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Mod-p Reduction for Quantum Groups

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Let $\mathscr{U}_\epsilon(\mathscr{G})$ be the quantized enveloping algebra associated to the Lie algebra $\mathscr{G} = sl(n+1)$ at a pth-root of unity ε and assume that p is a prime which does not divide $n+1$. It is known that the irreducible, finite dimensional representations of $\mathscr{U}_\varepsilon(\mathscr{G})$ are parametrized, up to isomorphisms, by the conjugacy classes of $\mathrm{SL}(n+1)$. In the paper we prove that the dimension of any $\mathscr{U}_\varepsilon(\mathscr{G})$-module M parametrized by a conjugacy class $\mathscr{O}$ is divided by $p^{1/2 \dim(\mathscr{O})}$. This result was conjectured by C. De Concini, V. G. Kac, and C. Procesi (*J. Amer. Math. Soc.* **5**, 1992, 151–190).

INTRODUCTION

Let $\mathscr{U}_q(\mathscr{G})$ be the quantized enveloping algebra associated to a simple, finite dimensional Lie algebra $\mathscr{G}$ according to the definition given by Drinfeld [D2] and Jimbo [J], and let $\mathscr{U}_\varepsilon(\mathscr{G})$ be the algebra over $\mathbf{C}$ obtained from $\mathscr{U}_q(\mathscr{G})$ by specializing q at a primitive lth-root of unity ε, l being a fixed odd integer strictly greater than 1. In the papers [DC-K1, DC-K-P1] it is shown how to parametrize the irreducible, finite dimensional representations of $\mathscr{U}_\varepsilon(\mathscr{G})$ by the conjugacy classes of the algebraic group G with Lie algebra $\mathscr{G}$ and trivial center. In particular it was conjectured that the dimension of any irreducible, finite dimensional $\mathscr{U}_\varepsilon(\mathscr{G})$-module, corresponding to a conjugacy class $\mathscr{O}$, is divisible by $l^{1/2 \dim(\mathscr{O})}$. The main purpose of this paper is to prove this conjecture for the irreducible representations of the quantized enveloping algebra associated to $\mathscr{G} = sl(n+1)$.

We recall that the regular case, that is, the case of the representations corresponding to a conjugacy class of maximal dimension, was proved by

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De Concini, Kac, and Procesi in [DC-K-P2]. According to [DC-K2], any irreducible representation of $\mathscr{U}_\varepsilon(sl(n+1))$ is either unipotent (see Section 2 for the definition) or induced by a unipotent one. Therefore one can reduce to the study of the unipotent representations (Section 2).

The above conjecture is the quantum-analogue of a well known conjecture which was formulated by Kac and Weisfeiler in [K-W1] and proved by Premet in [P]. It states that if G is a simple and simply connected algebraic group defined over an algebraically closed field, whose characteristic $p > 0$ is good for G (see Section 3) and such that $\mathscr{G} = \mathrm{Lie}(G)$ admits a non-degenerate G-invariant trace form, then any irreducible $\mathscr{G}$-module V has dimension divisible by $p^{1/2\dim(\Omega(\chi))}$, where $\chi \in \mathscr{G}^*$ is the p-character of V and $\Omega(\chi)$ is its coadjoint orbit in $\mathscr{G}^*$. We recall that, also for the Kac–Weisfeiler conjecture, a theorem of reduction to the nilpotent case, similar to the theorem mentioned above, had been proved in [K-W1].

We make use of Premet's theorem to obtain our result. More precisely, we fix a prime p and a primitive pth-root of unity ε and, starting from an irreducible $\mathscr{U}_\varepsilon(sl(n+1))$-module M parametrized by a unipotent conjugacy class $\mathscr{O}$ of $\mathrm{SL}(n+1)$, we construct a representation $\overline{M}$ of the Lie algebra $sl(n+1)$ over an algebraically closed field of characteristic p, such that $\dim(\overline{M}) = \dim(M)$ and the dimension of $\mathscr{O}$ is equal to the dimension of the coadjoint orbit of the p-character of $\overline{M}$. In order to realize this construction we will be considering the "limiting" specialization $\mathscr{U}_1$ of $\mathscr{U}_q(\mathscr{G})$ (Section 1). Besides, taking into account Premet's results [P], we shall assume $p \nmid n+1$. One fundamental step in our construction consists in proving that any irreducible, finite dimensional, unipotent $\mathscr{U}_\varepsilon(sl(n+1))$-module is a quotient of an induced module (Section 3).

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1. DEFINITIONS AND NOTATIONS

Let $(a_{ij})_{i,j=1,\dots,n}$ be a symmetric Cartan matrix and let $\mathscr{G}$ be the associated simple, finite dimensional Lie algebra. By Q and Q^+ we shall denote the sets of roots and positive roots, respectively, with $N = |Q^+|$, by R the root lattice, and by Δ a fixed set of simple roots.

Definition 1.1. The quantized enveloping algebra $\mathscr{U}_q(\mathscr{G})$ associated to $(a_{ij})_{i,j=1}^n$ is the associative algebra over $\mathbf{C}(q)$ with generators $E_i, F_i, K_i^{\pm 1}$, for $i = 1, \dots, n$, and the following defining relations:

$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}} \tag{1.1}$$

$$K_i K_j = K_j K_i = K_{i+j}, \qquad K_i K_i^{-1} = K_i^{-1} K_i = 1 \tag{1.2}$$

$$K_i E_j K_i^{-1} = q^{a_{ij}} E_j, \qquad K_i F_j K_i^{-1} = q^{-a_{ij}} F_j \tag{1.3}$$

$$\sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1 - a_{ij} \\ s \end{bmatrix} E_i^{1-a_{ij}-s} E_j E_i^s = 0 \qquad \text{if } i \neq j \tag{1.4}$$

$$\sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1 - a_{ij} \\ s \end{bmatrix} F_i^{1-a_{ij}-s} F_j F_i^s = 0 \qquad \text{if } i \neq j. \tag{1.5}$$

Here $\begin{bmatrix} 1 - a_{ij} \\ s \end{bmatrix}$ is the Gaussian binomial coefficient $\begin{bmatrix} 1 - a_{ij} \\ s \end{bmatrix}_d$ with $d = 1$ (see, for example, [DC-K1]).

Let $\mathscr{W}$ be the Weyl group associated to (a_{ij}) and $w_0 \in \mathscr{W}$ its element with maximal length: $l(w_0) = N$. By $T_{\mathscr{W}}$ we denote the corresponding Braid group whose generators T_i, with $i = 1, \dots, n$, act on $\mathscr{U}_q(\mathscr{G})$ according to [L2]. It is known that if we fix a reduced expression of w_0: $w_0 = s_{i_1} s_{i_2} \cdots s_{i_N}$, then we can define a total convex ordering in Q^+ by putting

$$\beta_j = s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j})$$

for $j = 1, \dots, N$. We define

$$E_{\beta_j} = T_{i_1} \cdots T_{i_{j-1}} E_{i_j}, \qquad F_{\beta_j} = T_{i_1} \cdots T_{i_{j-1}} F_{i_j}$$

and, for every $k = (k_1, \dots, k_N) \in \mathbf{Z}_+^N$,

$$E^k = E_{\beta_1}^{k_1} \cdots E_{\beta_N}^{k_N}, \qquad F^k = F_{\beta_N}^{k_N} \cdots F_{\beta_1}^{k_1}.$$

Let us fix a prime p and a primitive pth-root of unity ε. By $\mathscr{U}_\varepsilon(\mathscr{G})$ we denote the associative algebra over $\mathbf{C}$ with generators $E_i, F_i, K_i^{\pm 1}$ and relations (1.1)–(1.5) where q is replaced by ε.

Now put $\mathscr{A} = \mathbf{Z}[q, q^{-1}]$ and take the $\mathscr{A}$-subalgebra $\mathscr{U}_{\mathscr{A}}$ of $\mathscr{U}_q(\mathscr{G})$ generated by the elements E_i, F_i, K_i, and $H_i := [E_i, F_i]$, with the relations (1.1)–(1.5) and $K_i - K_i^{-1} = (q - q^{-1}) H_i$. We define

$$\mathscr{U}_1 := \mathscr{U}_{\mathscr{A}} / \big(\phi_q^p, q - 1\big) \mathscr{U}_{\mathscr{A}},$$

where ϕ_q^p is the pth cyclotomic polynomial in the variable q.

Notice that the following isomorphism of fields holds:

$$\mathbf{Z}\big[q, q^{-1}\big]/\big(\phi_q^p, q - 1\big) \simeq \mathbf{Z}/p\mathbf{Z}.$$

Furthermore we recall the following well known result:

PROPOSITION 1.2 [L1]. *$\mathcal{U}_1 \simeq \mathcal{U}_{\mathbf{Z}/p\mathbf{Z}}(\mathcal{G}) \otimes \mathbf{Z}/p\mathbf{Z}[(\mathbf{Z}/2\mathbf{Z})^{rk\mathcal{G}}]$ where $\mathcal{U}_{\mathbf{Z}/p\mathbf{Z}}(\mathcal{G})$ is the universal enveloping algebra of $\mathcal{G}$ over $\mathbf{Z}/p\mathbf{Z}$.*

2. REDUCTION TO THE EXCEPTIONAL CASE

Let $\mathrm{Rep}(\mathcal{U}_\varepsilon)$ be the set of the equivalence classes of the irreducible, finite dimensional representations of $\mathcal{U}_\varepsilon$. Let Z_ε be the center of $\mathcal{U}_\varepsilon$. Then (see [DC-K1]) the elements E_α^p, F_α^p, with $\alpha \in Q^+$, and K_i^p, with $i = 1, \dots, n$, lie in Z_ε. Besides, if Z_0 is the subalgebra of Z_ε generated by these elements, then $\mathcal{U}_\varepsilon$ is a free Z_0-module with a basis consisting of the vectors $\{F^k K_1^{m_1} \cdots K_n^{m_n} E^r \mid k = (k_1, \dots, k_N), r = (r_1, \dots, r_N) \in \mathbf{Z}_+^N,\ m_j \in \mathbf{Z}, 0 \le k_j, r_j, m_j \le p - 1\}$.

By G we shall denote the algebraic group whose Lie algebra is $\mathcal{G}$ and whose center is trivial. It is known [DC-K-P1] that the elements of $\mathrm{Rep}(\mathcal{U}_\varepsilon)$ are parametrized by the conjugacy classes of the group G as it is stated by the following theorem:

THEOREM 2.1 [DC-K-P1]. *There exists a map*

$$\varphi: \mathrm{Rep}(\mathcal{U}_\varepsilon) \to G$$

such that:

(a) $\mathrm{Im}(\varphi)$ *is the big cell of G;*

(b) *the representations corresponding to conjugated elements in G are all of the same "type" (i.e., they are equivalent, up to a twist, by the elements of a group $\tilde{G}$ of automorphisms of $\mathcal{U}_\varepsilon$), in particular they have the same dimension.*

In fact it has been shown by De Concini and Kac [DC-K2] that, in order to classify the irreducible, finite dimensional representations of $\mathcal{U}_\varepsilon$, it is sufficient to consider the representations corresponding to the conjugacy classes of some special elements called *exceptional.*

DEFINITION 2.2. A semisimple element g in G is called exceptional if the center of its centralizer $Z_G(g)$ in G is finite.

DEFINITION 2.3. An element g in G is called exceptional if its semisimple part, with respect to the Jordan decomposition, is exceptional.

DEFINITION 2.4. A representation σ in $\operatorname{Rep}(\mathscr{U}_\varepsilon)$ is called exceptional if $\varphi(\sigma)$ is an exceptional element of G.

Let us now take a non-exceptional representation σ in $\operatorname{Rep}(\mathscr{U}_\varepsilon)$ on a vector space V, with central character χ. Then we may regard σ as a representation of the algebra $\mathscr{U}_\chi := \mathscr{U}_\varepsilon/(z - \chi(z), z \in Z_0)$. As $g = \varphi(\sigma)$ is not exceptional, we may suppose, up to the action of some element of $\tilde{G}$ on σ, that if $g = g_s g_u$ is the Jordan decomposition of g, then g satisfies the following conditions:

—g_u belongs to the maximal unipotent subgroup N_- of G, corresponding to $-Q^+$;

—$h_{\mathscr{G}} := \operatorname{Lie}(\operatorname{center}(Z_G(g_s))) \neq 0$;

—$Q' := \{\alpha \in Q | \alpha \text{ vanishes on } h_{\mathscr{G}}\} = \mathbf{Z}\Delta' \cap Q$ where $\mathbf{Z}\Delta'$ is a sublattice of R spanned by a proper subset Δ' of Δ.

Let $\mathscr{U}'_\varepsilon$ be the subalgebra of $\mathscr{U}_\varepsilon$ generated by the K_i's, K_i^{-1}'s, with $i = 1, \ldots, n$ and by the elements E_j, F_j such that $\alpha_j \in \Delta'$. Put

$$\mathscr{U}'_\chi = \mathscr{U}'_\varepsilon/\big(z - \chi(z), z \in Z_0 \cap \mathscr{U}'_\varepsilon\big),$$

$$\mathscr{U}^g = \mathscr{U}'_\varepsilon \mathscr{U}^+, \qquad \mathscr{U}^g_\chi = \mathscr{U}^g/\big(z - \chi(z), z \in Z_0 \cap \mathscr{U}^g\big),$$

where $\mathscr{U}^+$ is the subalgebra of $\mathscr{U}_\varepsilon$ generated by the E_i's. Then the following theorem holds:

THEOREM 2.5 [DC-K2]. *There exists a unique irreducible $\mathscr{U}^g_\chi$-submodule V' of V such that*:

(a) *V' is a $\mathscr{U}'_\chi$-module.*

(b) *The $\mathscr{U}_\chi$-module V is induced by V',*

$$V = \mathscr{U}_\chi \otimes_{\mathscr{U}^g_\chi} V'.$$

In particular $\dim(V) = p^t \dim(V')$, *where* $2t = |Q \setminus Q'|$.

(c) *The map $V \mapsto V'$ is a bijection* $\operatorname{Rep}(\mathscr{U}_\varepsilon) \to \operatorname{Rep}(\mathscr{U}'_\chi)$.

Remark. If we restrict the representation of $\mathscr{U}'_\chi$ in V' to the subalgebra of $\mathscr{U}'_\chi$ generated by the elements $E_i, F_i, K_i^{\pm 1}$ such that $\alpha_i \in \Delta'$, we get an irreducible representation. In fact this is an exceptional representation of the quantum group $\mathscr{U}_\varepsilon(\mathscr{G}')$ where $\mathscr{G}'$ is the subalgebra of $\mathscr{G}$ generated by its Chevalley generators corresponding to $\alpha_i \in \Delta'$.

We recall that if $\mathscr{G} = sl(n+1)$ the exceptional elements are the unipotent ones. Furthermore every unipotent matrix in $\mathrm{SL}(n)$ is conjugated to one with the form

$$\Lambda = \begin{pmatrix} J_{h_1} & & & 0 \\ & J_{h_2} & & \\ & & \ddots & \\ 0 & & & J_{h_t} \end{pmatrix}, \tag{2.6}$$

where $h_1 \geq h_2 \geq \cdots \geq h_t$ is a partition of n and J_t is a $t \times t$ Jordan block:

$$J_t = \underbrace{\begin{pmatrix} 1 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 1 \end{pmatrix}}_{t}.$$

We underline that this classification does not depend on the characteristic of the field.

3. THE A_n-CASE

In this section we shall examine the case $\mathscr{G} = sl(n+1)$. We know that the Weyl group $\mathscr{W}$ is the symmetric group S_{n+1}.

From now on we shall assume that a reduced expression of w_0, say $w_0 = s_n s_{n-1} \cdots s_1 s_n \cdots s_2 s_n \cdots s_n$, and the associated ordering $\{\beta_1, \dots, \beta_N\}$ of the positive roots have been fixed.

Let $\mathscr{O}$ be a conjugacy class of SL$(n+1)$ parametrized by a unipotent matrix Λ of type (2.6) for some blocks $J_{h_1}, \dots, J_{h_t}$, with $h_1 \geq h_2 \geq \cdots \geq h_t$ a partition of $n+1$.

Take an irreducible $\mathscr{U}_\varepsilon(\mathscr{G})$-module M such that $\varphi(M)$ is conjugated to Λ. Our first aim is to show that every such a module is the quotient of an induced module. According to the definition of φ (see [DC-K-P1]), for every positive root α we have $E_\alpha^p = 0$, F_α^p either 0 or 1, and for every $i = 1, \dots, n$, $K_i^p = 1$ (by E_α, F_α, K_j we denote both the generators of $sl_\varepsilon(n+1)$ and their images through the representation).

We consider the set

$$B = \{x \in M | E_i(x) = 0 \; \forall i = 1, \dots, n\}.$$

Since the subalgebra of $sl_\varepsilon(n+1)$ generated by the E_i's acts nilpotently, B is a non-trivial set upon which the K_i's act diagonally; thus we can take a non-zero element $m \in B$ such that $K_i m = \varepsilon^{\sigma_i} m$ with $0 \le \sigma_i \le p-1$. Now, if $\mathscr{U}^*$ is the $\mathbf{C}$-subalgebra of $\mathscr{U}_\varepsilon(sl(n+1))$ generated by E_i, K_i, F_α^p for every $i = 1, \dots, n$, $\alpha \in Q^+$, then the line L generated by m is an $\mathscr{U}^*$-module. Since M is irreducible it is generated by m hence M is a quotient of the induced module

$$I := \mathscr{U}_\varepsilon(sl(n+1)) \otimes_{\mathscr{U}^*} L.$$

We have thus established the following proposition:

PROPOSITION 3.1. *Let M be an irreducible, finite dimensional $\mathscr{U}_\varepsilon(sl(n+1))$-module lying over a unipotent canonical element of $SL(n+1)$. Then there exist a proper subalgebra $\mathscr{U}^*$ of $\mathscr{U}_\varepsilon$ and an element $m \in M \setminus \{0\}$ such that, if L is the line generated by m, then*:

(1) *L is an $\mathscr{U}^*$-module*;

(2) *$M = I/S$ where $I := \mathscr{U}_\varepsilon(sl(n+1)) \otimes_{\mathscr{U}^*} L$ and S is a submodule of I.*

Remark. The induced module I has got a natural basis $\mathscr{C}$:

$$\mathscr{C} = \left\{ F_{\beta_1}^{h_1} \cdots F_{\beta_N}^{h_N} m \mid 0 \le h_j < p \right\}.$$

If $C = \mathbf{Z}[\varepsilon]$-span$\{\mathscr{C}\}$ then C is stable with respect to the action of the E_i's, F_i's, K_i's, H_i's for every $i = 1, \dots, n$. Indeed, for every z in C such that $K_j z = \varepsilon^{\lambda_j^z} z$ $(0 \le \lambda_j^z \le p-1)$ the following relations hold:

- $E_j F_j^k z = F_j^k E_j z + (1 - \varepsilon^{2k})\ (\varepsilon^{2-2k+\lambda_j^z} - \varepsilon^{-\lambda_j^z})/(1-\varepsilon^2)(\varepsilon - \varepsilon^{-1})\, F_j^{k-1} z$;
- $E_j F_{h\dots j}^k z = -(1-\varepsilon^{2k})/(1-\varepsilon^2)\varepsilon^{-\lambda_j^z} F_{h\dots j}^{k-1} F_{h\dots j-1} z + F_{h\dots j}^k E_j z$ if $h < j$;
- $E_j F_{j\dots t}^k z = (1-\varepsilon^{-2k})/(1-\varepsilon^{-2})\varepsilon^{\lambda_j^z+1} F_{j\dots t}^{k-1} F_{j+1\dots t} z + F_{j\dots t}^k E_j z$ if $t > j$;
- $E_j F_{h\dots t}^k = F_{h\dots t}^k E_j$ if $h < j < t$;
- $F_j F_{h\dots t}^k = F_{h\dots t}^k F_j$ if $h < j < t$;
- $F_j F_{j\dots t}^k = \varepsilon^k F_{j\dots t}^k F_j$ if $t > j$;
- $F_j F_{h\dots j}^k = \varepsilon^{-k} F_{h\dots j}^k F_j$ if $h < j$;
- $F_j F_{j+1\dots t}^k = \varepsilon^{-k} F_{j+1\dots t}^k F_j + \varepsilon^{-k}(1-\varepsilon^{2k})/(1-\varepsilon^2) F_{j+1\dots t}^{k-1} F_{j\dots t}$.

We notice that, since $\mathscr{U}_\varepsilon(sl(n+1))$ is a finitely generated algebra, the module M is defined over a finite extension K of $\mathbf{Q}(\varepsilon)$ and, in order to compute the dimension of M, we may reduce to K, instead of $\mathbf{C}$.

Let $\mathscr{L}$ be the localization of the ring of integers $\mathbf{Z}[\varepsilon]$ of $\mathbf{Q}(\varepsilon)$ at the ideal $\mathscr{P}_0$ generated by $\varepsilon - 1$, by $m_{\mathscr{P}_0}$ denote the maximal ideal of $\mathscr{L}$, and let

$$\mathscr{U}_{\mathscr{L}} = \mathscr{U}_{\mathscr{A}} / \phi_q^p \mathscr{U}_{\mathscr{A}} \otimes_{\mathbf{Z}[\varepsilon]} \mathscr{L}.$$

Then the induced module I is defined over $\mathscr{U}_{\mathscr{L}}$. Let us choose an ideal $\mathscr{P}$, in the ring of integers Z of K, lying over $\mathscr{P}_0$ and let $\mathscr{L}'$ be the localization of Z at $\mathscr{P}$, $m_{\mathscr{P}}$ its maximal ideal. Then $\mathscr{L}'/m_{\mathscr{P}}$ is a finite extension of $\mathscr{L}/m_{\mathscr{P}_0}$ and thus a finite field of characteristic p.

We put $\mathscr{U}_{\mathscr{L}'} = \mathscr{U}_{\mathscr{L}} \otimes_{\mathscr{L}} \mathscr{L}'$.

PROPOSITION 3.2. *Let M be an irreducible $\mathscr{U}_\varepsilon(sl(n+1))$-module such that $\varphi(M)$ is a unipotent canonical element in $SL(n+1)$. Then there exist a finite extension K of $\mathbf{Q}(\varepsilon)$, a local ring $\mathscr{L}'$ in K, and an $\mathscr{U}_{\mathscr{L}'}$-module M' such that M' is a free $\mathscr{L}'$-module and*

$$M = M' \otimes_{\mathscr{L}'} K.$$

Proof. We use the same notations as in Proposition 3.1; then we know that M is the quotient of the induced module I by a submodule S. According to the remark on the basis $\mathscr{C}$ of I, we have

$$I = I_{\mathscr{P}} \otimes_{\mathscr{L}'} K,$$

where $I_{\mathscr{P}}$ is the module, over the localization $\mathscr{L}'$ of Z in $\mathscr{P}$, naturally defined. Now, since $\mathscr{L}'$ is a discrete valuation ring, $S \cap I_{\mathscr{P}}$ is free and it is a direct factor of $I_{\mathscr{P}}$, hence, if we put $M' = I_{\mathscr{P}}/(S \cap I_{\mathscr{P}})$ we obtain the result. In particular, $\mathrm{rk}(S \cap I_{\mathscr{P}}) = \dim_K S$ and $\dim_K M = \mathrm{rk}(M')$.

Notice that the action of $\mathscr{U}_\varepsilon$ on M is obtained by extending that of $\mathscr{U}_{\mathscr{L}'}$. ∎

Before coming to a conclusion we recall the following theorem:

THEOREM 3.3 [P]. *Let G be a simple and simply connected algebraic group defined over an algebraically closed field whose characteristic $p > 0$ is good for G. Suppose that $\mathscr{G} = \mathrm{Lie}(G)$ admits a non-degenerate G-invariant trace form. Let V be an irreducible $\mathscr{G}$-module with p-character $\chi \in \mathscr{G}^*$. Then the dimension of V is divisible by $p^{1/2 \dim(\Omega(\chi))}$, where $\Omega(\chi)$ is the coadjoint orbit of χ.*

Here by a "good" prime we mean a prime greater than any coefficient of any positive root in its expression relative to a basis of simple roots. Therefore if $\mathscr{G} = sl(n+1)$ any prime p is good for $\mathscr{G}$. On the other hand, in order to guarantee the existence of a non-degenerate trace form on $\mathscr{G}$, we shall suppose $p \nmid n+1$. In fact in [P] it is shown that the

Kac–Weisfeiler conjecture is no longer true for $\mathscr{G} = sl(mp, \mathbf{k})$ when $m \geq 1$ (p is the characteristic of the field $\mathbf{k}$).

We are now ready to state our main result:

THEOREM 3.4. *Let ε be a primitive pth-root of unity with p prime, $p > 2$, $p \nmid n+1$. If M is an irreducible, finite dimensional $\mathscr{U}_\varepsilon(sl(n+1))$-module lying over a conjugacy class $\mathscr{O}$, then* $\dim(M) = kp^{1/2 \dim(\mathscr{O})}$ *for some positive integer k.*

Proof. As we explained in Section 2 we may reduce to the unipotent case and thus suppose that $\mathscr{O}$ is the conjugacy class of a unipotent element of $\mathrm{SL}(n+1)$. Then let us assume the construction of the module M' and, using the same notations as above, let us put $\tilde{M} = M'/m_{\mathscr{P}}M'$. As $\mathscr{L}'$ is a local ring, $\mathrm{rk}(M') = \dim_{\mathscr{L}'/m_{\mathscr{P}}}(\tilde{M})$. Besides, by its own definition, $\tilde{M}$ is a $\mathscr{U}_1$-module and therefore a $\mathscr{U}_{\mathbf{Z}/p\mathbf{Z}}(\mathscr{G})$-module (see Section 1). We finally denote the algebraic closure of the finite field $\mathscr{L}'/m_{\mathscr{P}}$ by $\overline{\mathscr{L}'/m_{\mathscr{P}}}$ and define

$$\overline{M} = \tilde{M} \otimes_{\mathscr{L}'/m_{\mathscr{P}}} \overline{\mathscr{L}'/m_{\mathscr{P}}}.$$

If χ is the p-character of $\overline{M}$, then

$$\dim_K(M) = \dim_{\overline{\mathscr{L}'/m_{\mathscr{P}}}}(\overline{M}) = \mathrm{k}\mathrm{p}^{1/2 \dim(\Omega(\chi))}$$

for some positive integer k. Indeed, since $\overline{M}$ is finite-dimensional, it has a composition series consisting of irreducible modules $\overline{M}_j$ satisfying the hypotheses of Theorem 3.3, whose dimension is therefore divided by $p^{1/2 \dim(\Omega(\chi))}$.

We point out that the canonical unipotent element Λ of type (2.6), corresponding to the Jordan partition λ, reduces "modulo p" to the canonical nilpotent element corresponding to λ, that is, to the nilpotent element with the same blocks as Λ but with all zeros on the diagonal. Therefore the dimension of the coadjoint orbit $\Omega(\chi)$ of χ is the dimension of the conjugacy class $\mathscr{O}$. ∎

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