \sigma(E_\alpha^\ell) = 0 \quad \text{for every } \alpha \in R^+;$$
$$\sigma(F_\alpha^\ell) = 1 \quad \text{for every } \alpha \neq \alpha_n; \quad \sigma(F_n^\ell) = 0.$$

We fix the following reduced expression of the longest element w_0 in the Weyl group $\mathcal{W}$:

$$w_0 = s_1 \ldots s_n s_1 \ldots s_{n-1} s_1 \ldots s_1$$

and, according to what we recalled in Sect. 1, we shall consider the corresponding root vectors in $\mathcal{U}_\varepsilon(sl(n+1))$.

3.1. Let λ_n be an integer such that $0 \leq \lambda_n \leq \ell - 1$, and let us embed $\mathcal{U}_\varepsilon(sl(2))$ into $\mathcal{U}_\varepsilon(sl(n+1))$ through the map

$$j \,:\, \mathcal{U}_\varepsilon(sl(2)) \hookrightarrow \mathcal{U}_\varepsilon(sl(n+1)) \,:$$

$J(E) = E_n$, $j(F) = F_n$, $j(K^{\pm 1}) = K_n^{\pm 1}$. Let V be an irreducible $(\lambda_n + 1)$-dimensional representation of $\mathcal{U}_\varepsilon(sl(2))$ such that $E_n^\ell = 0 = F_n^\ell$. Then there exists (see [DC-K 1]) a non zero vector $v \in V$ such that $E_n v = 0$, $K_n v = \varepsilon^{\lambda_n} v$, $F_n^{\lambda_n+1} v = 0$ and the set $\{v,\ F_n v, \ldots, F_n^{\lambda_n} v\}$ is a basis of V.

Let $\tilde{\mathcal{U}}$ be the subalgebra of $\mathcal{U}_\varepsilon(sl(n+1))$ generated by $F_n,\ E_n,\ K_j$ with $j = 1, \ldots, n$, F_α^ℓ with $\alpha \in R^+$, E_i with $i = 1, \ldots, n-1$, and extend the action of $\mathcal{U}_\varepsilon(sl(2))$ to an action of $\tilde{\mathcal{U}}$ on V as follows:

- $E_i F_n^r v = 0 \quad \forall i = 1, \ldots, n-1,\ (r = 0, \ldots, \lambda_n)$;
- $F_\alpha^\ell \equiv 1 \quad \forall \alpha \in R^+,\ \alpha \neq \alpha_n,\quad F_n^\ell = 0$;
- $K_{n-1} F_n^r v = \varepsilon^{r+\lambda_{n-1}} F_n^r v\,;\ K_i F_n^r v = \varepsilon^{\lambda_i} F_n^r v$ if $i \leq n-2$, $(r = 0, \ldots, \lambda_n)$, for some fixed integers λ_i such that $0 \leq \lambda_i \leq \ell - 1$ $(i = 1, \ldots, n-1)$.

Then V becomes a left $\tilde{\mathcal{U}}$-module with respect to this action and we can consider the induced representation

$$S_{\underline{\lambda}} := \mathcal{U}_\varepsilon(sl(n+1)) \otimes_{\tilde{\mathcal{U}}} V, \tag{3.4}$$

where $\underline{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$.

Notice that the dimension of $S_{\underline{\lambda}}$ is $(\lambda_n + 1)\ell^{N-1}$.

Indeed, by definition and according to the choice of the reduced expression of w_0, the set

$$\begin{aligned}\mathcal{B} = \{F_1^{p_1^{(1)}} \ldots F_{12\ldots n}^{p_1^{(n)}} F_2^{p_2^{(2)}} \ldots F_{2\ldots n}^{p_2^{(n)}} \ldots F_{n-1}^{p_{n-1}^{(n-1)}} F_{n-1n}^{p_{n-1}^{(n)}} F_n^{p_n^{(n)}} v :\\ 0 \le p_j^{(t)} \le \ell - 1,\ \forall\ j = 1, \ldots, n-1\ \forall\ t = j, \ldots, n,\ 0 \le p_n^{(n)} \le \lambda_n\}\end{aligned} \tag{3.5}$$

is a basis of $S_{\underline{\lambda}}$.

Remark. By $V_{12\ldots k}$ we shall denote the subspace of $S_{\underline{\lambda}}$ generated by the elements $\{F_1^{p_1^{(1)}} F_{12}^{p_1^{(2)}} \ldots F_{1\ldots k}^{p_1^{(k)}} F_2^{p_2^{(2)}} \ldots F_{k-1}^{p_{k-1}^{(k-1)}} F_{k-1k}^{p_{k-1}^{(k)}} F_k^{p_k^{(k)}} v\}$; $V_{1\ldots k}$ is then a regular $\mathcal{U}_\varepsilon(sl(k+1))$-module for every $k < n$. In particular we shall call $V_{12\ldots n-1}$ the regular part of $S_{\underline{\lambda}}$.

In the same way let $V_{k\ldots n}$ be the subspace of $S_{\underline{\lambda}}$ generated by the set $\{F_k^{p_k^{(k)}} F_{kk+1}^{p_k^{(k+1)}} \ldots \ldots F_{k\ldots n}^{p_k^{(n)}} F_{k+1}^{p_{k+1}^{(k+1)}} \ldots F_{n-1}^{p_{n-1}^{(n-1)}} F_{n-1n}^{p_{n-1}^{(n)}} F_n^{p_n^{(n)}} v\}$, then $V_{k\ldots n}$ is the subregular induced $\mathcal{U}_\varepsilon(sl(n-k+2))$-module $S_{(\lambda_k,\ldots,\lambda_n)}$.

3.2. As in 2.2, for every $k \in \mathbf{Z}$, by $[k]$ we denote the integer such that

$$0 \le [k] \le \ell - 1,$$
$$[k] \equiv k \pmod{\ell}.$$

Definition 3.1. *For every positive integer t, $1 \le t \le n-1$, we shall say that a weight $\underline{\lambda} = (\lambda_1, \ldots, \lambda_n)$, $0 \le \lambda_i \le \ell - 1$, is (t)-nice if either*

$$n - t + 1 + \lambda_t + \cdots + \lambda_n \equiv 0 \quad (mod\ \ell)$$

or

$$[n - t + 1 + \lambda_t + \cdots + \lambda_n] > \lambda_n.$$

Furthermore we shall call $\underline{\lambda}$ nice if it is (t)-nice for every $t = 1, \ldots, n-1$.

Remark. It is easy to see that if $n = 2$, Definition 3.1 can be restated as follows: a weight $\underline{\lambda} = (\lambda_1, \lambda_2)$ $(0 \le \lambda_1\, \lambda_2 \le \ell - 1)$ is nice if either

$$2 + \lambda_1 + \lambda_2 \le \ell$$

or

$$\lambda_1 = \ell - 1.$$

Indeed, if $2 + \lambda_1 + \lambda_2 \le \ell - 1$ then $[2 + \lambda_1 + \lambda_2] = 2 + \lambda_1 + \lambda_2 > \lambda_2$. If, instead, $\ell + 1 \le 2 + \lambda_1 + \lambda_2 \le 2\ell - 1$, then $[2 + \lambda_1 + \lambda_2] = 2 + \lambda_1 + \lambda_2 - \ell$ and $2 + \lambda_1 + \lambda_2 - \ell > \lambda_2$ if and only if $\lambda_1 = \ell - 1$.

Finally, $2 + \lambda_1 + \lambda_2 \equiv 0 \pmod{\ell}$ if and only if either $2 + \lambda_1 + \lambda_2 = \ell$, or $\lambda_1 = \lambda_2 = \ell - 1$.

We now want to compare the nice-conditions on a weight $\underline{\lambda}$ with those on $s_i \cdot \underline{\lambda}$ for any reflection s_i in $\mathcal{W}$. We establish the following lemma:

Lemma 3.2. *Consider a weight* $\underline{\lambda} = (\lambda_1, \ldots, \lambda_n)$, $0 \leq \lambda_j \leq \ell - 1$, *and a reflection* s_i *with* $1 \leq i \leq n-2$. *Suppose that* $\lambda_i \neq \ell - 1$, *then:*

1) *for every* $t \geq i+2$ $\underline{\lambda}$ *is* (t)*-nice if and only if* $s_i \cdot \underline{\lambda}$ *is* (t)*-nice;*
2) $\underline{\lambda}$ *is* $(i+1)$*-nice if and only if* $s_i \cdot \underline{\lambda}$ *is* (i)*-nice;*
3) $\underline{\lambda}$ *is* (i)*-nice if and only if* $s_i \cdot \underline{\lambda}$ *is* $(i+1)$*-nice;*
4) *for every* $t \leq i-1$ $\underline{\lambda}$ *is* (t)*-nice if and only if* $s_i \cdot \underline{\lambda}$ *is* (t)*-nice.*

Proof. The statement follows by direct computation once we have noticed that

$$s_i \cdot \underline{\lambda} = (\lambda_1, \ldots, \lambda_{i-2}, [1 + \lambda_i + \lambda_{i-1}], \ell - \lambda_i - 2, [1 + \lambda_i + \lambda_{i+1}], \lambda_{i+2}, \ldots, \lambda_n). \quad \square$$

Corollary 3.3. *A weight* $\underline{\lambda}$ *is nice if and only if* $w \cdot \underline{\lambda}$ *is nice for every* $w \in \mathcal{W}_{1,\ldots,n-2}$.

Lemma 3.4. *Suppose* $\underline{\lambda}$ *is a nice weight, then either*

1) $\lambda_{n-1} = \ell - 1$ *(i.e.* $s_{n-1} \cdot \underline{\lambda} = \underline{\lambda}$*) or*
2) $s_{n-1} \cdot \underline{\lambda}$ *is not nice.*

Proof. 1) is obvious since $s_{n-1} \cdot \underline{\lambda} = (\lambda_1, \ldots, \lambda_{n-3}, [1 + \lambda_{n-2} + \lambda_{n-1}], [-2 - \lambda_{n-1}], [1 + \lambda_n + \lambda_{n-1}])$.

If $\lambda_{n-1} \neq \ell - 1$ then, according to the above remark, the $(n-1)^{\text{th}}$-nice condition for $\underline{\lambda}$ is: $2 + \lambda_{n-1} + \lambda_n \leq \ell$ hence $s_{n-1} \cdot \underline{\lambda} = (\lambda_1, \ldots, \lambda_{n-3}, [1 + \lambda_{n-2} + \lambda_{n-1}], \ell - 2 - \lambda_{n-1}, 1 + \lambda_n + \lambda_{n-1}) := \underline{\sigma}$ is not $(n-1)$-nice. Indeed

$$2 + \sigma_{n-1} + \sigma_n = \ell + 1 + \lambda_n > \ell. \quad \square$$

Corollary 3.5. *Let* $\underline{\lambda}$ *be a nice weight and* w *an element in* $\mathcal{W}_{1,\ldots,n-1}$. *Then* $w \cdot \underline{\lambda}$ *is nice if and only if* w *lies in* $\mathcal{W}_{1,\ldots,n-2}$.

Proof. We have already noticed that if w lies in $\mathcal{W}_{1,\ldots,n-2}$ and $\underline{\lambda}$ is nice then $w \cdot \underline{\lambda}$ is nice. Conversely suppose that $w \in \mathcal{W}_{1,\ldots,n-1} \setminus \mathcal{W}_{1,\ldots,n-2}$. Then we can choose a reduced expression of w in which s_{n-1} appears only once:

$$w = r s_{n-1} \ldots s_t$$

with $r \in \mathcal{W}_{1,\ldots,n-2}$, $1 \leq t \leq n-2$. Therefore

$$w \cdot \underline{\lambda} = r s_{n-1} \ldots s_t \cdot \underline{\lambda} = r s_{n-1} \cdot \underline{\mu}$$

with $\underline{\mu}$ a nice weight. Since $r \in \mathcal{W}_{1,\ldots,n-2}$, $w \cdot \underline{\lambda}$ is nice if and only if $s_{n-1} \cdot \underline{\mu}$ is nice, so we can conclude using Lemma 3.4. $\square$

Proposition 3.6. *Let* $\underline{\lambda}$ *be a* (t)*-nice weight for every* $1 \leq t \leq n-2$ *and suppose that it is not nice. Then*

1) $\underline{\tau} = s_{n-1} \cdot \underline{\lambda}$ *is a nice weight;*
2) $\underline{\sigma} = s_{n-1} s_n s_{n-1} \cdot \underline{\lambda}$ *is a nice weight.*

Proof. First of all we notice that $\underline{\lambda}$ is not $(n-1)$-nice, therefore $\lambda_{n-1} \neq \ell - 1$ and $2+\lambda_{n-1}+\lambda_n > \ell$. Besides, for every $t = 1, \ldots, n-2$, either $n-t+1+\lambda_t+\cdots+\lambda_n \equiv 0$ or $[n-t+1+\lambda_t+\cdots+\lambda_n] > \lambda_n$, (in particular, in the latter case, $\lambda_n \neq \ell - 1$).

Since $\lambda_{n-1} \neq \ell - 1$, $\underline{\tau} = s_{n-1} \cdot \underline{\lambda} = (\lambda_1, \ldots, \lambda_{n-3}, [1+\lambda_{n-2}+\lambda_{n-1}], \ell - 2 - \lambda_{n-1}, 1+\lambda_n+\lambda_{n-1}-\ell) \neq \underline{\lambda}$ and for every $1 \leq t \leq n-2$,

$$[n-t+1+\tau_t+\cdots+\tau_n] = [n-t+1+\lambda_t+\cdots+\lambda_n].$$

Thus either $[n-t+1+\tau_t+\cdots+\tau_n] = 0$ or

$$[n-t+1+\tau_t+\cdots+\tau_n] = [n-t+1+\lambda_t+\cdots+\lambda_n] > \lambda_n > 1+\lambda_n+\lambda_{n-1}-\ell = \tau_n.$$

The $(n-1)$-nice condition is also satisfied. Indeed:

$$[2+\tau_{n-1}+\tau_n] = 1+\lambda_n > 1+\lambda_n+\lambda_{n-1}-\ell = \tau_n.$$

This concludes the proof of part 1).

Let us prove part 2): a direct computation shows that

$$\begin{aligned}\underline{\sigma} &= s_{n-1}s_n s_{n-1} \cdot \underline{\lambda} \\ &= (\lambda_1, \ldots, \lambda_{n-3}, [2+\lambda_{n-2}+\lambda_{n-1}+\lambda_n], [\ell-2-\lambda_n], \ell-2-\lambda_{n-1}),\end{aligned}$$

and that, for every $1 \leq s \leq n-2$, the (s)-nice condition for $\underline{\sigma}$ is the following:

$$\begin{aligned}[n-s+1+\sigma_s+\cdots+\sigma_n] &= [n-s-1+\lambda_s+\cdots+\lambda_{n-2}] \\ &> \ell-2-\lambda_{n-1} = \sigma_n.\end{aligned} \tag{3.6}$$

Let $1 \leq s \leq n-2$. If

$$[n-s+1+\lambda_s+\cdots+\lambda_n] = n-s+1+\lambda_s+\cdots+\lambda_n - k\ell$$

for some $k \in \mathbf{N}$, then, since $\underline{\lambda}$ is (s)-nice, either

a) $n-s+1+\lambda_s+\cdots+\lambda_n - k\ell = 0$ or
b) $\lambda_n < n-s+1+\lambda_s+\cdots+\lambda_n - k\ell \leq \ell - 1$.

Therefore we distinguish the corresponding 2 cases:
a) $n-s-1+\lambda_s+\cdots+\lambda_{n-2} - k\ell = -2-\lambda_{n-1}-\lambda_n$ which implies:
$[n-s-1+\lambda_s+\cdots+\lambda_{n-2}] = 2\ell-2-\lambda_{n-1}-\lambda_n > \ell-2-\lambda_{n-1}$.
b) $\ell-2-\lambda_{n-1} < n-s-1+\lambda_s+\cdots+\lambda_{n-2}-k\ell+\ell \leq 2\ell-3-\lambda_{n-1}-\lambda_n \leq \ell-1$ *i.e.* $[n-s-1+\lambda_s+\cdots+\lambda_{n-2}] = n-s-1+\lambda_s+\cdots+\lambda_{n-2}-k\ell+\ell$ and (3.6) is proved.

Finally, the $(n-1)$-condition for σ is also satisfied. Indeed

$$\begin{aligned}[2+\sigma_{n-1}+\sigma_n] &= [-\lambda_n-\lambda_{n-1}-2] \\ &= 2\ell-2-\lambda_{n-1}-\lambda_n > \ell-2-\lambda_{n-1} = \sigma_n. \qquad \square\end{aligned}$$

Proposition 3.7. *For every weight* $\underline{\lambda} = (\lambda_1, \ldots, \lambda_n)$ *there exists an element* $w \in \mathcal{W}_{1,\ldots,n-1}$ *such that* $w \cdot \underline{\lambda}$ *is a nice weight.*

Proof. We use induction on n.

If $n = 2$ and $\underline{\lambda} = (\lambda_1, \lambda_2)$ is not nice, then it is easy to verify that $s_1 \cdot \underline{\lambda} = (\ell-2-\lambda_1, 1+\lambda_1+\lambda_2-\ell)$ is nice.

Now let $n > 2$ and suppose that the weight $\underline{\lambda} = (\lambda_1, \ldots, \lambda_n)$ is not nice. By the inductive hypothesis we can also assume that $(\lambda_2, \ldots, \lambda_n)$ is a nice weight. Then Lemma 3.2 implies that the weight $s_{n-2} \ldots s_2 s_1 \cdot \underline{\lambda}$ is (t)-nice for every $t \leq n-2$ but it is not $(n-1)$-nice. Therefore, according to Proposition 3.6, $s_{n-1}s_{n-2} \ldots s_1 \cdot \underline{\lambda}$ is a nice weight. $\square$

3.3. As in the previous section let $S_{\underline{\lambda}}$ be the subregular induced $\mathcal{U}_\varepsilon(sl(n+1))$-module defined by (3.4).

Theorem 3.8. *Let $\underline{\lambda} = (\lambda_1, \dots, \lambda_n)$ be a nice weight. If $x \in S_{\underline{\lambda}}$ is a highest weight vector then x lies in the regular part of $S_{\underline{\lambda}}$.*

The proof of this theorem will be given in Subsect. 3.4. Before going into the details of the proof we want to state its main consequences:

Corollary 3.9. *If $\underline{\lambda}$ is a nice weight then $S_{\underline{\lambda}}$ is an irreducible module.*

Proof. Let J be a submodule of $S_{\underline{\lambda}}$. Since $E_\alpha^\ell = 0$ for every $\alpha \in \mathrm{R}^+$ the subalgebra of $\mathcal{U}_\varepsilon(sl(n+1))$ generated by the E_i's acts nilpotently, therefore the subspace $\{y \in J \mid E_i(y) = 0\}$ is nontrivial and the K_i's act diagonally on it. Hence there exists a highest weight vector $z \in J \subset S_{\underline{\lambda}}$ which, according to Theorem 3.8, lies in the regular part of $S_{\underline{\lambda}}$. In particular, by Proposition 2.7, z is equal, up to a scalar, to the vector y_w for some w in $\mathcal{W}_{1,\dots,n-1}$. Since F_j is invertible for every $j \neq n$, z generates $S_{\underline{\lambda}}$ as a $\mathcal{U}_\varepsilon(sl(n+1))$-module. Thus $J = S_{\underline{\lambda}}$. □

Lemma 3.10. *Suppose that the weight $\underline{\lambda} = (\lambda_1, \dots, \lambda_n)$ is not $(n-1)$-nice. Let $\underline{\sigma} = s_{n-1}s_ns_{n-1} \cdot \underline{\lambda}$ and consider the vector y_w of weight $\underline{\sigma}$, as in Proposition 2.7, with $w = s_{n-1}s_ns_{n-1}$. Then*

1) $y_w \neq 0$;
2) $F_n^{\sigma_n+1} y_w = 0$.

Proof. Notice that, since $\underline{\lambda}$ is not $(n-1)$-nice, then $\lambda_{n-1} \neq \ell - 1$ and $2 + \lambda_{n-1} + \lambda_n > \ell$.

Using the definition of y_w introduced in 2.1 we have:

$$y_w = F_{n-1}^{1+\lambda_n} F_n^{2+\lambda_{n-1}+\lambda_n-\ell} F_{n-1}^{1+\lambda_{n-1}} v$$

where v is a fixed highest weight vector in $S_{\underline{\lambda}}$ of weight $\underline{\lambda}$. Then, since F_{n-1} is invertible, $y_w \neq 0$ if and only if $F_n^{2+\lambda_{n-1}+\lambda_n-\ell} F_{n-1}^{1+\lambda_{n-1}} v \neq 0$. Using the commutation relation between F_n and F_{n-1} and the inequality $2 + \lambda_n + \lambda_{n-1} - \ell < 1 + \lambda_{n-1}$, we can write

$$F_n^{2+\lambda_{n-1}+\lambda_n-\ell} F_{n-1}^{1+\lambda_{n-1}} v = \sum_{k=0}^{2+\lambda_n+\lambda_{n-1}-\ell} \alpha_k F_{n-1}^{1+\lambda_{n-1}-k} F_{n-1n}^k F_n^{2+\lambda_{n-1}+\lambda_n-\ell-k} v$$

for some $\alpha_k \in \mathbf{C}^\times$. Since the vectors in the sum are linearly independent and the power of F_n is less than or equal to λ_n, we can conclude part 1).

Now,

$$F_n^{\sigma_n+1} y_w := F_n^{\ell-1-\lambda_{n-1}} F_{n-1}^{1+\lambda_n} F_n^{2+\lambda_{n-1}+\lambda_n-\ell} F_{n-1}^{1+\lambda_{n-1}} v = F_{n-1}^{2+\lambda_{n-1}+\lambda_n-\ell} F_n^{1+\lambda_n} v = 0$$

and this proves part 2). □

Theorem 3.11. *Every subregular $\mathcal{U}_\varepsilon(sl(n+1))$-module M is isomorphic to $S_{\underline{\lambda}}$ for some nice weight $\underline{\lambda}$.*

Proof. Let M be an irreducible subregular $\mathcal{U}_\varepsilon(sl(n+1))$-module. Since $E_\alpha^\ell = 0$ for every $\alpha \in R^+$, the subalgebra of $\mathcal{U}_\varepsilon(sl(n+1))$ generated by the E_i's acts nilpotently, therefore the subspace

$$X := \{x \in M \mid E_i(x) = 0 \ \forall i = 1, \dots, n\}$$

is nontrivial and the K_i's act diagonally on it. Let us take $u \in X$, $u \neq 0$ such that $K_i u = \varepsilon^{x_i} u$ with $0 \leq x_i \leq \ell - 1$ and let us consider the subspace V of M spanned by the set $\{F_n^r u \mid 0 \leq r \leq \ell - 1\}$. Since $u \neq 0$ and M is irreducible, u generates M as a $\mathcal{U}_\varepsilon(sl(n+1))$-module.

We can notice that V is stable under the action of E_n, F_n, K_j, with $j = 1, \dots, n$ and in particular it defines a representation of the subalgebra $\tilde{\mathcal{U}}$ of $\mathcal{U}_\varepsilon(sl(n+1))$ with generators $E_n, F_n, K_n, K_j \ (j \leq n-2), K_{n-1}^2 K_n$. Let V' be an irreducible $\tilde{\mathcal{U}}$-submodule of V: then V' is an irreducible $\mathcal{U}_\varepsilon(sl(2))$-module since K_j, with $j \leq n-2$, and $K_{n-1}^2 K_n$ are central in $\tilde{\mathcal{U}}$; besides V' is stable under the action of K_j for every $j = 1, \dots, n$.

Let us fix a basis $\{z, F_n z, .., F_n^{\lambda_n} z\}$ of V' with $z \in V'$, $z \neq 0$, $E_n z = 0$, $F_n^{\lambda_n + 1} z = 0$, $K_i z = \varepsilon^{\lambda_i} z$, for some fixed integers λ_i such that $0 \leq \lambda_i \leq \ell - 1$, $i = 1, \dots, n$.

Let $S_{\underline{\lambda}}$ be the subregular representation induced from V' in the natural way. From the irreducibility of M it follows that M is a quotient of $S_{\underline{\lambda}}$. Therefore if $\underline{\lambda}$ is nice then $M = S_{\underline{\lambda}}$.

Suppose that $\underline{\lambda}$ is not nice. Then Lemma 3.2 and Proposition 3.7 show that there exists $\tau \in \mathcal{W}_{1\dots n-1}$ such that $\tau \cdot \underline{\lambda}$ is (t)-nice for every $t = 1, \dots, n-2$ but not $(n-1)$-nice. Therefore the weights $\sigma \cdot \underline{\lambda} = s_{n-1}\tau \cdot \underline{\lambda}$ and $w \cdot \underline{\lambda} = s_{n-1}s_n s_{n-1}\tau \cdot \underline{\lambda}$ are nice (see Proposition 3.6). Let us use the notation introduced in the previous paragraphs: if T is the submodule of $S_{\underline{\lambda}}$ generated by y_w then T is the subregular induced module $S_{w\cdot\underline{\lambda}}$ and the quotient $S_{\underline{\lambda}}/T$ is the subregular induced module $S_{\sigma\cdot\underline{\lambda}}$ generated by the vector y_σ (see Lemma 3.10). Notice that T and $S_{\underline{\lambda}}/T$ are both irreducible since $w \cdot \underline{\lambda}$ and $\sigma \cdot \underline{\lambda}$ are nice weights. Therefore, if we show that either $M \cong T$ or $M \cong S_{\underline{\lambda}}/T$, the proof will be completed. Indeed, let φ be the surjective map

$$\varphi : S_{\underline{\lambda}} \longrightarrow M.$$

If T does not intersect the kernel of φ then the restriction

$$\varphi|_T : T \longrightarrow M$$

is an injective map and therefore an isomorphism since M is irreducible. On the contrary, if $T \cap \mathrm{Ker}\varphi \neq 0$, then $T \cap \mathrm{Ker}\varphi = T$ and φ induces a surjective map

$$\varphi' : S_{\underline{\lambda}}/T \longrightarrow M.$$

The map φ' is an isomorphism since $S_{\underline{\lambda}}/T$ is irreducible. □

Theorem 3.12. *The subregular modules $S_{\underline{\lambda}}$ and $S_{\underline{\mu}}$ with $\underline{\lambda}$, $\underline{\mu}$ nice weights, are isomorphic if and only if $\underline{\lambda} = w \cdot \underline{\mu}$ for some $w \in \mathcal{W}_{1,\dots,n-2}$.*

Proof. Let $\underline{\lambda}$ and $\underline{\mu}$ be nice weights. If $S_{\underline{\lambda}} \cong S_{\underline{\mu}}$ then $S_{\underline{\mu}}$ contains a highest weight vector v of weight $\underline{\lambda}$. In fact, by Theorem 3.8, v lies in the regular part of $S_{\underline{\mu}}$, therefore, by Proposition 2.5, $\underline{\lambda}$ is linked to $\underline{\mu}$ through an element $w \in \mathcal{W}_{12\dots n-1}$. Finally Corollary 3.5 implies $w \in \mathcal{W}_{12\dots n-2}$.

Conversely, if $\underline{\lambda} = w \cdot \underline{\mu}$ for some $w \in \mathcal{W}_{12\ldots n-2}$, then $S_{\underline{\mu}}$ contains the highest weight vector y_w of weight $\underline{\lambda}$ and the map

$$F_1^{p_1^{(1)}} \ldots F_n^{p_n^{(n)}} v \longmapsto F_1^{p_1^{(1)}} \ldots F_n^{p_n^{(n)}} y_w,$$

where v is the unique, (up to a scalar), highest weight vector of weight $\underline{\lambda}$ in $S_{\underline{\lambda}}$, is an isomorphism between $S_{\underline{\lambda}}$ and $S_{\underline{\mu}}$. □

The results of this paragraph are summarized in the following theorem:

Theorem 3.13. *The irreducible subregular $\mathcal{U}_\varepsilon(sl(n+1))$-modules are parametrized, up to isomorphisms, by the $\mathcal{W}_{1,\ldots,n-2}$-orbits of nice weights with respect to the "dot"-action. Besides, every unipotent subregular module can be induced from an irreducible $\mathcal{U}_\varepsilon(sl(2))$-module.*

At this point, using the De Concini–Kac theorem of reduction to the unipotent case, the De Concini–Kac-Procesi conjecture for subregular modules follows from our construction of the unipotent subregular modules as induced modules:

Corollary 3.14. *Any subregular $\mathcal{U}_\varepsilon(sl(n+1))$-module has dimension divisible by ℓ^{N-1}.*

3.4. The proof of Theorem 3.8. In this paragraph we prove Theorem 3.8 by induction on n. For $n = 2$ the theorem is proved in [C1].

Now let $n > 2$ and suppose that for every $k < n$ if $\underline{\lambda} = (\lambda_1, \ldots, \lambda_k)$ is a nice weight, $S_{\underline{\lambda}}$ an induced subregular $\mathcal{U}_\varepsilon(sl(k+1))$-module and $x \in S_{\underline{\lambda}}$ a highest weight vector then x lies in the regular part of $S_{\underline{\lambda}}$. We shall adopt the same notation as in the previous paragraphs.

Definition 3.15. *Let $\underline{h} = (h_1, \ldots, h_n)$, $\underline{k} = (k_1, \ldots, k_n)$ in $\mathbf{Z}^n$. We shall define*

$$\underline{h} \le \underline{k}$$

if

$$\sum_{i=1}^{t} h_i \le \sum_{i=1}^{t} k_i$$

for every $t = 1, \ldots, n$.

Definition 3.16. *Let $\underline{h} = (h_1, \ldots, h_n) \in \mathbf{N}^n$ with $0 \le h_j \le \ell - 1$. We define*

$$F^{\underline{h}} = F_{12\ldots n}^{h_1} F_{2\ldots n}^{h_2} \cdots F_{n-1n}^{h_{n-1}} F_n^{h_n}$$

and

$$V^{\underline{h}} = \sum_{\underline{k} \le \underline{h}} \mathcal{U}_\varepsilon^-(sl(n)) F^{\underline{k}} v.$$

Proposition 3.17. *$V^{\underline{h}}$ is stable under the action of $\mathcal{U}_\varepsilon(sl(n))$.*

Proof. One can easily show, by direct computation, that $E_i F^{\underline{k}} v \in V^{\underline{h}}$ for every $i = 1, \ldots, n-1$ whenever $\underline{k} \le \underline{h}$. □

Remark. If we put $Z_{\underline{h}} = \sum_{\underline{k}<\underline{h}} V^{\underline{k}}$ and $W^{\underline{h}} = V^{\underline{h}}/Z_{\underline{h}}$ then $W^{\underline{h}}$ is the regular $\mathcal{U}_\varepsilon(sl(n))$-module generated by the vector $F^{\underline{h}}v$.

As before, let us fix the reduced expression of $w_0 \in \mathcal{W}$, $w_0 = s_n \dots s_1 s_n \dots s_2 s_n \dots s_n$, and the corresponding basis $\mathcal{B}$ of $S_{\underline{\lambda}}$ (3.5). From now on we shall consider a highest weight vector $x \in S_{\underline{\lambda}}$ with $\underline{\lambda}$ nice and its expression as a linear combination of the vectors in $\mathcal{B}$, where we shall omit the coefficients:

$$x = \sum_{p_j^{(t)}} F_1^{p_1^{(1)}} \dots F_{12\dots n}^{p_1^{(n)}} F_2^{p_2^{(2)}} \dots F_{2\dots n}^{p_2^{(n)}} \dots F_{n-1}^{p_{n-1}^{(n-1)}} F_{n-1n}^{p_{n-1}^{(n)}} F_n^{p_n^{(n)}} v.$$

Let $\bar{p}_n$ be the highest power with which $F_{12\dots n}$ appears in the expression of x.

Lemma 3.18. *The highest weight vector x lies in $V^{\underline{h}}$ for $\underline{h} = (\bar{p}_n, 0, \dots, 0)$.*

Proof. In order to prove the lemma we will show that

$$x = \sum_{p_j^{(t)}, h_i} F_1^{p_1^{(1)}} \dots F_{n-1}^{p_{n-1}^{(n-1)}} F_{12\dots n}^{\bar{p}_n - h_1} F_{2\dots n}^{h_1 - h_2} \dots F_{n-1n}^{h_{n-2}-h_{n-1}} F_n^{h_{n-1}} v,$$

where $0 \le h_{n-1} \le h_{n-2} \le \dots \le h_1 \le \bar{p}_n$ *i.e.* we will show, using induction on t, that, for every $t = 2, \dots, n$, h_{t-1} is the highest power with which $F_{t\dots n}$ appears in the terms containing the monomial $F_{12\dots n}^{\bar{p}_n - h_1} \dots F_{t-1\dots n}^{h_{t-2}-h_{t-1}}$, where $0 \le h_{t-1} \le h_{t-2} \le \dots \le h_1 \le \bar{p}_n$.

Let $t = 2$. By definition of $\bar{p}_n$, $F_{12\dots n}$ appears in the expression of x with powers $\bar{p}_n - h_1$ for $0 \le h_1 \le \bar{p}_n$. Besides, since $E_1(x) = 0$ and

$$[E_1, F_{12\dots n}^r] = -(1 - \varepsilon^{2r})/(1 - \varepsilon^2) F_{12\dots n}^{r-1} F_{2\dots n} K_1^{-1},$$

all the powers of $F_{12\dots n}$ between 0 and $\bar{p}_n$ appear in the expression of x. Then let us use induction on h_1.

If $h_1 = 0$ let us denote by $\bar{p}_h$ ($h = 1, \dots, n-1$) the highest power with which $F_{12\dots h}$ appears, in the expression of x, in the terms containing $F_{12\dots t}$ at the power $\bar{p}_t$ for every $t > h$. Then

$$x = F_1^{\bar{p}_1} F_{12}^{\bar{p}_2} \dots F_{12\dots n-1}^{\bar{p}_{n-1}} F_{12\dots n}^{\bar{p}_n} \omega + \dots$$

for some $\omega \in V_{2\dots n}$.

Since $E_i(x) = 0$ for every $i = 1, \dots, n$ the maximality of the $\bar{p}_j$'s implies $E_i(\omega) = 0$ for every $i = 2, \dots, n$, therefore, by the inductive hypothesis, $\omega \in V_{2\dots n-1}$. This means in particular (see Lemma 2.9) that $\bar{p}_n$ is uniquely determined by the weight of x.

Now let M, $0 \le M \le \ell - 1$, be the highest power with which $F_{2\dots n}$ appears in the terms containing $F_{12\dots n}^{\bar{p}_n}$. Then we can isolate the terms containing $F_{12\dots n}^{\bar{p}_n}$ and $F_{2\dots n}^M$ and write

$$x = \sum_{\underline{k}=(k_i^{(j)})} F_1^{k_1^{(1)}} \dots F_{12\dots n-1}^{k_1^{(n-1)}} F_2^{k_2^{(2)}} \dots F_{2\dots n-1}^{k_2^{(n-1)}} F_{12\dots n}^{\bar{p}_n} F_{2\dots n}^M \omega_{\underline{k}} + \dots,$$

where $\omega_{\underline{k}} \in V_{3\dots n}$.

Using the maximality of $\bar{p}_n$ and M, we see that $E_n(x) = 0$ implies $E_n(\omega_{\underline{k}}) = 0$ for every $\underline{k}$. Furthermore we notice that there exists one $\underline{\tilde{k}} = (\tilde{k}_1^{(1)}, \dots, \tilde{k}_2^{(n-1)})$ such that $E_i(\omega_{\underline{\tilde{k}}}) = 0$ for all $i = 3, \dots, n-1$. Indeed it is sufficient to take:

$\tilde{k}_2^{(n-1)} = \max\{k_2^{(n-1)}\}$,
$\tilde{k}_1^{(n-1)} = \max\{k_1^{(n-1)} \mid F_{1\ldots n-1}^{k_1^{(n-1)}}$ appears in the terms containing $F_{2\ldots n-1}^{\tilde{k}_2^{(n-1)}}\}$,
then, recursively,
$\tilde{k}_2^{(h)} = \max\{k_2^{(h)} \mid F_{2\ldots h}^{k_2^{(h)}}$ appears in the terms containing $F_{12\ldots t}^{\tilde{k}_1^{(t)}}$ and $F_{2\ldots t}^{\tilde{k}_2^{(t)}}$ for every $t > h\}$,
$\tilde{k}_1^{(h)} = \max\{k_1^{(h)} \mid F_{1\ldots h}^{k_1^{(h)}}$ appears in the terms containing $F_{12\ldots t}^{\tilde{k}_1^{(t)}}\ \forall t > h$ and $F_{2\ldots t}^{\tilde{k}_2^{(t)}}$ for every $t \geq h\}$.

Therefore $\omega_{\underline{\tilde{k}}}$ lies in $V_{3\ldots n-1}$ and, by Lemma 2.9, $\bar{p}_n + M \equiv \bar{p}_n$, thus $M = 0$.

We point out that the same argument shows that the terms containing $F_{12\ldots n}^{\bar{p}_n}$ do not contain $F_{j\ldots n}$ for any $j \geq 2$.

Let us now fix $0 < h_1 \leq \bar{p}_n$ and assume the inductive hypothesis. Suppose that $h_1 < M \leq \ell - 1$ is the highest power of $F_{2\ldots n}$ in the terms containing $F_{12\ldots n}^{\bar{p}_n - h_1}$:

$$x = \sum_{\underline{k}} F_1^{k_1^{(1)}} \ldots F_{2\ldots n-1}^{k_2^{(n-1)}} F_{12\ldots n}^{\bar{p}_n - h_1} F_{2\ldots n}^{M} \omega_{\underline{k}} + \ldots$$

with $\omega_{\underline{k}} \in V_{3\ldots n}$. Then $E_n(\omega_{\underline{k}}) = 0$ by the maximality of M and the inductive hypothesis. Furthermore, following the same procedure used for $h_1 = 0$, we see that there exists one $\underline{\tilde{k}}$ such that $E_i(\omega_{\underline{\tilde{k}}}) = 0$ for every $i = 3, \ldots, n-1$. Thus $\omega_{\underline{\tilde{k}}} \in V_{3\ldots n-1}$ and $\bar{p}_n - h_1 + M \equiv \bar{p}_n \pmod{\ell}$ which is a contradiction.

Now let $t > 2$ and let us assume that, for any $r < t$, h_{r-1} is the highest power of $F_{r\ldots n}$ in the terms containing the monomial $F_{1\ldots n}^{\bar{p}_n - h_1} \ldots F_{r-1\ldots n}^{h_{r-2} - h_{r-1}}$ with $h_{r-1} \leq h_{r-2} \leq \cdots \leq h_1 \leq \bar{p}_n$.

We use induction on the set $\{(h_1, \ldots, h_{t-1}) : 0 \leq h_{t-1} \leq \cdots \leq h_1 \leq \bar{p}_n\}$ ordered as in Definition 3.15.

When $(h_1, \ldots, h_{t-1}) = (0, \ldots, 0)$ we are dealing with the terms containing $F_{1\ldots n}^{\bar{p}_n}$ that we have already considered in the case $t = 2$.

Let us fix $(h_1, \ldots, h_{t-1}) \neq (0, \ldots, 0)$ and let M, with $h_{t-1} < M \leq \ell - 1$, be the highest power of $F_{t\ldots n}$ in the terms containing the monomial
$F_{1\ldots n}^{\bar{p}_n - h_1} \ldots F_{t-1\ldots n}^{h_{t-2} - h_{t-1}}$:

$$x = \sum F_1^{k_1^{(1)}} \ldots F_{t\ldots n-1}^{k_t^{(n-1)}} F_{12\ldots n}^{\bar{p}_n - h_1} \ldots F_{t-1\ldots n}^{h_{t-2} - h_{t-1}} F_{t\ldots n}^{M} \omega_{\underline{k}} + \cdots$$

with $\omega_{\underline{k}} \in V_{t+1\ldots n}$. By the maximality of M and the inductive hypothesis, $E_n(x) = 0$ implies $E_n(\omega_{\underline{k}}) = 0$. Moreover, following the same procedure used for $t = 2$, we see that there exists $\underline{\tilde{k}}$ such that $E_i(\omega_{\underline{\tilde{k}}}) = 0$ for every $i = t+1, \ldots, n$.

Thus $\omega_{\underline{\tilde{k}}} \in V_{t+1\ldots n-1}$ and $\bar{p}_n - h_{t-1} + M \equiv \bar{p}_n \pmod{\ell}$, which is a contradiction. The proof is concluded. □

Proposition 3.19. *Let $S_{\underline{\lambda}}$ be a subregular induced module and let x, y be highest weight vectors with the same weight $\underline{\mu}$. Then $y = ax$ for some $a \in \mathbf{C}^{\times}$.*

Proof. Let x be a highest weight vector of weight $\underline{\mu}$ in a subregular induced $\mathcal{U}_\varepsilon(sl(n+1))$-module $S_{\underline{\lambda}}$ and let $v \in S_{\underline{\lambda}}$ be a highest weight vector of weight $\underline{\lambda}$. If we use the same notation as before we can write x with respect to the basis $\mathcal{B}$ and isolate the terms containing $F_{12\ldots n}^{\bar{p}_n}$ so that

$$x = a\tilde{F}F_{12\ldots n}^{\bar{p}_n}v + \{\text{terms with powers of } F_{12\ldots n} < \bar{p}_n\},$$

where $a \in \mathbf{C}^\times$, $\tilde{F}$ is a sum of monomials in the F_α's with $\alpha \in R^+_{1,\ldots,n-1}$. In fact $\tilde{F}$ is a monomial in the F_j's with $1 \le j \le n-1$. Indeed conditions $E_i(x) = 0$ with $i = 1, \ldots, n-1$ imply what follows: let z be a highest weight vector with weight $\underline{\sigma}$ equal to the weight of $F_{12\ldots n}^{\bar{p}_n}v$ and let $W_{\underline{\sigma}}$ be the regular $\mathcal{U}_\varepsilon(sl(n))$-module generated by z. Then $\tilde{F}z$ is a highest weight vector in $W_{\underline{\sigma}}$ and, by Proposition 2.7, $\tilde{F}z = cy_w$ for some $c \in \mathbf{C}^\times$ and $w \in \mathcal{W}$ such that $w \cdot \underline{\sigma}$ is equal to the weight of x. In particular, since $\bar{p}_n$ is uniquely determined by the weight of x, also $\tilde{F}$ is.

Let y be another highest weight vector in $S_{\underline{\lambda}}$ with weight $\underline{\mu}$. Then

$$x = a\tilde{F}F_{12\ldots n}^{\bar{p}_n}v + \{\text{terms with powers of } F_{12\ldots n} < \bar{p}_n\},$$
$$y = b\tilde{F}F_{12\ldots n}^{\bar{p}_n}v + \{\text{terms with powers of } F_{12\ldots n} < \bar{p}_n\}$$

with $b \in \mathbf{C}^\times$.

Therefore if $x/a - y/b \neq 0$ then it is a highest weight vector with weight $\underline{\mu}$ which contains only powers of $F_{12\ldots n}$ strictly less than $\bar{p}_n$. This is a contradiction.

Notice that we can write $\tilde{F} = GF_1^{p_1} \ldots F_t^{p_t}$ where $1 \le t \le n-1$ and G is a monomial in the F_j's with $2 \le j \le n-1$. Indeed, according to Lemma 1.1, we can choose a reduced expression of w in which s_1 appears at most once and consider the corresponding vector y_w. □

Up to now we have proved that if x is a highest weight vector in $S_{\underline{\lambda}}$ then we can write

$$x = GF_1^{p_1} \ldots F_t^{p_t} F_{12\ldots n}^{\bar{p}_n}v + \{\text{terms with powers of } F_{12\ldots n} < \bar{p}_n\}$$

for some $p_1, \ldots, p_t$ with $1 \le t \le n-1$, and a monomial G in the F_j's with $2 \le j \le n-1$.

Remark. Let $x = F_r^{\lambda_r^z+1}z + y$ be a weight vector in $S_{\underline{\lambda}}$ of weight $\underline{\sigma}$, where $r \neq n$ and $\underline{\lambda}^z = (\lambda_1^z, \ldots, \lambda_n^z)$ is the weight of z. Then x is a highest weight vector if and only if $z + F_r^{-\lambda_r^z-1}y$ is a highest weight vector. Indeed $\sigma_r \equiv -\lambda_r^z - 2 \pmod{\ell}$ and $F_r^{\sigma_r+1}x = z + F_r^{-\lambda_r^z-1}y$. Since F_r is invertible for every $r \neq n$ we can conclude. □

Now notice that, by Proposition 2.5, any highest weight in $S_{\underline{\lambda}}$ is linked to $\underline{\lambda}$. Therefore let $\underline{\mu} = w \cdot \underline{\lambda}$ be the weight of x: from Proposition 3.19 we immediately see that if $w \in \mathcal{W}_{12\ldots n-1}$ (that is, by Lemma 2.9, if and only if $\bar{p}_n = 0$) then $x = \sigma y_w$ for some $\sigma \in \mathbf{C}$, therefore x lies in the regular part of $S_{\underline{\lambda}}$.

Now let $w \notin \mathcal{W}_{12\ldots n-1}$ and choose a reduced expression of w containing only one s_n:

$$w = rs_n \ldots s_k$$

with $r \in \mathcal{W}_{12\ldots n-1}$, $1 \le k \le n-1$. We can assume $r = id$ and distinguish two cases:

Case 1: $k \neq 1$. Under this hypothesis $p_1 = \ell - \bar{p}_n$ by Lemma 2.9, and we can write:

$$x = GF_1^{\ell-\bar{p}_n} F_{12\ldots n}^{\bar{p}_n} w + \{\text{terms with powers of } F_{12\ldots n} < \bar{p}_n\}$$

for a highest weight vector $w \in V_{2\ldots n-1}$. According to the above remark and since the action of the F_j's on x, for $j = 2, \ldots, n-1$, does not modify the powers of $F_{12\ldots n}$, the existence of x is equivalent to the existence of a highest weight vector

$$x' = F_1^{\ell-\bar{p}_n} F_{12\ldots n}^{\bar{p}_n} w + \{\text{terms with powers of } F_{12\ldots n} < \bar{p}_n\}.$$

We then have, up to some non-zero coefficient,

$$E_1(x') = F_1^{\ell-\bar{p}_n} F_{12\ldots n}^{\bar{p}_n-1} F_{2\ldots n} w + \ldots.$$

Let us set $w_1 = F_{12\ldots n}^{\bar{p}_n-1} F_{2\ldots n} w$. Since $x' \in V^{\underline{h}}$ with $\underline{h} = (\bar{p}_n, 0, \ldots, 0)$, in order to have $E_1(x') = 0$ the expression of x' must contain a vector $z_1 \in W^{(\bar{p}_n-1,1,0,\ldots,0)}$ such that $E_1(z_1) = F_1^{\ell-\bar{p}_n} w_1$ and $E_i(z_1) = 0$ for every $i > 1$. By Lemma 2.10, $z_1 = \sigma_1 F_1^{\ell-\bar{p}_n+1} w_1$ with σ_1 as in the lemma.

In the same way, if we compute $E_1(z_1)$ we find that the expression of x' must contain a term $z_2 = \sigma_2 \sigma_1 F_1^{\ell-\bar{p}_n+2} F_{12\ldots n}^{\bar{p}_n-2} F_{2\ldots n}^2 w$, with σ_2 as in Lemma 2.10. We repeat the same argument till we find that $E_1(x')$ contains the term $F_1^{\ell-1} F_{2\ldots n}^{\bar{p}_n} w \in W^{(0,\bar{p}_n,0,\ldots,0)}$ which can be annihilated only by an element z in $W^{(0,\bar{p}_n,0,\ldots,0)}$ such that $E_i(z) = 0$ for every $i \neq 1$ and $E_1(z) = F_1^{\ell-1} w_{\bar{p}_n}$ with $w_{\bar{p}_n} = F_{2\ldots n}^{\bar{p}_n} w$. Since by Lemma 2.10 such a vector z does not exist, we conclude that $S_{\underline{\lambda}}$ does not contain any highest weight vector of weight w as in our assumptions.

Case 2: $k = 1$. We follow the same procedure as in Case 1 by taking into account what follows: there exists a highest weight vector with weight $s_n \ldots s_1 \cdot \underline{\lambda}$ if and only if there exists a highest weight vector with weight $s_1 \ldots s_{n-2} s_n \ldots s_1 \cdot \underline{\lambda} = s_n s_{n-1} \ldots s_1 s_2 \ldots s_{n-1} \cdot \underline{\lambda}$. In this case $\bar{p}_n \equiv n + \lambda_1 + \cdots + \lambda_n \pmod{\ell}$ and, by Lemma 2.9, $p_1 \equiv -1 - \lambda_n \pmod{\ell}$. Therefore the existence of x is equivalent to that of a highest weight vector

$$x' = F_1^{\ell-1-\lambda_n} F_{12\ldots n}^{\bar{p}_n} w + \{\text{terms with powers of } F_{12\ldots n} < \bar{p}_n\},$$

where w is a highest weight vector in $V_{2\ldots n-1}$. Since $\underline{\lambda}$ is nice, $\bar{p}_n = [n+\lambda_1+\cdots+\lambda_n] > \lambda_n$ thus, by arguing as in case 1, we find a contradiction using Lemma 2.10.

Thus we conclude that, if $\underline{\lambda}$ is nice, $S_{\underline{\lambda}}$ does not contain any highest weight vector with weight $w \cdot \underline{\lambda}$ such that $w \in \mathcal{W}_{1\ldots n} \setminus \mathcal{W}_{12\ldots n-1}$.

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