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A CLASSIFICATION OF $\mathbb{Z}$-GRADED LIE SUPERALGEBRAS OF INFINITE DEPTH

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In 1998 Victor Kac classified infinite-dimensional, transitive, irreducible $\mathbb{Z}$-graded Lie superalgebras of finite depth. Here we classify bitransitive, irreducible $\mathbb{Z}$-graded Lie superalgebras of infinite depth and finite growth which are not contragredient. In particular we show that the growth of every such superalgebra is equal to one.

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1. Introduction

Since the late 60s supersymmetry has attracted increasing attention among physicists, becoming a fundamental concept in physics. In particular the role of Lie superalgebras has become more and more central to solving problems in the quantum field theory and string theory.

A Lie superalgebra is a $\mathbb{Z}_2$-graded vector space $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ on which a product $[\cdot\,,\cdot]$, satisfying the following properties, is defined: If $a \in \mathfrak{g}_i$, $b \in \mathfrak{g}_j$, i, $j \in \mathbb{Z}_2$, then:

(1) $[a,b] \in \mathfrak{g}_{i+j}$;
(2) $[a,b] = -(-1)^{ij}[b,a]$;
(3) $[a,[b,c]] = [[a,b],c] + (-1)^{ij}[b,[a,c]]$.

In particular the subalgebra $\mathfrak{g}_{\bar{0}}$ of $\mathfrak{g}$ is a Lie algebra.

A Lie superalgebra is $\mathbb{Z}$-graded when it is decomposed into the direct sum of $\mathbb{Z}_2$-graded, finite-dimensional subspaces $\mathfrak{g}_i$, $i \in \mathbb{Z}$, such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$. The sum $\bigoplus_{i<0} \mathfrak{g}_i$ is called the negative part of $\mathfrak{g}$ and is usually denoted by $\mathfrak{g}^-$. If the negative part of $\mathfrak{g}$ contains infinite non-zero terms then we say that the depth of $\mathfrak{g}$ is infinite, otherwise we say that $\mathfrak{g}$ has finite depth.

The classification of irreducible, transitive $\mathbb{Z}$-graded Lie superalgebras of finite depth (see Definitions 2.4 and 2.5 below) over an algebraically closed field of characteristic zero is due to Victor Kac, both in the finite [5] and in the infinite-dimensional case [6]. A part of the classification in [5] is devoted to finite-dimensional contragredient Lie superalgebras which are there introduced for the first time. Using the same method van de Leur classifies the symmetrizable contragredient Lie superalgebras of finite growth [10].

Let us recall that the growth (or Gelfand–Kirillov dimension) of a $\mathbb{Z}$-graded Lie (super)algebra $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i$ is the following limit:

$$r(\mathfrak{g}) = \lim_{n\to\infty} \ln\left(\sum_{i=-n}^{n} \dim \mathfrak{g}_i\right)\Big/ \ln(n)\,.$$

$\mathbb{Z}$-graded Lie superalgebras of finite depth always have finite growth, as one can show by applying the Guillemin–Sternberg theorem for filtered Lie algebras [2] to the filtered Lie algebra associated to $\mathfrak{g}$.

The complete classification of $\mathbb{Z}$-graded infinite-dimensional Lie algebras of finite growth was given by O. Mathieu [8]. A significant achievement had already been made by Victor Kac [4]:

Theorem 1.1. *Let $\mathfrak{g}$ be a simple, infinite-dimensional, $\mathbb{Z}$-graded Lie algebra such that:*

(1) $\mathfrak{g}$ *is generated by its local part* $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$;
(2) $\mathfrak{g}_{-1}$ *is an irreducible* $\mathfrak{g}_0$*-module.*

Then $\mathfrak{g}$ has finite growth if and only if it is isomorphic to an affine Kac–Moody Lie algebra or to a Cartan algebra.

In the same paper [4] Kac introduces contragredient Lie algebras associated to any matrix and obtains Kac–Moody algebras as a special case of this construction. In particular, Kac shows that the affine Kac–Moody Lie algebras can be constructed as covering Lie algebras (see Sec. 3.1) of finite-dimensional Lie algebras and that their growth is therefore equal to 1. In the same way contragredient Lie superalgebras of finite growth can be constructed as covering Lie superalgebras of finite-dimensional contragredient Lie superalgebras, and consequently their growth is equal to 1.

Our main goal is to study $\mathbb{Z}$-graded Lie superalgebras of infinite depth and finite growth which are not contragredient. We already gave [1] three examples of such superalgebras constructed as covering superalgebras of finite-dimensional Lie superalgebras. We prove here that these three superalgebras classify, up to isomorphism, all simple, irreducible Lie superalgebras generated by their local parts, with a consistent $\mathbb{Z}$-gradation. It turns out that $\mathbb{Z}$-graded Lie superalgebras of infinite depth and finite growth necessarily have growth 1.

The paper is organized as follows: The basic definitions and the main results concerning $\mathbb{Z}$-graded Lie superalgebras of finite depth are recalled in Sec. 2 where we also state the above mentioned classification theorems due to Victor Kac. In Sec. 3 we recall the afore-mentioned examples of non-isomorphic $\mathbb{Z}$-graded Lie superalgebras of infinite depth and finite growth constructed in [1]. These examples satisfy the following properties:

(1) $\mathfrak{g}$ has a consistent $\mathbb{Z}$-gradation of infinite depth;
(2) $\mathfrak{g}$ is irreducible, simple and generated by its local part;
(3) $\mathfrak{g}_0$ is a simple Lie algebra;
(4) $\mathfrak{g}_1$ is an irreducible $\mathfrak{g}_0$-module and is not contragredient to $\mathfrak{g}_{-1}$.

In [1] we show that a Lie superalgebra $\mathfrak{g}$ satisfying properties (1)–(4) has finite growth if and only if it is isomorphic to one of the afore-mentioned examples. Here we generalize this result to any Lie superalgebra satisfying properties (1) and (2) only, namely, we show that whenever $\mathfrak{g}_0$ is not a simple Lie algebra or $\mathfrak{g}_1$ is not an irreducible $\mathfrak{g}_0$-module, then $\mathfrak{g}$ has infinite growth. Theorem 2.1 and Lemma 2.1, by Victor Kac, play an essential role in establishing necessary conditions under which the growth of $\mathfrak{g}$ is finite. Our analysis is essentially based on the classification of non-contragredient $\mathbb{Z}$-graded Lie superalgebras of finite depth and of contragredient Lie superalgebras.

Throughout the paper the base field is assumed to be the field of complex numbers. The classification remains valid if we replace $\mathbb{C}$ with any algebraically closed field of characteristic zero.

2. The General Setting

2.1. *Lie superalgebras*

Definition 2.1. A superalgebra is a $\mathbb{Z}_2$-graded algebra $A = A_{\bar{0}} \oplus A_{\bar{1}}$. $A_{\bar{0}}$ is called the even part of A and $A_{\bar{1}}$ is called the odd part of A.

Definition 2.2. A Lie superalgebra is a superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ whose product $[\cdot\,,\cdot]$ satisfies the following axioms:

(1) $[a, b] = -(-1)^{\mathrm{deg(a)deg(b)}}[b, a]$;
(2) $[a, [b, c]] = [[a, b], c] + (-1)^{\mathrm{deg(a)deg(b)}}[b, [a, c]]$

for a, b, c homogeneous elements in $\mathfrak{g}$.

Definition 2.3. A $\mathbb{Z}$-grading of a Lie superalgebra $\mathfrak{g}$ is a decomposition of $\mathfrak{g}$ into a direct sum of finite-dimensional $\mathbb{Z}_2$-graded subspaces $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i$ for which $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$. A $\mathbb{Z}$-grading is said to be consistent if $\mathfrak{g}_{\bar{0}} = \oplus\mathfrak{g}_{2i}$ and $\mathfrak{g}_{\bar{1}} = \oplus\mathfrak{g}_{2i+1}$.

By definition, if $\mathfrak{g}$ is a $\mathbb{Z}$-graded Lie superalgebra, then $\mathfrak{g}_0$ is a subalgebra of $\mathfrak{g}$ and $[\mathfrak{g}_0, \mathfrak{g}_i] \subseteq \mathfrak{g}_i$; therefore the restriction of the adjoint representation to $\mathfrak{g}_0$ induces linear representations of $\mathfrak{g}_0$ on the subspaces $\mathfrak{g}_i$.

Example 2.1. Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a $\mathbb{Z}_2$-graded, finite-dimensional vector space. Let m and n be the dimensions of $V_{\bar{0}}$ and $V_{\bar{1}}$, respectively. Then the algebra of endomorphisms of V is an associative superalgebra with the following gradation:

$$\mathrm{End}(V) = \mathrm{End}_{\bar{0}}(V) \oplus \mathrm{End}_{\bar{1}}(V)\,,$$

where $\mathrm{End}_\alpha(V) = \{a \in \mathrm{End}(V) \mid a(V_s) \subseteq V_{\alpha+s}\}$.

$\mathrm{End}(V)$ can be made a Lie superalgebra, denoted by $\ell(V)$ or by $\ell(m,n)$, by the following definition:

$$[a,b] = ab - (-1)^{(\deg \mathrm{a})(\deg \mathrm{b})} ba\,.$$

If we regard the decomposition $V = V_{\bar{0}} \oplus V_{\bar{1}}$ as a $\mathbb{Z}$-grading of V, then the same construction provides a consistent $\mathbb{Z}$-grading of $\ell(V)$.

Example 2.2 (The Lie superalgebra $A(m, n)$). Let us choose a basis of the superspace V consisting of the union of a basis of $V_{\bar{0}}$ and a basis of $V_{\bar{1}}$. In this basis the matrix of an element a in $\ell(V)$ can be written in the form $\left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right)$, where α is an $(m \times m)$-, δ an $(n \times n)$-, β an $(m \times n)$-, and γ an $(n \times m)$-matrix.

We can now define (see [5]) a linear function

$$\mathrm{str} : \ell(V) \longrightarrow \mathbb{C}\,,$$

called supertrace, as follows:

$$\mathrm{str}(a) = \mathrm{str}\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} = \mathrm{tr}(\alpha) - \mathrm{tr}(\delta)\,.$$

Then, in particular,

$$\mathrm{str}([a,b]) = 0 \quad \text{for every } a,\ b \in \ell(V)\,.$$

As a consequence the subspace

$$\mathrm{sl}(m,n) = \{a \in \ell(m,n) \mid \mathrm{str}(a) = 0\}$$

is an ideal of $\ell(m,n)$ of codimension 1. With respect to the above mentioned basis we can describe the $\mathbb{Z}$-gradation of $\mathrm{sl}(m,n)$ as follows:

$$\mathrm{sl}(m,n)_0 = \mathrm{sl}(m,n)_{\bar{0}} = \left\{ \begin{pmatrix}\alpha & 0 \\ 0 & \delta\end{pmatrix} \,\middle|\, \mathrm{tr}(\alpha) = \mathrm{tr}(\delta) \right\};$$

$$\mathrm{sl}(m,n)_1 = \left\{ \begin{pmatrix}0 & \beta \\ 0 & 0\end{pmatrix} \right\};$$

$$\mathrm{sl}(m,n)_{-1} = \left\{ \begin{pmatrix}0 & 0 \\ \gamma & 0\end{pmatrix} \right\}.$$

The superalgebra $\mathrm{sl}(n,n)$ contains the one-dimensional ideal consisting of scalar matrices. We define:

$$A(m,n) = \mathrm{sl}(m+1, n+1)\,, \quad \text{for } m \neq n\,, \quad m,n \geq 0\,,$$

$$A(n,n) = \mathrm{sl}(n+1, n+1)/\langle I_{2n+2}\rangle\,, \quad \text{for } n \geq 0\,.$$

Example 2.3 (The Lie superalgebra $Q(n)$, $n\geq 2$). Let $\tilde{Q}(n)$ ($n \geq 2$) be the subalgebra of $\mathrm{sl}(n+1, n+1)$ consisting of matrices of the form $\binom{a\ b}{b\ a}$, where $\mathrm{tr}(b) = 0$. Then $\tilde{Q}(n)$ has a one-dimensional center $C = \langle I_{2n+2}\rangle$ and we define $Q(n) = \tilde{Q}(n)/C$. The even part of $Q(n)$ is isomorphic to the Lie algebra of type A_n and its odd part is isomorphic to the $\mathrm{sl}(n+1)$-adjoint module. Therefore $Q(n)$ has dimension $2(n^2 + 2n)$.

Definition 2.4. A $\mathbb{Z}$-graded Lie superalgebra $\mathfrak{g}$ is called irreducible if $\mathfrak{g}_{-1}$ is an irreducible $\mathfrak{g}_0$-module.

Definition 2.5. A $\mathbb{Z}$-graded Lie superalgebra $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i$ is called transitive if it satisfies the following property:

$$\text{if } a \in \mathfrak{g}_i,\ i \geq 0 \quad \text{and} \quad [a, \mathfrak{g}_{-1}] = 0\,, \quad \text{then } a = 0\,.$$

A transitive $\mathbb{Z}$-graded Lie superalgebra is called bitransitive if, in addition, it satisfies the following property:

$$\text{if } a \in \mathfrak{g}_i,\ i \leq 0 \quad \text{and} \quad [a, \mathfrak{g}_{1}] = 0\,, \quad \text{then } a = 0\,.$$

Observe that if $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i$ is transitive then $\mathfrak{g}_{-1}$ is a faithful $\mathfrak{g}_0$-module and if $\mathfrak{g}$ is bitransitive then also $\mathfrak{g}_1$ is a faithful $\mathfrak{g}_0$-module.

Definition 2.6. If, for some positive integer d, $\mathfrak{g} = \bigoplus_{i\geq -d} \mathfrak{g}_i$, then d is called the depth of $\mathfrak{g}$. Thus the depth of $\mathfrak{g}$ is infinite if the negative part $\mathfrak{g}^- = \bigoplus_{i<0} \mathfrak{g}_i$ of $\mathfrak{g}$ contains an infinite number of non-zero terms.

Let $\hat{\mathfrak{g}}$ be a $\mathbb{Z}_2$-graded space, decomposed into the direct sum of $\mathbb{Z}_2$-graded subspaces, i.e., $\hat{\mathfrak{g}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Suppose that whenever $|i+j| \leq 1$ a bilinear operation is defined: $\mathfrak{g}_i \times \mathfrak{g}_j \to \mathfrak{g}_{i+j}$, $(x, y) \mapsto [x, y]$, satisfying the axiom of anticommutativity and the Jacobi identity for Lie superalgebras, provided that all the commutators in this identity are defined. Then $\hat{\mathfrak{g}}$ is called a *local* Lie superalgebra.

If $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i$ is a $\mathbb{Z}$-graded Lie superalgebra then $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a local Lie superalgebra called the *local part* of $\mathfrak{g}$. A $\mathbb{Z}$-graded Lie superalgebra $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i$ with local part $\hat{\mathfrak{g}}$ is said to be *minimal* if for any $\mathbb{Z}$-graded Lie superalgebra $\mathfrak{g}'$ an isomorphism of the local parts $\hat{\mathfrak{g}}$ and $\hat{\mathfrak{g}}'$ extends to an epimorphism of $\mathfrak{g}'$ onto $\mathfrak{g}$. Let us recall the following well known facts [5]:

Proposition 2.1. *Let $\hat{\mathfrak{g}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a local Lie superalgebra. Then there is a minimal $\mathbb{Z}$-graded Lie superalgebra whose local part is isomorphic to $\hat{\mathfrak{g}}$.*

Proposition 2.2. (1) *A bitransitive $\mathbb{Z}$-graded Lie superalgebra is minimal.*

(2) *A minimal $\mathbb{Z}$-graded Lie superalgebra with bitransitive local part is bitransitive.*

(3) *Two bitransitive $\mathbb{Z}$-graded Lie superalgebras are isomorphic if and only if their local parts are isomorphic.*

Proposition 2.3. *Let $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be an irreducible, transitive Lie superalgebra such that the representation of $\mathfrak{g}_0$ on $\mathfrak{g}_1$ is faithful. Then $\mathfrak{g}$ is bitransitive.*

Proof. Clearly $V = \{a \in \mathfrak{g}_{-1} \mid [a, \mathfrak{g}_1] = 0\}$ is a submodule of the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$. By the transitivity of $\mathfrak{g}$ we have $[\mathfrak{g}_{-1}, \mathfrak{g}_1] \neq 0$, therefore $V \neq \mathfrak{g}_{-1}$; consequently, $V = 0$. □

Definition 2.7. A Lie superalgebra is called simple if it contains no nontrivial ideals.

Proposition 2.4. *If in a simple $\mathbb{Z}$-graded Lie superalgebra $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ the subspace $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ generates $\mathfrak{g}$ then $\mathfrak{g}$ is bitransitive.*

Let $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ be a transitive, irreducible $\mathbb{Z}$-graded Lie superalgebra and suppose that the representation of $\mathfrak{g}_0$ on $\mathfrak{g}_1$ is irreducible. Let F_Λ be a highest weight vector of $\mathfrak{g}_{-1}$ and let E_M be a lowest weight vector of $\mathfrak{g}_1$. Then, as it is shown in [5, Proposition 1.2.10], either $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ are contragredient $\mathfrak{g}_0$-modules and $[F_\Lambda, E_M] = h \neq 0$ lies in the Cartan subalgebra of $\mathfrak{g}_0$, or $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ are not contragredient and $[F_\Lambda, E_M] = e_{-\alpha} \neq 0$ where $\alpha = -\Lambda - M$ is a root of the Lie algebra $[\mathfrak{g}_0, \mathfrak{g}_0]$. If α is a positive root then the transitivity of $\mathfrak{g}$ implies $[E_M, E_M] = 0$, since $[[E_M, E_M], F_\Lambda] = 2[E_M, e_{-\alpha}] = 0$. Consequently, for every positive root ρ of $\mathfrak{g}_0$, $[[E_M, e_\rho], E_M] = 0$ as it is clear applying the Jacobi identity for Lie superalgebras. In the same way, if $\mathfrak{g}$ is bitransitive and $(\Lambda, \alpha) = 0$, then $[F_\Lambda, F_\Lambda] = 0$ and, consequently, $[[F_\Lambda, e_{-\tau}], F_\Lambda] = 0$ for every positive root τ.

2.2. *On the growth of* $\mathfrak{g}$

Definition 2.8. Let $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ be a $\mathbb{Z}$-graded Lie superalgebra. The limit

$$r(\mathfrak{g}) = \lim_{n \to \infty} \ln \left(\sum_{i=-n}^{n} \dim \mathfrak{g}_i \right) \Big/ \ln(n)$$

is called the growth of $\mathfrak{g}$. If $r(\mathfrak{g})$ is finite we say that $\mathfrak{g}$ has finite growth.

Proposition 2.5. [4, Proposition 3] *If the growth of a subsuperalgebra or of a factor superalgebra of a Lie superalgebra $\mathfrak{g}$ is infinite, then the growth of $\mathfrak{g}$ is infinite.*

Theorem 2.1. [4, Theorem 2] *Let $L = \bigoplus L_i$ be a $\mathbb{Z}$-graded Lie algebra with the following properties*:

(a) *the Lie algebra L_0 has no center*;
(b) *the representations ϕ_{-1} and ϕ_1 of L_0 on L_{-1} and L_1 are irreducible*;
(c) $[L_{-1}, L_1] \neq 0$;
(d) $\Lambda + M = -\alpha$ *where Λ is the highest weight of ϕ_{-1}, M is the lowest weight of ϕ_1 and α is a positive root of L_0*;
(e) *the representations ϕ_{-1} and ϕ_1 are faithful*;
(f) *the growth of L is finite.*

Then L_0 is isomorphic to one of the Lie algebras A_n or C_n, ϕ_{-1} is the corresponding standard representation and α is the highest root of L_0.

Lemma 2.1. [4, Lemma 24] *Let $L = \bigoplus L_i$ be a graded Lie algebra, where L_0 is semisimple. Assume that there exist weight vectors x_λ and x_μ corresponding to the weights λ and μ of the adjoint representation of L_0 on L, and a root vector e_γ, corresponding to the root γ of L_0, which satisfy the following relations:*

$$[x_\mu, x_\lambda] = e_\gamma\,,$$
$$[x_\lambda, e_{-\gamma}] = 0 = [x_\mu, e_\gamma]\,,$$
$$\lambda(h_\gamma) \neq -1\,, \quad (\lambda, \gamma) \neq 0\,.$$

Then the growth of L is infinite.

Lemma 2.2. [4] *Let L be a Lie algebra containing elements $H \neq 0$, E_i, F_i, $i = 1, 2$, connected by the equations $[E_i, F_j] = \delta_{ij} H$, $[H, E_1] = aE_1$, $[H, E_2] = bE_2$, $[H, F_1] = -aF_1$, $[H, F_2] = -bF_2$, where $a \neq -b$, $b \neq -2a$, and $a \neq -2b$, then the growth of L is infinite.*

Lemma 2.3. [1, Lemma 1.13] *Let $\mathfrak{g}$ be a consistent, ℤ-graded Lie superalgebra and suppose that $\mathfrak{g}_0$ is a semisimple Lie algebra. Let E_i, F_i $(i = 1, 2)$ be odd elements and H a non zero element in $\mathfrak{g}_{\bar{0}}$ such that:*

$$[E_i, F_j] = \delta_{ij} H\,, \quad [H, E_i] = a_i E_i\,, \quad [H, F_i] = -a_i F_i\,, \tag{1}$$

where $a_1 \neq -a_2$, $a_1 \neq -2a_2$ and $a_2 \neq -2a_1$. Then the growth of $\mathfrak{g}$ is infinite.

Proof. Suppose first that $a_1 \neq 0 \neq a_2$. Then the elements $\tilde{E}_1 = a_1^{-1/2}[E_1, E_1]$, $\tilde{E}_2 = a_2^{-1/2}[E_2, E_2]$, $\tilde{F}_1 = {a_1}^{-1/2}[F_1, F_1]$, $\tilde{F}_2 = a_2^{-1/2}[F_2, F_2]$, $K = -4H$ satisfy the hypotheses of Lemma 2.2 in the Lie algebra $\mathfrak{g}_{\bar{0}}$. Thus, the growth of $\mathfrak{g}_{\bar{0}}$ is infinite and we get the thesis.

If, let us say, $a_1 \neq 0$, $a_2 = 0$ then the elements $E_1' = [E_1, E_1]$, $E_2' = [E_1, E_2]$, $F_1' = -(4a_1)^{-1}[F_1, F_1]$, $F_2' = a_1^{-1}[F_1, F_2]$, H satisfy the hypotheses of Lemma 2.2 in $\mathfrak{g}_{\bar{0}}$, thus we conclude. □

2.3. *Contragredient Lie superalgebras*

Let $I = \{1, 2, \ldots, n\}$, let $A = (a_{ij})_{i,j \in I}$ be a complex matrix, $\mathfrak{h}$ a complex vector space of dimension $n + \operatorname{corank}(A)$ and $\mathfrak{h}^*$ its dual space. Then there exist linearly independent indexed sets

$$\Pi = \{\alpha_i \mid i \in I\} \subseteq \mathfrak{h}^* \quad \text{and} \quad \Pi^\vee = \{h_i \mid i \in I\} \subseteq \mathfrak{h}$$

such that $\alpha_j(h_i) = a_{ij}$; these sets are determined by A up to isomorphism.

Let τ be a subset of I. We denote by $\tilde{\mathfrak{g}}(A, \tau)$ the Lie superalgebra generated by $\mathfrak{h}$ and $\{e_i, f_i\}_{i \in I}$ with the following defining relations:

$$[e_i, f_j] = \delta_{ij} h_i\,, \qquad [h, h'] = 0\,,$$
$$[h, e_i] = \alpha_i(h) e_i\,, \quad [h, f_i] = -\alpha_i(h) f_i\,,$$

$$\deg e_i = \deg f_i = \bar{0} \quad \text{for } i \notin \tau\,, \quad \deg e_i = \deg f_i = \bar{1} \quad \text{for } i \in \tau\,, \quad \deg h = \bar{0}\,,$$

where i, $j \in I$ and h, $h' \in \mathfrak{h}$. Let us recall the following well known fact [5, 11]:

Proposition 2.6. *Among the ideals of* $\tilde{\mathfrak{g}}(A,\tau)$ *intersecting* $\mathfrak{h}$ *trivially there exists a unique maximal ideal* $\mathfrak{r}$.

Let us define the contragredient Lie superalgebra $\mathfrak{g}(A,\tau)$ as the quotient $\tilde{\mathfrak{g}}(A,\tau)/\mathfrak{r}$. We denote $\mathfrak{h}$, e_i, f_i and their images in this quotient by the same letters. We call e_i and f_i ($i \in I$) the Chevalley generators of $\mathfrak{g}(A,\tau)$ and A its Cartan matrix.

We say that A is decomposable if I can be decomposed into a disjoint union of non-empty sets I_1 and I_2 such that $a_{ij} = a_{ji} = 0$ for $i \in I_1$ and $j \in I_2$, and indecomposable otherwise. If A is decomposable then $\mathfrak{g}(A,\tau)$ decomposes into the direct sum of contragredient Lie superalgebras associated to the indecomposable components of A.

Let $\mathfrak{c}$ be the center of $\mathfrak{g}(A,\tau)$. It is well known [11] that the derived superalgebra $\mathfrak{g}'(A,\tau) = [\mathfrak{g}(A,\tau), \mathfrak{g}(A,\tau)]$ of $\mathfrak{g}(A,\tau)$ contains $\mathfrak{c}$. In the following by a contragredient Lie superalgebra we shall mean any of the superalgebras $\mathfrak{g}(A,\tau)$, $\mathfrak{g}'(A,\tau)$, $\mathfrak{g}(A,\tau)/\mathfrak{c}$ and $\mathfrak{g}'(A,\tau)/\mathfrak{c}$.

Definition 2.9. [11] Let I and τ be as above. A complex matrix $A = (a_{ij})_{i,j\in I}$ is called a generalized Cartan matrix if it satisfies the following properties:

(1) if $a_{ii} = 0$ then $i \in \tau$;
(2) if $a_{ii} \neq 0$ then $a_{ii} = 2$;
(3) if $a_{ii} = 2$ then a_{ij} (resp. $\frac{1}{2}a_{ij}$) are nonpositive integers for $i \neq j$ and $i \notin \tau$ (resp. $i \in \tau$);
(4) $a_{ij} = 0$ implies $a_{ji} = 0$.

The matrix A is called symmetrizable if there exists an invertible diagonal matrix D such that DA is symmetric. In the following we shall only consider symmetrizable Cartan matrices.

The complete list of contragredient Lie superalgebras of finite growth can be found in [11]. Instead of describing the pairs (A,τ) it is more convenient, as in the Lie algebra case, to introduce Dynkin diagrams. We define three types of nodes, namely, white: ○, grey: ⊗ and black: ●. The i-th vertex of a Dynkin diagram is white or black if $a_{ii} = 2$ and $i \notin \tau$ or $i \in \tau$, respectively; it is grey if $a_{ii} = 0$ and $i \in \tau$. Two vertices are not joined if $a_{ij} = a_{ji} = 0$, otherwise they are joined by n_{ij} edges, where n_{ij} is defined as follows:

$$n_{ij} = \begin{cases} \max(-a_{ij}, -a_{ji}) & \text{if } a_{ii} = a_{jj} = 2\,; \\ 1 & \text{if } a_{ii} = a_{jj} = 0\,; \\ -a_{ij} & \text{if } a_{ii} = 2 \text{ and } a_{jj} = 0\,. \end{cases}$$

An arrow pointing from vertex j to vertex i is added in the following cases:

(1) $n_{ij} > 1$;
(2) $a_{ii} = a_{jj} = 2$ and $-a_{ij} > -a_{ji}$, or $a_{jj} = 0$.

These conditions do not define all Dynkin diagrams. Some extra-diagrams corresponding to 3×3-matrices need to be introduced. For the definition of these diagrams and the complete list of the Dynkin diagrams of contragredient Lie superalgebras of finite growth we refer to [11, Tables 1, 2, 3a, 3b, 3c].

Theorem 2.2. [11, Theorem 5.4] *Let A be an indecomposable symmetrizable matrix. Then $\mathfrak{g}(A,\tau)$ has finite growth if and only if (A,τ) is equivalent to $((0),\emptyset)$, $((0),\{1\})$, $(A,\emptyset)$, where A is a Cartan matrix either of a finite-dimensional simple Lie algebra or of an affine Lie algebra, or to one of the (A,τ), $\tau \neq \emptyset$, which can be deduced from one of the Dynkin diagrams in Tables* 2 *and* 3.

Remark 2.1. Let $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i$ be an irreducible, transitive, consistent $\mathbb{Z}$-graded Lie superalgebra such that $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ are contragredient $\mathfrak{g}_0$-modules. Then the Lie algebra $\mathfrak{g}_0$ is a direct sum of a semisimple Lie algebra $[\mathfrak{g}_0,\mathfrak{g}_0]$ and of a center $\mathbb{C}c$ which is at most one-dimensional. Let e_i, f_i, h_i, for $i = 1,\dots,n$, be the Chevalley generators of $[\mathfrak{g}_0,\mathfrak{g}_0]$, e_0 the lowest weight vector of the $[\mathfrak{g}_0,\mathfrak{g}_0]$-module $\mathfrak{g}_1$, f_0 the highest weight vector of the $[\mathfrak{g}_0,\mathfrak{g}_0]$-module $\mathfrak{g}_{-1}$, and $h_0 = [e_0,f_0]$. Then the elements e_i, f_i, h_i, for $i = 0,\dots,n$, generate a contragredient Lie superalgebra $\mathfrak{g}'$ such that $\mathfrak{g} = \mathfrak{g}' + \mathbb{C}c$. Under these assumptions the Dynkin diagram of $\mathfrak{g}'$ has a unique non-white node and the Dynkin diagram of the Lie algebra $[\mathfrak{g}_0,\mathfrak{g}_0]$ is obtained by removing this node.

2.4. *Classification theorems for $\mathbb{Z}$-graded Lie superalgebras of finite depth*

Let us recall the classification of $\mathbb{Z}$-graded Lie superalgebras of finite depth. We already noticed in the introduction that a $\mathbb{Z}$-graded Lie superalgebra of finite depth always has finite growth. For the notation and the definitions of the involved superalgebras we refer to [6]. Notice that a consistent $\mathbb{Z}$-graded Lie superalgebra $\mathfrak{g}$ of depth 1 is finite-dimensional since it can be embedded in $W(0,\dim\mathfrak{g}_{-1})$.

We shall adopt the following notation: Given a semisimple Lie algebra L, by $V(\omega)$ we shall denote its finite-dimensional highest weight module of highest weight ω. ω_i will be the fundamental weights. It is well known that if λ is a weight of a finite-dimensional representation of L and β is a root of L, then the set of weights of the form $\lambda + s\beta$ gives rise to a continuous string:

$$\lambda - p\beta,\ \lambda-(p-1)\beta,\ \dots,\ \lambda-\beta,\ \lambda,\ \lambda+\beta,\ \dots,\ \lambda+q\beta\,,$$

where p and q are nonnegative integers and $p - q = 2(\lambda,\beta)/(\beta,\beta)$. Let us define $2(\lambda,\beta)/(\beta,\beta) = \lambda(h_\beta)$. The numbers $\lambda(h_{\alpha_i})$, for a fixed basis of simple roots α_i, are called the *numerical marks* of the weight λ. For what concerns Lie algebras we will use the notation of [3, §11, §12]. In particular we shall adopt the same enumeration of the vertices of the Dynkin diagrams of simple Lie algebras and, where needed, we will refer to the bases of simple roots described by Humphreys [3].

sl_n, sp_n and so_n will denote the standard modules of these Lie algebras, sl_n^* will denote the module contragredient to the standard module and 1 the trivial 1-dimensional representation. Given two Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$, a $\mathfrak{g}_1$-module V_1 and a $\mathfrak{g}_2$-module V_2, we will denote by $V_1 \boxtimes V_2$ the outer tensor product of V_1 and V_2, i.e., the $\mathfrak{g}_1 \oplus \mathfrak{g}_2$-module $V_1 \otimes V_2$. As usual $S^k V$ and $\Lambda^k V$ will denote the k-th symmetric and exterior powers of the $\mathfrak{g}$-module V, and $\mathbb{C}(a)$ the 1-dimensional module over $\mathbb{C}$ such that $1 \mapsto a$.

Theorem 2.3. [5, Theorem 4] *The following is a complete list of transitive irreducible consistent $\mathbb{Z}$-graded Lie superalgebras $\mathfrak{g} = \bigoplus_{j=-1}^{k} \mathfrak{g}_j$ of depth 1 and $k \geq 1$:*

(1) *The $\mathfrak{g}_0$-modules $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ are contragredient and $k = 1$:*
   (a) $\mathrm{sl}(m,n)$, $m \neq n$, m, $n \geq 1$, ($\mathfrak{g}_0$-*module* $\mathfrak{g}_{-1} = \mathrm{gl}_m \boxtimes \mathrm{sl}_n$),
   (b) $\mathrm{sl}(n,n)$, $n \geq 2$, ($\mathfrak{g}_0$-*module* $\mathfrak{g}_{-1} = \mathrm{sl}_n \boxtimes \mathrm{sl}_n$),
   (c) $\mathrm{spo}(m,2)$, $m \geq 2$ *even*, ($\mathfrak{g}_0$-*module* $\mathfrak{g}_{-1} = \mathrm{csp}_m$),
   (d) $\mathbb{C}_0 + \mathfrak{g}$, *where* $\mathfrak{g}$ *is of type* 1(b).

(2) *The $\mathfrak{g}_0$-modules $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ are not contragredient and $k = 1$:*
   (a) $p(n)$, $n \geq 3$, ($\mathfrak{g}_0$-*module* $\mathfrak{g}_{\pm 1}$ (*resp.* $\mathfrak{g}_{\mp 1}$) $= S^2 \mathrm{sl}_n$, $\mathfrak{g}_{\pm 1} = S^2 \mathrm{sl}_n$ (*resp.* $\Lambda^2 \mathrm{sl}_n^*$)),
   (b) $\mathcal{P}[\xi] + \mathbb{C}(\frac{d}{d\xi})$, *where* $\mathcal{P}$ *is a simple Lie algebra* ($\mathfrak{g}_0 = \mathcal{P}$, $\mathfrak{g}_0$-*module* $\mathfrak{g}_{-1} = ad\,\mathcal{P}$ *and* $\mathfrak{g}_1 = 1$),
   (c) $\mathbb{C}_0 + \mathfrak{g}$ *where* $\mathfrak{g}$ *is of type* 2(a), (b).

(3) $k > 1$:
   (a) $W(0,n)$, $n \geq 3$, *with principal gradation* ($\mathfrak{g}_0$-*module* $\mathfrak{g}_{-1} = \mathrm{gl}_n$),
   (b) $S(0,n)$, $n \geq 4$, *with principal gradation* ($\mathfrak{g}_0$-*module* $\mathfrak{g}_{-1} = \mathrm{sl}_n$),
   (c) $H(0,n)$, $n \geq 5$, *with principal gradation* ($\mathfrak{g}_0$-*module* $\mathfrak{g}_{-1} = \mathrm{so}_n$),
   (d) $H'(0,n)$, $n \geq 4$, *with principal gradation* ($\mathfrak{g}_0$-*module* $\mathfrak{g}_{-1} = \mathrm{so}_n$),
   (e) $\mathbb{C}_0 + \mathfrak{g}$ *where* $\mathfrak{g}$ *is of type* (3)-(b), (d).

Theorem 2.4. [6, Theorem 5.3] *An even transitive irreducible consistent $\mathbb{Z}$-graded Lie superalgebra $\mathfrak{g} = \bigoplus_{j \geq -d} \mathfrak{g}_j$ of depth $d \geq 2$ is isomorphic to one of the following $\mathbb{Z}$-graded Lie superalgebras*:

(a) $K(1,n)$, $n \geq 1$, $n \neq 2$,
(b) $E(1,6)$ *and* $\mathrm{sl}_2 + S(1,2)$,
(c) $E(3,6)$,
(d) $E(3,8)$,
(e) $E(5,10)$,
(f) $E'(3,6)$ *and* $E'(3,8)$,
(g) $\mathbb{C}_0 + \mathfrak{g}$ *where* $\mathfrak{g}$ *is of type* (e) *or* (f).

3. Infinite Depth

3.1. *Preliminary results*

Let L be a finite-dimensional Lie superalgebra and let σ be an automorphism of L of finite order k. Then

$$L = \bigoplus_{i=0}^{k-1} L_i \tag{2}$$

where $L_i = \{x \in L \mid \sigma(x) = \epsilon^i x, \epsilon = e^{2\pi i/k}\}$. Decomposition (2) is a mod-k gradation of L.

Consider the Lie superalgebra $\mathbb{C}[x, x^{-1}] \otimes L = \bigoplus_{i\in\mathbb{Z}} x^i \otimes L$ and its subalgebra

$$G^k(L,\sigma) := \bigoplus_{i\in\mathbb{Z}} x^i \otimes L_{i(\mathrm{mod}\, k)}$$

called the *covering superalgebra* of L (see [4]). Then $G^k(L,\sigma)$ is a $\mathbb{Z}$-graded Lie superalgebra of infinite depth and growth 1.

Example 3.1 ([1], The Lie superalgebra $Q(n)^{(4)}_\sigma$). Let us consider the Lie superalgebra $Q(n)$ in Example 2.3 and the following automorphism σ of $Q(n)$ of order 4:

$$\sigma \begin{pmatrix} a & b \\ b & a \end{pmatrix} = \begin{pmatrix} -a^t & -ib^t \\ -ib^t & -a^t \end{pmatrix}.$$

We define $\mathfrak{g} = Q(n)^{(4)}_\sigma = G^4(Q(n),\sigma)$. Then $\mathfrak{g}_0 = \mathrm{so}_{n+1}$ and $\mathfrak{g}_{-1} = \mathrm{ad}\,\mathrm{so}(n+1)$. If $n = 2$ the $\mathfrak{g}_0$-module $\mathfrak{g}_1$ is the sl(2)-irreducible module of dimension 5 and if $n > 2$ it is the irreducible so$(n+1)$-module of highest weight $2\omega_1$.

Example 3.2 ([1], The Lie superalgebra $Q(2n-1)^{(4)}_\tau$). Let $m = 2n - 1$ and consider the following automorphism τ of order 4 of $Q(m)$:

$$\tau \left(\begin{array}{cc|cc} a & b & r & s \\ c & d & v & w \\ \hline r & s & a & b \\ v & w & c & d \end{array}\right) = \left(\begin{array}{cc|cc} -d^t & b^t & iw^t & -is^t \\ c^t & -a^t & -iv^t & ir^t \\ \hline iw^t & -is^t & -d^t & b^t \\ -iv^t & ir^t & c^t & -a^t \end{array}\right),$$

where a, b, c, d, r, s, v, w are $n \times n$-blocks and $\mathrm{tr}(r) + \mathrm{tr}(w) = 0$.

Let $\mathfrak{g} = Q(n)^{(4)}_\tau = G^4(Q(n),\tau)$. Then $\mathfrak{g}_0 = \mathrm{sp}(2n)$, $\mathfrak{g}_1 = \mathrm{ad}\,\mathrm{sp}(2n)$ and $\mathfrak{g}_{-1} = \Lambda^2_0\,\mathrm{sp}_{2n}$ (i.e., the highest component of the sp$(2n)$-module $\Lambda^2\,\mathrm{sp}_{2n}$).

Example 3.3 ([1], The Lie superalgebra $D(2,1;\alpha)^{(6)}$). Let $\alpha \in \mathbb{C}\backslash\{0,-1\}$ and let $D(2,1;\alpha)$ be the contragredient Lie superalgebra associated to the matrix

$$\begin{pmatrix} 0 & 1 & -1-\alpha \\ 1/\alpha & 0 & 1 \\ 1 & -\alpha/(1+\alpha) & 0 \end{pmatrix}$$

(see [5, Section 2.5.5]). Then the even part of $D(2,1;\alpha)$ is isomorphic to $A_1 \oplus A_1 \oplus A_1$ and its odd part is isomorphic to $\mathrm{sl}_2 \otimes \mathrm{sl}_2 \otimes \mathrm{sl}_2$.

If $\alpha^2+\alpha+1=0$ we can define the following automorphism φ of $D(2,1;\alpha)$ of order 6:

$$\begin{aligned} &\varphi(e_1)=-e_2\,, \quad \varphi(f_1)=-f_2\,, \quad \varphi(h_1)=h_2\,,\\ &\varphi(e_2)=-e_3\,, \quad \varphi(f_2)=-f_3\,, \quad \varphi(h_2)=h_3\,,\\ &\varphi(e_3)=-e_1\,, \quad \varphi(f_3)=-f_1\,, \quad \varphi(h_3)=h_1\,. \end{aligned}$$

We denote by $D(2,1;\alpha)^{(6)}$ the covering superalgebra $\mathfrak{g}=G^6(D(2,1;\alpha),\varphi)$. It turns out that $\mathfrak{g}_0=\mathrm{sl}(2)$, the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ is the $\mathrm{sl}(2)$-irreducible module of dimension 2 and $\mathfrak{g}_1$ is the $\mathrm{sl}(2)$-irreducible module of dimension 4.

Theorem 3.1. [1, Theorem 3.1] *Let $\mathfrak{g}=\bigoplus_{i\in\mathbb{Z}}\mathfrak{g}_i$ be a $\mathbb{Z}$-graded Lie superalgebra. Suppose that*:

(1) $\mathfrak{g}$ *is simple and generated by its local part*,
(2) *the $\mathbb{Z}$-gradation is consistent and has infinite depth*,
(3) $\mathfrak{g}_0$ *is simple*,
(4) $\mathfrak{g}_{-1}$ *and* $\mathfrak{g}_1$ *are irreducible $\mathfrak{g}_0$-modules which are not contragredient.*

Then $\mathfrak{g}$ has finite growth if and only if it is isomorphic to one of the following Lie superalgebras:

(a) $Q(n)^{(4)}_\sigma$, *where* $n\geq 2$, $n\neq 3$;
(b) $Q(2n-1)^{(4)}_\tau$, *where* $n\geq 3$;
(c) $D(2,1;\alpha)^{(6)}$, *where* $\alpha^2+\alpha+1=0$.

3.2. *The classification*

Let $\mathfrak{g}=\bigoplus_{i\in\mathbb{Z}}\mathfrak{g}_i$ be a consistent, irreducible, transitive, $\mathbb{Z}$-graded Lie superalgebra of infinite depth. Since the representation of $\mathfrak{g}_0$ on $\mathfrak{g}_{-1}$ is faithful and irreducible, $\mathfrak{g}_0=\mathfrak{g}_0'\oplus C$, where $\mathfrak{g}_0'=[\mathfrak{g}_0,\mathfrak{g}_0]$ is semisimple, C is the center of $\mathfrak{g}_0$, with $\dim(C)\leq 1$. Therefore the representation of $\mathfrak{g}_0$ on $\mathfrak{g}_1$ is completely reducible, i.e., $\mathfrak{g}_1$ can be decomposed into a direct sum of irreducible $\mathfrak{g}_0$-submodules: $\mathfrak{g}_1=\bigoplus_{s=1}^n\mathfrak{g}_1^{[s]}$. If C has dimension one then $C=\mathbb{C}c$ where $[c,x]=kx$ for $x\in\mathfrak{g}_k$. In the following we will denote by $\mathfrak{g}^{[s]}$ the minimal Lie subsuperalgebra of $\mathfrak{g}$ with local part $\mathfrak{g}_{-1}\oplus\mathfrak{g}_0'\oplus\mathfrak{g}_1^{[s]}$.

Proposition 3.1. *Suppose that*:

(1) $\mathfrak{g}$ *is simple and generated by its local part*;
(2) $\mathfrak{g}_0$ *is simple*;
(3) $\mathfrak{g}_1^{[s]}$ *is not contragredient to* $\mathfrak{g}_{-1}$ *for any* s;
(4) $\mathfrak{g}$ *has infinite depth and finite growth.*

Then $\mathfrak{g}_1$ is an irreducible $\mathfrak{g}_0$-module.

Table 1. Non-contragredient, transitive, irreducible, consistent, $\mathbb{Z}$-graded Lie superalgebras $\mathfrak{g}$ of finite growth such that $\mathfrak{g}_0$ is simple and $\mathfrak{g}_1$ is an irreducible $\mathfrak{g}_0$-module.

$\mathfrak{g}$	$\mathfrak{g}_0$	$\mathfrak{g}_{-1}$	$\mathfrak{g}_1$
$p(n)$ $(n \geq 2)$	A_n	$\Lambda^2\,\mathrm{sl}^*_{n+1}$	$S^2\,\mathrm{sl}_{n+1}$
$\mathcal{P}[\xi] + \mathbb{C}\dfrac{d}{d\xi}$	$\mathcal{P}$ simple	ad	1
$S(0, n+1)$ $(n \geq 3)$	A_n	sl_{n+1}	$V(\omega_1 + \omega_{n-1})$
$H(0, m)$ $(m \geq 5, m \neq 6)$	B_n $(m = 2n+1)$, D_n $(m = 2n)$	so_m	$\Lambda^3\,\mathrm{so}_m$
$E(5, 10)$	A_4	$\Lambda^2\,\mathrm{sl}_5$	$V(\omega_1 + \omega_2)$
$Q(2n)^{(4)}_\sigma$ $(n \geq 2)$	B_n	ad	$V(2\omega_1)$
$Q(2n-1)^{(4)}_\sigma$ $(n \geq 3)$	D_n	ad	$V(2\omega_1)$
$Q(2n-1)^{(4)}_\tau$ $(n \geq 3)$	C_n	$\Lambda^2_0\,\mathrm{sp}_{2n}$	ad
$Q(2)^{(4)}_\sigma$	A_1	ad	$V(4\omega)$
$D(2,1;\alpha)^{(6)}$ $(\alpha^2 + \alpha + 1 = 0)$	A_1	$V(\omega)$	$V(3\omega)$

Proof. Suppose that $\mathfrak{g}_1$ is not irreducible. Let F_Λ be a highest weight vector of the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$, and E_{M_s} a lowest weight vector of $\mathfrak{g}_1^{[s]}$. Since $\mathfrak{g}_1^{[s]}$ is not contragredient to $\mathfrak{g}_{-1}$ for any s, $\Lambda + M_s \neq 0$ and $[F_\Lambda, E_{M_s}] = e_{-\alpha_s}$, where $e_{-\alpha_s}$ is a root vector attached to the root $-\alpha_s$ of $\mathfrak{g}_0$. By the results recalled in Sec. 1, either the superalgebra $\mathfrak{g}^{[s]}$ is a bitransitive Lie superalgebra or it is isomorphic to the Lie superalgebra $\mathfrak{g}_0[\xi] + \mathbb{C}(\frac{d}{d\xi})$. Therefore the subalgebras $\mathfrak{g}^{[s]}$ are non-contragredient, consistent, $\mathbb{Z}$-graded Lie superalgebras of either finite or infinite depth with the same simple term of degree 0 and the same term of degree -1. We claim that all the $\mathfrak{g}_0$-modules $\mathfrak{g}_1^{[s]}$ are pairwise inequivalent. Suppose that among the $\mathfrak{g}_1^{[s]}$ there are two equivalent $\mathfrak{g}_0$-modules, say $\mathfrak{g}_1^{[1]}$ and $\mathfrak{g}_1^{[2]}$. Then $[F_\Lambda, E_{M_1}]$ and $[F_\Lambda, E_{M_2}]$ are root vectors of $\mathfrak{g}_0$ corresponding to the same root and are, therefore, proportional. Consequently, $[F_\Lambda, E_{M_1} - tE_{M_2}] = 0$ for some $t \in \mathbb{C}$, thus $[\mathfrak{g}_{-1}, E_{M_1} - tE_{M_2}] = 0$, and this contradicts the transitivity of $\mathfrak{g}$.

The possibilities for the subalgebras $\mathfrak{g}^{[s]}$ are collected in Table 1, where $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ can be interchanged. Therefore the only possible cases are the following:

(1) $\mathfrak{g}_1 = \mathfrak{g}_1^{[1]} \oplus \mathfrak{g}_1^{[2]}$, where $\mathfrak{g}^{[1]}$ and $\mathfrak{g}^{[2]}$ are isomorphic to $p(3)$, namely, $\mathfrak{g}_0$ is of type D_3, $\mathfrak{g}_{-1} \cong \mathrm{so}_6 \cong \Lambda^2\,\mathrm{sl}_4$, $\mathfrak{g}_1^{[1]} \cong V(2\omega_1) \cong S^2\,\mathrm{sl}_4$, $\mathfrak{g}_1^{[2]} \cong V(2\omega_3) \cong S^2\,\mathrm{sl}_4^*$. In this case $\mathfrak{g}$ is isomorphic to $H(0,6)$ (see [5, Proposition 3.3.6]) and, in particular, has finite depth.

(2) $\mathfrak{g}_1 = \mathfrak{g}_1^{[1]} \oplus \mathfrak{g}_1^{[2]}$, $\mathfrak{g}^{[1]}$ is isomorphic to $E(5,10)$ and $\mathfrak{g}^{[2]}$ is isomorphic to $p(4)$, namely the superalgebra $\mathfrak{g}$ has the following local part:

— $\mathfrak{g}_0$ of type A_4;
— $\mathfrak{g}_{-1} = \Lambda^2\,\mathrm{sl}_5 = V(\omega_2)$;
— $\mathfrak{g}_1 = \mathfrak{g}_1^{[1]} \oplus \mathfrak{g}_1^{[2]}$, where $\mathfrak{g}_1^{[1]}$ is the highest component of $\mathrm{sl}_5 \otimes \Lambda^2\,\mathrm{sl}_5$ and $\mathfrak{g}_1^{[2]} = S^2\,\mathrm{sl}_5^*$, i.e., $\mathfrak{g}_1^{[1]} = V(\omega_1 + \omega_2)$, $\mathfrak{g}_1^{[2]} = V(2\omega_4)$.

Let us denote by E_{M_1} a lowest weight vector of $\mathfrak{g}_1^{[1]}$, by E_{M_2} a lowest weight vector of $\mathfrak{g}_1^{[2]}$ and by F_Λ a highest weight vector of $\mathfrak{g}_{-1}$. We therefore have:

$$M_1 = -\omega_3 - \omega_4\,, \qquad \Lambda + M_1 = -\alpha_3 - \alpha_4$$
$$M_2 = -2\omega_1\,, \qquad \Lambda + M_2 = -\alpha_1\,.$$

Under these assumptions the superalgebra $\mathfrak{g}$ has infinite growth. In fact there exist a lowest weight vector x in $\mathfrak{g}_2$ and a highest weight vector y in $\mathfrak{g}_{-2}$, both of non-zero weight, such that $[x,y] = e_{-\alpha_1-\alpha_2-\alpha_3}$ and, since $\alpha_1+\alpha_2+\alpha_3$ is not the highest root of sl(5), Theorem 2.1 allows us to conclude. Vectors x and y can be chosen as follows:

$$x = [[E_{M_1}, e_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}], E_{M_2}] - \frac{1}{2}\,[[[E_{M_1}, e_{\alpha_1+\alpha_2+\alpha_3}], e_{\alpha_4}], E_{M_2}]\,,$$

$$y = -\frac{2}{3}\,[[[F_\Lambda, e_{-\alpha_2}], e_{-\alpha_1-\alpha_2-\alpha_3}], F_\Lambda]\,.$$

(3) $\mathfrak{g}_1 = \mathfrak{g}_1^{[1]} \oplus \mathfrak{g}_1^{[2]}$, where $\mathfrak{g}^{[1]}$ is the Lie superalgebra that we obtain from $H(0,5)$ exchanging $\mathfrak{g}_1$ with $\mathfrak{g}_{-1}$, and $\mathfrak{g}^{[2]}$ is $\mathrm{so}(5)[\xi] + \mathbb{C}\frac{d}{d\xi}$. More precisely $\mathfrak{g}$ has the following $\mathbb{Z}$-gradation: $\mathfrak{g}_0$ is of type B_2, $\mathfrak{g}_{-1} = \Lambda^3\,\mathrm{so}_5 \cong \mathrm{ad}\,\mathrm{so}_5$, $\mathfrak{g}_1^{[1]} = \mathrm{so}_5$ and $\mathfrak{g}_1^{[2]} = 1 = \langle\frac{d}{d\xi}\rangle$. Therefore, if we denote by F_Λ a highest weight vector of $\mathfrak{g}_{-1}$ (of weight $\Lambda = 2\omega_2$) and by E_M a lowest weight vector of $\mathfrak{g}_1^{[1]}$ (of weight $M = -\omega_1$), we have

$$[F_\Lambda, E_M] = \left[e_{\alpha_2}, \left[F_\Lambda, \frac{d}{d\xi}\right]\right] = e_{\alpha_1+2\alpha_2}\,.$$

In particular $\mathfrak{g}_2$ is isomorphic to the $\mathfrak{g}_0$-standard module and is generated by the lowest weight vector $[E_M, \frac{d}{d\xi}]$. Notice that

$$\left[F_\Lambda, \left[E_M, \frac{d}{d\xi}\right]\right] = [e_{\alpha_1+2\alpha_2}, E_M] \neq 0\,,$$

therefore $[\mathfrak{g}_{-1}, \mathfrak{g}_2] \subseteq \mathfrak{g}_1^{[1]}$ and, consequently, the subspace

$$\cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1^{[1]} \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3 \oplus \cdots$$

is a proper ideal of $\mathfrak{g}$. (We recall that, by definition of the superalgebra $\mathcal{P}[\frac{d}{d\xi}] + \mathbb{C}\frac{d}{d\xi}$, $[\mathfrak{g}_0, \mathfrak{g}_1^{[2]}] = 0$.) In particular $\mathfrak{g}$ is not simple.

(4) $\mathfrak{g}_1 = \mathfrak{g}_1^{[1]} \oplus \mathfrak{g}_1^{[2]}$, where $\mathfrak{g}^{[1]}$ is the Lie superalgebra that we obtain from $H(0,5)$ exchanging $\mathfrak{g}_1$ with $\mathfrak{g}_{-1}$, and $\mathfrak{g}^{[2]}$ is isomorphic to $Q(4)_\sigma^{(4)}$.

The $\mathbb{Z}$-gradation of $\mathfrak{g}$ is the following: $\mathfrak{g}_0$ is of type B_2, $\mathfrak{g}_{-1}$ is the $\mathfrak{g}_0$-adjoint module, $\mathfrak{g}_1^{[1]}$ is the so(5)-standard module and $\mathfrak{g}_1^{[2]}$ is the so(5)-module of highest weight $2\omega_1$. Let F_Λ be a highest weight vector of $\mathfrak{g}_{-1}$ and E_{M_1}, E_{M_2} lowest weight vectors of $\mathfrak{g}_1^{[1]}$ and $\mathfrak{g}_1^{[2]}$, respectively. Let us consider the following vectors:

$$x = [[F_\Lambda, e_{-\alpha_2}], [F_\Lambda, e_{-\alpha_2}]]$$

and

$$y = [[E_{M_1}, e_{\alpha_1}], E_{M_2}] \,.$$

Notice that x is a highest weight vector in $\mathfrak{g}_{-2}$ and y is a lowest weight vector in $\mathfrak{g}_2$. Besides, $[x,y]$ is proportional to $e_{-\alpha_2}$. Then Theorem 2.1 implies that the even part of $\mathfrak{g}$ has infinite growth since α_2 is not the highest root of the Lie algebra of type B_2; therefore $\mathfrak{g}$ itself has infinite growth.

(5) $\mathfrak{g}_1 = \mathfrak{g}_1^{[1]} \oplus \mathfrak{g}_1^{[2]}$, $\mathfrak{g}^{[1]}$ is of type $Q(m)_\sigma^{(4)}$ and $\mathfrak{g}^{[2]}$ is isomorphic to $\mathcal{P}[\xi]+\mathbb{C}\frac{d}{d\xi}$. In this case, $\mathcal{P} = \mathfrak{g}_0$ is of type B_n (if $m = 2n$), C_n (if $m = 2n-1$) or A_1 (if $m = 2$), $\mathfrak{g}_{-1}$ is the $\mathfrak{g}_0$-adjoint module and $\mathfrak{g}_1^{[2]} = 1$. Let us denote by F_Λ the highest weight vector of $\mathfrak{g}_{-1}$, by E_M the lowest weight vector of $\mathfrak{g}_1^{[1]}$ and by $\frac{d}{d\xi}$ the generator of $\mathfrak{g}_1^{[2]}$. We have that $\Lambda + M = -\alpha_1$, $\Lambda = \rho$, where ρ is the highest root of $\mathfrak{g}_0$, and $[\frac{d}{d\xi}, F_\Lambda] = e_\rho$. We notice that $\frac{d}{d\xi}$ acts as a derivation on the negative part of $\mathfrak{g}$ and this action can be extended to the whole algebra $\mathfrak{g}$ as follows: If $\mathfrak{g}_0$ is either of type B_n or D_n, then $(\mathfrak{g}_1^{[1]})^2$ is the highest weight module $V(2\omega_1)$ with lowest weight vector $y = [[[E_M, e_{\alpha_1+\alpha_2}], e_{\rho-\alpha_2}], E_M]$ (see [1]). If we compute the commutators $[y, F_\Lambda]$ and $[[E_M, \frac{d}{d\xi}], F_\Lambda]$, we find, using the transitivity of $\mathfrak{g}$, that $[E_M, \frac{d}{d\xi}]$ is proportional to y. Exactly the same procedure works when $\mathfrak{g}^{[1]}$ is of type $Q(2)_\sigma^{(4)}$. Therefore $\mathfrak{g}$ is the semidirect product of $\mathbb{C}\frac{d}{d\xi}$ with $\mathfrak{g}^{[1]}$. In particular it is not simple.

(6) $\mathfrak{g}_1 = \mathfrak{g}_1^{[1]} \oplus \mathfrak{g}_1^{[2]}$, $\mathfrak{g}^{[2]}$ is isomorphic to $\mathcal{P}[\xi] + \mathbb{C}\frac{d}{d\xi}$ with $\mathcal{P}$ of type C_n, and $\mathfrak{g}^{[1]}$ is the Lie superalgebra we obtain from $Q(2n-1)_\tau^{(4)}$ $(n \geq 3)$ by interchanging the $\mathfrak{g}_0$-modules $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ indicated in Table 1.

Using the same arguments as in point (5), one can show that $\mathfrak{g}$ is the semidirect product of $\mathbb{C}\frac{d}{d\xi}$ with $\mathfrak{g}^{[1]}$. We observe that in this case $\Lambda + M = \alpha_1$ is a positive root of $\mathfrak{g}_0$. □

The following results take into account the possibility that $\mathfrak{g}_1^{[s]}$ is contragredient to $\mathfrak{g}_{-1}$ for some s, i.e., that $\mathfrak{g}^{[s]}$ is a contragredient Lie superalgebra.

Lemma 3.1. *At most one of the $\mathfrak{g}_0$-modules $\mathfrak{g}_1^{[s]}$ is contragredient to $\mathfrak{g}_{-1}$, unless the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ is isomorphic to* cso_4 (*in this case $\mathfrak{g}$ has finite depth*).

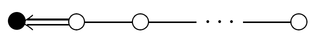
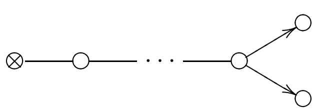
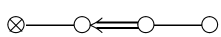
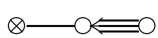
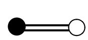
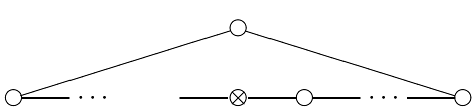
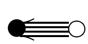
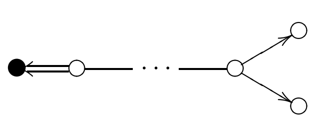
Proof. Let us suppose, by contradiction, that two modules, let us say $\mathfrak{g}_1^{[1]}$ and $\mathfrak{g}_1^{[2]}$, are contragredient to $\mathfrak{g}_{-1}$. Let $E'_{-\Lambda}$ and $E''_{-\Lambda}$ be their lowest weight vectors, let F_Λ be a highest weight vector of $\mathfrak{g}_{-1}$ and $h' = [E'_{-\Lambda}, F_\Lambda]$, $h'' = [E''_{-\Lambda}, F_\Lambda]$. If $h' = th''$, for some $t \in \mathbb{C}$, then $[E'_{-\Lambda} - tE''_{-\Lambda}, F_\Lambda] = 0$, i.e., by transitivity, $E'_{-\Lambda} = tE''_{-\Lambda}$. We conclude that $\mathfrak{g}_1^{[1]}$ and $\mathfrak{g}_1^{[2]}$ are inequivalent $\mathfrak{g}_0$-modules. We therefore have to compare non-isomorphic contragredient Lie superalgebras $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$ having the same terms of degree 0 and -1. We distinguish two cases:

Case 1: $[\mathfrak{g}_0, \mathfrak{g}_0]$ is simple.

Since we are assuming that $[\mathfrak{g}_0, \mathfrak{g}_0]$ is simple, among all the diagrams in [11, Tables 2, 3] we must select those with a non-white end point and no other non-white node. The complete list of these diagrams is given in Table 2.

Let us analyze the Dynkin diagrams contained in Table 2: The Dynkin diagram of $\mathfrak{g}_0'$ is obtained by removing the non-white node and the highest weight of the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ has non-zero labels corresponding to the nodes connected to the non-white node. Using the Cartan matrices corresponding to the listed diagrams, one can easily verify that $\mathfrak{g}^{[1]}$ and $\mathfrak{g}^{[2]}$ can be chosen from the following lists:

Table 2. Contragredient Lie superalgebras of finite growth with a non-white end-point. Every diagram has $n+1$ vertices indexed from 0 to n (in particular $n = 4$ for $F(4)$, $n = 3$ for $G(3)$, $n = 2$ for $A(0,2)^{(4)}$ and $B(0,1)^{(1)}$).

	$A(0,n)$
	$B(0,n+1)$
	$B(1,n)$
	$D(1,n)$
	$C(n+1)\quad (n>1)$
	$F(4)$
	$G(3)$
	$A(0,2)^{(4)}$
	$A(k-1,\ell)^{(1)}\quad (k \geq 1,\ k+\ell=n)$
	$B(0,1)^{(1)}$
	$B(0,n)^{(1)}$
	$A(0,2n-1)^{(2)}$
	$A(0,2n)^{(4)}$

(1) (i) $A(0,n)$
(ii) $B(0,n+1)$
(with $n>1$)
(2) (i) $A(0,1)$
(ii) $B(0,2)$
(iii) $B(0,1)^{(1)}$
(3) (i) $B(1,1)$
(ii) $A(0,2)^{(4)}$
(4) (i) $C(n+1)$ with $n>1$
(ii) $B(0,n)^{(1)}$
(5) (i) $B(1,n)$
(ii) $A(0,2n)^{(4)}$
(6) (i) $D(1,n)$
(ii) $A(0,2n-1)^{(2)}$.

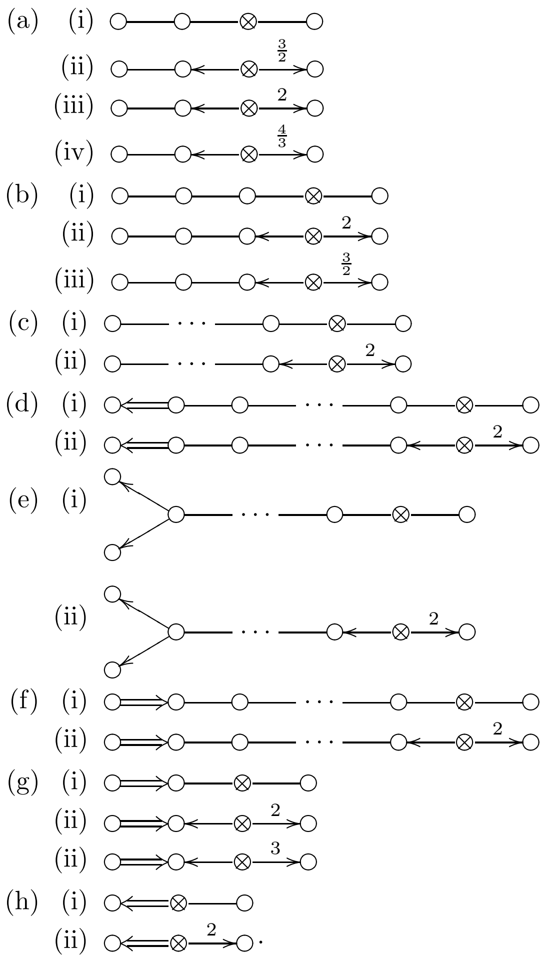
Thus, for any possible choice of $\mathfrak{g}^{[1]}$ and $\mathfrak{g}^{[2]}$, at least one of their diagrams, say the diagram of $\mathfrak{g}^{[2]}$, will have a black node. Consequently, if F_Λ is a lowest weight vector of $\mathfrak{g}_{-1}$, $[F_\Lambda, F_\Lambda]$ is a highest weight vector of $\mathfrak{g}_{-2}$. Indeed, let us suppose that the Dynkin diagrams in Table 2 are indexed from 0 to n, the index 0 corresponding to the non-white node of the diagram, and let us denote by $E_{-\Lambda}^{[i]}$ $(i=1,2)$ the lowest weight vector of $\mathfrak{g}_1^{[i]}$ such that $[F_\Lambda, E_{-\Lambda}^{[i]}] = h_0^{[i]}$. Then $[[F_\Lambda, F_\Lambda], E_{-\Lambda}^{[2]}] = 2[F_\Lambda, h_0^{[2]}] = -2\Lambda(h_0^{[2]})F_\Lambda = -4F_\Lambda \neq 0$, since the entry of the Cartan matrix corresponding to a black node is equal to 2. For the same reason the vectors $[E_{-\Lambda}^{[2]}, E_{-\Lambda}^{[2]}]$ and $[E_{-\Lambda}^{[1]}, E_{-\Lambda}^{[2]}]$ are lowest weight vectors in $\mathfrak{g}_2$.

Now we can show that $\mathfrak{g}$ has infinite growth in all the afore-listed cases using the following argument: We consider the Lie subalgebra $\bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_{4i}$ of $\mathfrak{g}_{\bar{0}}$ and its vectors $x = [F_\Lambda, [[F_\Lambda, F_\Lambda], [F_\Lambda, e_{-\alpha_1}]]]$ and $y = [[E_{-\Lambda}^{[1]}, E_{-\Lambda}^{[2]}], [E_{-\Lambda}^{[2]}, E_{-\Lambda}^{[2]}]]$. Then x is a highest weight vector in $\mathfrak{g}_{-4}$, y is a lowest weight vector in $\mathfrak{g}_4$ and $[x,y] = ae_{-\alpha_1}$, for some $a \in \mathbb{C}^*$, where either the rank of $\mathfrak{g}_0'$ is greater than one, and thus α_1 is not the highest root of $\mathfrak{g}_0'$, or $\mathfrak{g}_0'$ has rank one but $4\Lambda - \alpha_1$ is not the highest weight of the $\mathfrak{g}_0'$-standard module. Thus, by Theorem 2.1, $\mathfrak{g}$ has infinite growth.

Case 2: $[\mathfrak{g}_0, \mathfrak{g}_0]$ is semisimple (and not simple).

Let us decompose $[\mathfrak{g}_0, \mathfrak{g}_0]$ in a direct sum of simple Lie algebras: $[\mathfrak{g}_0, \mathfrak{g}_0] = \bigoplus_{i=1}^r \mathfrak{g}_{0i}$, for some $r>1$. As above we have to compare the diagrams of contragredient Lie superalgebras with a unique non-white node but in this case the non-white node is not an end-point of the diagram. Let us denote by $E_{-\Lambda}^{[i]}$ a lowest weight vector of $\mathfrak{g}_1^{[i]}$ and by F_Λ a highest weight vector of $\mathfrak{g}_{-1}$ such that $[F_\Lambda, E_{-\Lambda}^{[i]}] = h_\Lambda^{[i]}$. The following cases need to be studied:

(1) $\mathfrak{g}^{[1]}$ and $\mathfrak{g}^{[2]}$ are Lie superalgebras in the following lists (the number of vertices is strictly greater than 3):



Lie superalgebras (a)–(g) correspond to Cartan matrices of the following type:

$$\begin{pmatrix} \ddots & & \dots & \\ & 2 & -1 & 0 \\ \vdots & -\frac{1}{k} & 0 & 1 \\ & 0 & -1 & 2 \end{pmatrix}$$

and the Cartan matrices of superalgebras (h) are the following:

$$\begin{pmatrix} 2 & -2 & 0 \\ -\frac{1}{t} & 0 & 1 \\ 0 & -1 & 2 \end{pmatrix}$$

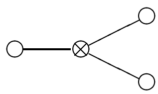
where $t = 1, 2$.

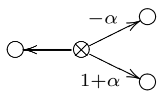
In all these cases we define

$$x = [[E'_{-\Lambda}, e_{\alpha_n}], E''_{-\Lambda}]\,,$$

$$y = [[F_\Lambda, e_{-\alpha_{n-2}}], [F_\Lambda, e_{-\alpha_n}]]\,,$$

where n is the number of vertices of the diagrams and $h_\Lambda = h_{n-1}$. The restriction of the weight of y to $\mathfrak{g}_{01}$ is $-a_{n-3n-2}\omega_{n-3}$ in cases (a)–(g) and it is equal to $2\omega_{n-2}$ in case (h). Then Theorem 2.1 implies that $\mathfrak{g}$ has infinite growth since x is a lowest weight vector in $\mathfrak{g}_2$, y is a highest weight vector in $\mathfrak{g}_{-2}$, and $[x, y] = (a'_{n-1n-2} - a''_{n-1n-2})e_{-\alpha_{n-2}}$.

(2) $\mathfrak{g}^{[1]} = D(2,2)$: 

$\mathfrak{g}^{[2]} = D(2,1;\alpha)^{(1)}$, $(\alpha \neq 0, -1)$: 

(the grey node is indexed by 2).

We recall that the Cartan matrices of the Lie superalgebras $D(2,2)$ and $D(2,1;\alpha)^{(1)}$ are

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 0 & 1 & 1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 0 & -\alpha & 1+\alpha \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix},$$

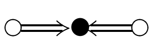
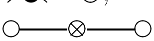
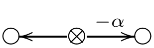
respectively.

We apply Theorem 2.1 with

$$x = [[[E''_{-\Lambda}, e_{\alpha_1}], [E''_{-\Lambda}, e_{\alpha_4}]], [[E'_{-\Lambda}, e_{\alpha_1}], E''_{-\Lambda}]]\,,$$

$$y = [[[F_\Lambda, e_{-\alpha_1}], [F_\Lambda, e_{-\alpha_3}]], [[F_\Lambda, e_{-\alpha_1}], [F_\Lambda, e_{-\alpha_4}]]]\,.$$

Then $[x, y]$ is proportional to the root vector $e_{-\alpha_3}$. Notice that the weight of y is $2\omega_3 + 2\omega_4$.

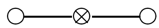
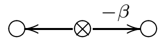
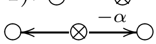
(3) $\mathfrak{g}^{[1]} = A(0,3)^{(2)}$: ,
$\mathfrak{g}^{[2]}$ is either $A(1,1)$:  or $D(2,1;\alpha)$: 

(the grey vertex is indexed by 2).

The Cartan matrices corresponding to $\mathfrak{g}^{[1]}$ and $\mathfrak{g}^{[2]}$ are

$$\begin{pmatrix} 2 & -1 & 0 \\ -2 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & -1 & 0 \\ \frac{1}{\alpha} & 0 & 1 \\ 0 & -1 & 2 \end{pmatrix}$$

respectively, where $\alpha = -1$ if $\mathfrak{g}^{[2]}$ is of type $A(1,1)$. Since the non-white node of $\mathfrak{g}^{[1]}$ is black the vectors $[F_\Lambda, F_\Lambda]$, $[E'_{-\Lambda}, E''_{-\Lambda}]$, $[E'_{-\Lambda}, E'_{-\Lambda}]$ are different from zero. Therefore we apply Theorem 2.1 with $x = [[E'_{-\Lambda}, E''_{-\Lambda}], [E'_{-\Lambda}, E'_{-\Lambda}]]$ and $y = [[F_\Lambda, F_\Lambda], [[F_\Lambda, e_{-\alpha_1}], F_\Lambda]]$ since we have $[x, y] = \sigma e_{-\alpha_1}$ for some $\sigma \neq 0$.

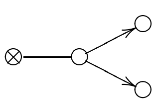
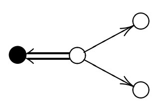
(4) $\mathfrak{g}^{[1]}$ is either $A(1,1)$:  or $D(2,1;\beta)$:  ;
$\mathfrak{g}^{[2]} = D(2,1;\alpha)$:  .
Then $\mathfrak{g}_{-1} = \mathrm{cso}_4$ and $\mathfrak{g}$ has finite depth, i.e., $\mathfrak{g} \cong K(1,4)$: This case is discussed in [6, Lemma 5.1]. □

Our next goal is to study the following situation: $\mathfrak{g}'_0$ is simple, $\mathfrak{g}_1 = \bigoplus_s \mathfrak{g}_1^{[s]}$ and $\mathfrak{g}_1^{[s]}$ is contragredient to $\mathfrak{g}_{-1}$ for exactly one s.

Theorem 3.2. *Let $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i$ be a consistent, irreducible, simple, $\mathbb{Z}$-graded Lie superalgebra of infinite depth and finite growth, generated by its local part and assume that $\mathfrak{g}$ is not contragredient. If $[\mathfrak{g}_0, \mathfrak{g}_0]$ is a simple Lie algebra then $\mathfrak{g}_1$ is an irreducible $\mathfrak{g}_0$-module.*

Proof. Suppose, by contradiction, that $\mathfrak{g}_1$ is not irreducible, i.e., that it can be decomposed into a direct sum of $\mathfrak{g}_0$-irreducible submodules: $\mathfrak{g}_1 = \bigoplus_{s=1}^t \mathfrak{g}_1^{[s]}$. By Proposition 3.1 and Lemma 3.1 exactly one of the $\mathfrak{g}_1^{[s]}$ is then contragredient to $\mathfrak{g}_{-1}$, say $\mathfrak{g}_1^{[2]}$. Therefore we need to compare the list of contragredient Lie superalgebras given in Table 2 with the list of non-contragredient Lie superalgebras given in Table 1. Let us denote by F_Λ a highest weight vector of $\mathfrak{g}_{-1}$, by E_M a lowest weight vector of $\mathfrak{g}_1^{[1]}$ and by $E_{-\Lambda}$ a lowest weight vector of $\mathfrak{g}_1^{[2]}$ so that $[F_\Lambda, E_{-\Lambda}] = h_0$ and $[F_\Lambda, E_M] = e_{-\alpha}$, where α is a negative root of $\mathfrak{g}'_0$ if $\mathfrak{g}^{[1]}$ is isomorphic to $\mathfrak{g}'_0[\xi] + \mathbb{C}\frac{d}{d\xi}$, and a positive root of $\mathfrak{g}'_0$ in all the other cases. The following situations need to be examined:

(1) $\mathfrak{g}^{[1]} \cong p(3)$, $\mathfrak{g}^{[2]}$ is a contragredient Lie superalgebra of type:

(a) $D(1,3)$:  or (b) $A(0,5)^{(2)}$: 

If $\mathfrak{g}^{[2]}$ is of type $D(1,3)$ then $\mathfrak{g}$ is isomorphic to $E(1,6)$.
If $\mathfrak{g}^{[2]}$ is of type $A(0,5)^{(2)}$ we notice that:

— $x = [F_\Lambda, F_\Lambda]$ is a highest weight vector in $\mathfrak{g}_{-2}$;
— $y = [E_M, E_{-\Lambda}]$ is a lowest weight vector in $\mathfrak{g}_2$;
— $[x, y] = -2\Lambda(h_0)e_{-\alpha_3} = -4e_{-\alpha_3}$, since the non-white node of $A(0,5)^{(2)}$ is black.

Therefore $\mathfrak{g}$ has infinite growth by Theorem 2.1.

(2) $\mathfrak{g}^{[1]} \cong \mathcal{P}[\xi] + \mathbb{C}\frac{d}{d\xi}$ and $\mathfrak{g}^{[2]}$ is a contragredient Lie superalgebra as follows:

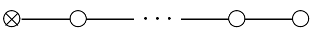
(a) $\mathcal{P}$ is of type A_n and $\mathfrak{g}^{[2]}$ is of type $A(k-1,\ell)^{(1)}$, with $k+\ell = n$, $k \geq 1$. Then, as in Proposition 3.1, cases (5) and (6), $\mathfrak{g}$ is isomorphic to the semidirect product of $\mathbb{C}\frac{d}{d\xi}$ with $\mathfrak{g}^{[2]}$.

(b) $\mathcal{P}$ is of type A_1 and $\mathfrak{g}^{[2]}$ is of type $B(1,1)$. Under these assumptions $\mathfrak{g}_{-1}$ is the sl(2)-adjoint module and, consequently, $\mathfrak{g}$ has finite depth. In fact

$\mathfrak{g}_{-2} \subseteq S^2\mathfrak{g}_{-1} = V(4\omega) + 1 = 1$ since the non-white node is grey and, by bitransitivity, $[F_\Lambda, F_\Lambda]$ is therefore equal to 0. Then $\mathfrak{g}$ is isomorphic to $K(1,3)$ (see [6, Lemma 5.5]).

(c) $\mathcal{P}$ is of type A_1 and $\mathfrak{g}^{[2]}$ is of type $A(0,2)^{(4)}$. In this case $[F_\Lambda, F_\Lambda]$ is a highest weight vector of $\mathfrak{g}_{-2}$ since the non-white node is black. Besides, $[E_{-\Lambda}, \frac{d}{d\xi}]$ is a lowest weight vector in $\mathfrak{g}_2$. Therefore we can consider the irreducible $\mathfrak{g}_0$-submodule $\mathfrak{h}_{-2}$ of $\mathfrak{g}_{-2}$ with highest weight vector $[F_\Lambda, F_\Lambda]$ and the irreducible $\mathfrak{g}_0$-submodule $\mathfrak{h}_2$ of $\mathfrak{g}_2$ with lowest weight vector $[E_{-\Lambda}, \frac{d}{d\xi}]$. Then Theorem 2.1 implies that the bitransitive Lie superalgebra with local part $\mathfrak{h}_{-2} + \mathfrak{g}'_0 + \mathfrak{h}_2$ has infinite growth (observe that, in order to apply Theorem 2.1, it is sufficient to interchange $\mathfrak{h}_{-2}$ with $\mathfrak{h}_2$).

(3) $\mathfrak{g}^{[1]} \cong S(0, n+1)$ $(n \geq 3)$, $\mathfrak{g}^{[2]}$ is a contragredient Lie superalgebra of type:

(a) $A(0,n)$: 

or

(b) $B(0,n+1)$: .

If $\mathfrak{g}^{[2]}$ is of type $A(0,n)$ then $\mathfrak{g}$ has depth 1. In fact $\mathfrak{g}_{-2} \subseteq S^2\mathfrak{g}_{-1} = V(2\omega_1) = 0$ since, by bitransitivity, $[F_\Lambda, F_\Lambda] = 0$. This case is discussed in [6, Lemma 5.5]. In particular $\mathfrak{g}$ has finite dimension.

If $\mathfrak{g}^{[2]}$ is of type $B(0, n+1)$ then $\mathfrak{g}$ has infinite growth by the same argument as in case (1)-(b) above, where $[x, y] = 4e_{-\alpha}$, $\alpha = \alpha_2 + \cdots + \alpha_n$ (here $\mathfrak{g}_0$ is of type A_n).

(4) $\mathfrak{g}^{[1]} \cong H(0,m)$ with $2n+1 = m \geq 5$ (resp. $2n = m \geq 5$, $m \neq 6$) and $\mathfrak{g}^{[2]}$ is a contragredient Lie superalgebra of type:

(a) $B(1,n)$ (resp. $D(1,n)$): 

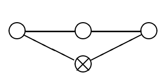
or

(b) $A(0,2n)^{(4)}$ (resp. $A(0,2n-1)^{(2)}$): .

If $\mathfrak{g}^{[2]}$ is of type $B(1,n)$ or $D(1,n)$ then $\mathfrak{g}_{-2} \subseteq S^2\mathfrak{g}_{-1} = 1$ and $\mathfrak{g}$ has finite depth, namely $\mathfrak{g}$ is isomorphic to $K(1,n)$ (see [6, Lemma 5.5]).

If $\mathfrak{g}^{[2]}$ is of type $A(0,2n)^{(4)}$ then $\mathfrak{g}$ has infinite growth by the same argument as in case (1)-(b) above, with $\alpha = \alpha_2 + 2\alpha_3 + \cdots + 2\alpha_n$ (here $\mathfrak{g}_0$ is of type B_n (resp. D_n)).

(5) $\mathfrak{g}^{[1]} \cong Q(5)^{(4)}_\sigma$ and $\mathfrak{g}^{[2]}$ is the contragredient Lie superalgebra of type $A(0,2)^{(1)}$:



Let 0 be the index of the grey node, and let us consider the following vectors:

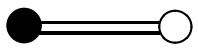
$$x_1 = [[F_\Lambda, e_{-\alpha_1-\alpha_2}], [F_\Lambda, e_{-\alpha_3}]]\,,$$

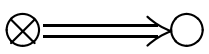
$$x_2 = [[F_\Lambda, e_{-\alpha_1}], [F_\Lambda, e_{-\alpha_2-\alpha_3}]]\,,$$

$$y = [[E_M, e_{\alpha_2}], [E_{-\Lambda}, e_{\alpha_1}]]\,.$$

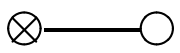
Then y is a lowest weight vector in $\mathfrak{g}_2$, $z = x_1 + x_2$ is a highest weight vector in $\mathfrak{g}_{-2}$ and $[z, y]$ is proportional to $e_{-\alpha_2-\alpha_3}$. Thus, by Theorem 2.1, $\mathfrak{g}$ has infinite growth.

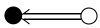
(6) (a) $\mathfrak{g}^{[1]} \cong Q(2)^{(4)}_{\sigma}$, $\mathfrak{g}^{[2]}$ is one of the following contragredient Lie superalgebras:

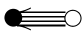
$A(0,2)^{(4)}$: 

$B(1,1)$:  .

(b) $\mathfrak{g}^{[1]} \cong D(2,1;\alpha)^{(6)}$, $\mathfrak{g}^{[2]}$ is one of the following contragredient Lie superalgebras:

$A(0,1)$: 

$B(0,2)$: 

$B(0,1)^{(1)}$:  .

In all these cases the growth of $\mathfrak{g}$ is infinite. In fact:

— $x = [F_\Lambda, F_\Lambda]$ is a highest weight vector in $\mathfrak{g}_{-2}$ by bitransitivity, since $\Lambda(h_\alpha) \neq 0$ (see Example 3.3);
— $y = [E_M, E_{-\Lambda}]$ is a lowest weight vector in $\mathfrak{g}_2$,
— $[x, y] = \tau e_{-\alpha}$ for some $\tau \in \mathbb{C}$, $\tau \neq 0$, where α is the highest root of $\mathfrak{g}'_0$ but the irreducible $\mathfrak{g}'_0$-submodule of $\mathfrak{g}_{-2}$ generated by x is not isomorphic to the standard module. □

Lemma 3.2. *Let* $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i$ *be a transitive* $\mathbb{Z}$*-graded Lie algebra with a consistent gradation. Suppose that*:

(1) $\mathfrak{g}_{-1}$ *and* $\mathfrak{g}_1$ *are irreducible* $\mathfrak{g}_0$*-modules*;
(2) F_Λ *is a highest weight vector of* $\mathfrak{g}_{-1}$, E_M *is a lowest weight vector of* $\mathfrak{g}_1$ *and* $\Lambda + M = -\alpha$ *for some root* α *of* $[\mathfrak{g}_0, \mathfrak{g}_0]$, *i.e.*, $\mathfrak{g}_1$ *and* $\mathfrak{g}_{-1}$ *are not contragredient* $[\mathfrak{g}_0, \mathfrak{g}_0]$*-modules.*

Then, *if* $(\Lambda, \alpha) \neq 0$ *and* $\mathfrak{g}$ *has finite growth*, $[\mathfrak{g}_0, \mathfrak{g}_0]$ *is a simple Lie algebra.*

Proof. Let $[\mathfrak{g}_0, \mathfrak{g}_0] = \bigoplus_{i=1}^k \mathfrak{g}_{0i}$ and assume that α is a positive root. Let α be a root of $\mathfrak{g}_{01}$ and let β be a simple root of $\mathfrak{g}_{0i}$ for some $i \neq 1$. Suppose $\Lambda(h_\beta) \neq 0$ and consider the following vectors:

$$E_1 = [e_\alpha, E_M], \quad E_2 = [[E_M, e_\alpha], e_\beta],$$
$$F_1 = F_\Lambda, \quad F_2 = \Lambda(h_\beta)^{-1}[F_\Lambda, e_{-\beta}],$$
$$H = h_\alpha.$$

Then the conditions of Lemma 2.3 are satisfied since $\alpha + \beta$ and $\alpha - \beta$ are not roots, and $a_1 = a_2 = -\Lambda(h_\alpha)$, thus $\mathfrak{g}$ has infinite growth. Therefore $\Lambda_i := \Lambda|_{\mathfrak{g}_{0i}} = 0$ for every $i \neq 1$, i.e., by the transitivity of $\mathfrak{g}$, $[\mathfrak{g}_0, \mathfrak{g}_0]$ is simple. □

Lemma 3.3. *If $\mathfrak{g}$ satisfies hypotheses* (1) *and* (2) *of Lemma* 3.2 *and* $(\Lambda, \alpha) = 0$ *then α is the highest root of one of the parts of the Dynkin diagram of $\mathfrak{g}_0'$ into which it is divided by the numerical marks of Λ.*

Proof. See [5, Lemma 4.1.6(b)] and [1, Lemma 2.4]. □

Lemma 3.4. *Let $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ be a bitransitive Lie superalgebra satisfying hypotheses* (1) *and* (2) *of Lemma* 3.2. *Then either $[\mathfrak{g}_0, \mathfrak{g}_0]$ is simple or $\mathfrak{g} \cong E'(3, 6)$.*

Proof. Decompose $[\mathfrak{g}_0, \mathfrak{g}_0]$ in a direct sum of simple Lie algebras and consider the corresponding decompositions of weights Λ and M:

$$[\mathfrak{g}_0, \mathfrak{g}_0] = \bigoplus_{i=1}^{r} \mathfrak{g}_{0i}\,, \quad \Lambda = \sum_{i=1}^{r} \Lambda_i\,, \quad M = \sum_{i=1}^{r} M_i\,.$$

Let $\alpha = -\Lambda - M$ be a positive root of $\mathfrak{g}_{01}$. Then either $\Lambda_i = 0$ for every $i \neq 1$, i.e., by transitivity, $[\mathfrak{g}_0, \mathfrak{g}_0]$ is simple, or $r > 1$ and for some $i \neq 1$, say $i = 2$, $\Lambda_i \neq 0$. Then, by Lemmas 3.2 and 3.3, $(\Lambda, \alpha) = 0$ and α is the highest root of one of the parts of the Dynkin diagram of $\mathfrak{g}_{01}$ into which it is divided by the numerical marks of Λ. Again, using the transitivity of $\mathfrak{g}$, we can suppose that $\Lambda_1 \neq 0$, and this implies that α is not the highest root of $\mathfrak{g}_{01}$. Then, let β and γ be simple roots of $\mathfrak{g}_{01}$ and $\mathfrak{g}_{02}$ respectively, such that $\Lambda(h_\beta) \neq 0 \neq \Lambda(h_\gamma)$ and $\alpha + \beta$ is a root of $\mathfrak{g}_{01}$.

Consider $x = [[[F_\Lambda, e_{-\beta}], e_{-\gamma}], F_\Lambda]$. Then x is a highest weight vector in $\mathfrak{g}_{-2}$. Indeed, $[F_\Lambda, F_\Lambda] = 0$, by the bitransitivity of $\mathfrak{g}$, since $(\Lambda, \alpha) = 0$ and, as a consequence of the Jacobi identity for Lie superalgebras, $[[F_\Lambda, e_\tau], F_\Lambda] = 0$ for every positive root τ. Besides, $y = [[[E_M, e_\alpha], e_\gamma], E_M]$ is a lowest weight vector in $\mathfrak{g}_2$ and $[x, y] = 2\Lambda(h_\gamma)e_{-\alpha-\beta}$. Notice that the weights of x and y restricted to $\mathfrak{g}_{01}$ are different from 0, indeed: $(2\Lambda - \beta)(h_\alpha) = -\beta(h_\alpha) \neq 0$ and $(2M + \alpha)(h_\alpha) = -\alpha(h_\alpha) = -2$. Let $\mathfrak{h}_{-2}$ and $\mathfrak{h}_2$ the $[\mathfrak{g}_0, \mathfrak{g}_0]$-irreducible submodules of $\mathfrak{g}_{-2}$ and $\mathfrak{g}_2$ with highest and lowest weight vectors x and y, respectively, and let $\mathfrak{h}$ be the bitransitive Lie superalgebra with local part $\mathfrak{h}_{-2} \oplus \mathfrak{g}_{01} \oplus \mathfrak{h}_2$. Then $\mathfrak{h}$ has infinite growth unless $\mathfrak{g}_{01}$ is either of type A_n or C_n, $\alpha + \beta$ is the highest root of $\mathfrak{g}_{01}$, $\mathfrak{h}_{-2}$ is the $\mathfrak{g}_{01}$-standard module. Now, since $\alpha + \beta$ must be the highest root of $\mathfrak{g}_{01}$, the root β must correspond to an end point of the Dynkin diagram of $\mathfrak{g}_{01}$. A brief analysis of the possibilities shows that $[\mathfrak{g}_0, \mathfrak{g}_0] = \mathrm{sl}(3) \oplus \mathrm{sl}(2)$, $\mathfrak{g}_{-1} = \mathrm{sl}_3 \boxtimes \mathrm{sl}_2$ and $\mathfrak{g}_1 = S^2\, \mathrm{sl}_3 \boxtimes \mathrm{sl}_2^*$, therefore $\mathfrak{g} \cong E'(3, 6)$ (see [6]). □

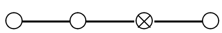
Theorem 3.3. *Let $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ be a consistent, irreducible, simple, $\mathbb{Z}$-graded Lie superalgebra of finite growth, generated by its local part. Suppose that $\mathfrak{g}$ is not contragredient. Then either $\mathfrak{g}$ has finite depth or $\mathfrak{g}_0$ is simple, $\mathfrak{g}_1$ is an irreducible $\mathfrak{g}_0$-module and $\mathfrak{g}$ is isomorphic to one of the Lie superalgebras listed in Theorem* 3.1.

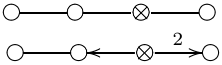
Proof. Due to Theorem 3.2 and the classifications in [1, 6], we only need to show that if the depth of $\mathfrak{g}$ is infinite and $\mathfrak{g}_0'$ is not simple then the growth of $\mathfrak{g}$ is infinite. Let us then suppose that $\mathfrak{g}_0'$ is not simple.

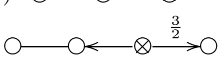
Let $\mathfrak{g}_1 = \bigoplus_{s=1}^{t} \mathfrak{g}_1^{[s]}$ be the decomposition of $\mathfrak{g}_1$ in a direct sum of irreducible $\mathfrak{g}_0$-modules. Let F_Λ be a highest weight vector of the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ and E_{M_s} a lowest weight vector of the $\mathfrak{g}_0$-module $\mathfrak{g}_1^{[s]}$. Since $\mathfrak{g}_1$ is not contragredient to $\mathfrak{g}_{-1}$ there exists at least one s such that $\Lambda + M_s \neq 0$ therefore, by Lemma 3.4 and its proof, applied to the Lie superalgebra $\mathfrak{g}^{[s]}$ with local part $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1^{[s]}$, we have $[\mathfrak{g}_0, \mathfrak{g}_0] = \mathrm{sl}(3) \oplus \mathrm{sl}(2)$ and $\mathfrak{g}_{-1} = \mathrm{sl}_3 \boxtimes \mathrm{sl}_2$.

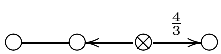
Now, if $\mathfrak{g}_1$ is irreducible, then $\mathfrak{g}$ is isomorphic to $E'(3,6)$ and has therefore finite depth; otherwise $\mathfrak{g}_1^{[s]}$ is contragredient to $\mathfrak{g}_{-1}$ for one s, namely, $[\mathfrak{g}_0, \mathfrak{g}_0] = \mathrm{sl}(3) \oplus \mathrm{sl}(2)$, $\mathfrak{g}_{-1} = \mathrm{sl}_3 \boxtimes \mathrm{sl}_2$ and $\mathfrak{g}_1 = \mathfrak{g}_1^{[1]} \oplus \mathfrak{g}_1^{[2]}$, where $\mathfrak{g}_1^{[1]} = S^2\,\mathrm{sl}_3 \boxtimes \mathrm{sl}_2$ and $\mathfrak{g}^{[2]}$ is a contragredient Lie superalgebra. Let us denote by E_M a lowest weight vector in $\mathfrak{g}_1^{[1]}$ and by $E_{-\Lambda}$ a lowest weight vector in $\mathfrak{g}_1^{[2]}$.

The following possibilities for the Lie superalgebra $\mathfrak{g}^{[2]}$ may then occur:

(1) $A(2,1)$: 

(2) $D(3,1)$: 

(3) $F(4)$: 

(4) $G(3)^{(2)}$: .

In cases (1) and (2) the Lie superalgebra $\mathfrak{g}$ is isomorphic to $E(3,8)$ and $E(3,6)$, respectively, and has therefore finite depth. Now we want to show, using Lemma 2.1, that in case $\mathfrak{g}^{[2]}$ is of type $F(4)$ or $G(3)^{(2)}$ then the growth of $\mathfrak{g}$ is infinite.

We recall that the Cartan matrix associated to the Lie superalgebra $\mathfrak{g}^{[2]}$ is:

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -\frac{1}{k} & 0 & 1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

where $k = \frac{3}{2}$ for $F(4)$ and $k = \frac{4}{3}$ for $G(3)^{(2)}$. Besides, we notice that $[F_\Lambda, F_\Lambda] = 0 = [E_M, E_{-\Lambda}]$ since the non-white node of the Dynkin diagram of $\mathfrak{g}^{[2]}$ is always grey.

We now apply Lemma 2.1 as follows:

(i) $\mathfrak{g}^{[2]}$ of type $F(4)$.

We define the following vectors:

$$A = [[F_\Lambda, e_{-\alpha_4}], [F_\Lambda, e_{-\alpha_2}]], \qquad B = [[F_\Lambda, e_{-\alpha_4}], [F_\Lambda, e_{-\alpha_1-\alpha_2}]],$$
$$x_1 = [[E_M, e_{\alpha_4}], E_{-\Lambda}], \qquad x_3 = [[E_{-\Lambda}, e_{\alpha_2}], [E_{-\Lambda}, e_{\alpha_4}]].$$

Using bitransitivity one can show that A is a highest weight vector in $\mathfrak{g}_{-2}$, $[B, A]$ is a highest weight vector in $\mathfrak{g}_{-4}$, and x_1, x_3 are lowest weight vectors in $\mathfrak{g}_2$. We define, using the same notation as in Lemma 2.1,

$$x_\lambda = [A, [B, A]]$$
$$x_\mu = [x_3, [x_1, [e_{\alpha_1}, x_3]]].$$

We notice that $[x_\lambda, x_\mu] = \tau e_{-\alpha_1-\alpha_2}$ for some $\tau \neq 0$, therefore we can apply Lemma 2.1 to the Lie algebra $[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{0}}]$ thus obtaining that $\mathfrak{g}$ has infinite growth, since $\lambda(h_{\alpha_1+\alpha_2}) = (6\Lambda - \alpha_1 - 3\alpha_2 - 3\alpha_4)(\alpha_1 + \alpha_2) = 2$.

(ii) $\mathfrak{g}^{[2]}$ of type $G(3)^{(2)}$.

Using the same notation as in the previous case, we define:

$$x_{\lambda'} = [x_\lambda, A]\,,$$

$$x_{\mu'} = [x_3, x_\mu]\,.$$

Then $[x_{\lambda'}, x_{\mu'}] = \sigma e_{-\alpha_1-\alpha_2}$, for some $\sigma \neq 0$, and the hypotheses of Lemma 2.1 are satisfied. As in the previous case $\mathfrak{g}$ has infinite growth since $\lambda'(h_{\alpha_1+\alpha_2}) = (8\Lambda - \alpha_1 - 4\alpha_2)(\alpha_1 + \alpha_2) = 3$. □

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References

[1] N. Cantarini, ℤ-graded Lie superalgebras of infinite depth and finite growth, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (2002), to appear.

[2] V. W. Guillemin and S. Sternberg, An algebraic model of transitive differential geometry, *Bull. Amer. Math. Soc.* **70** (1964) 16–47.

[3] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Graduate Texts in Mathematics **9** (Springer-Verlag, New York-Berlin, 1978).

[4] V. G. Kac, Simple irreducible graded Lie algebras of finite growth, *Math. USSR — Izvestija* **2**(6) (1968) 1271–1311.

[5] V. G. Kac, Lie superalgebras, *Adv. Math.* **26** (1977) 8–96.

[6] V. G. Kac, Classification of infinite-dimensional simple linearly compact Lie superalgebras, *Adv. Math.* **139** (1998) 1–55.

[7] V. G. Kac, *Infinite Dimensional Lie Algebras* (Cambridge University Press, Boston, 1990).

[8] O. Mathieu, Classification of simple graded Lie algebras of finite growth, *Invent. Math.* **108** (1992) 455–519.

[9] A. L. Onishchik and E. B. Vinberg, *Lie Groups and Algebraic Groups* (Springer-Verlag, New York-Berlin, 1990).

[10] J. W. van de Leur, Contragredient Lie superalgebras of finite growth, Ph.D Thesis, University of Utrecht, The Netherlands (1986).

[11] J. W. van de Leur, A classification of contragredient Lie superalgebras, *Comm. Algebra* **17**(8) (1989) 1815–1841.