Easter Exercises

1. Let $Q_8$ be the quaternion group, that is, the group of order 8 generated by $i$ and $j$ with relations $i^4 = j^4 = 1, i^2 = j^2, ij = ji^3$.

   (a) Describe the conjugacy classes of $Q_8$ and compute its character table. Compare it with the character table of the dihedral group $D_4$ of order 8.

   (b) Consider the subgroup $H = \langle i \rangle$ of $Q_8$, isomorphic to the cyclic group of order 4. Induce all its irreducible representations to $Q_8$ and decompose the induced representations into irreducible ones.

   (c) Let $\rho$ be any irreducible representation of $Q_8$. Consider $\text{Res}_{Q_8}^{Q_8}(\rho)$. Tell if it is irreducible and, if not, decompose it into irreducible representations of $H$.

   (d) Let $V$ and $W$ range among all irreducible representations of $Q_8$. Decompose $V \otimes W$ into irreducible representations of $Q_8$.

2. Let $V$ and $W$ be completely reducible finite dimensional representations of a group $G$ over an algebraically closed field. Prove that

   (a) $\text{Hom}_G(V, W) = 0$ if and only if $V$ and $W$ have no common irreducible component.

   (b) $\dim \text{Hom}_G(V, W) = 1$ if and only if they have only one common irreducible component $U$ and $U$ has multiplicity exactly 1 in $V$ and $W$.

   (c) If $V$ and $W$ have a unique common irreducible component of multiplicity $m$ in $V$ and $n$ in $W$, then $\dim \text{Hom}_G(V, W) = mn$.

   (d) If $V = \bigoplus_{i \in I} U_i^{m_i}$ and $W = \bigoplus_{i \in I} U_i^{n_i}$ are decompositions into irreducibles of $V$ and $W$ respectively then $\dim \text{Hom}_G(V, W) = \sum_{i \in I} m_i n_i$. Can you figure out how $\text{Hom}_G(V, W)$ looks like?

   Hint: use for all cases a decomposition into irreducibles of both representations, injections of an irreducible into $V$ and projections of $W$ onto an irreducible.

3. Let $V$ be a 1-dimensional representation of $G$ with character $\chi$. Show that $FS(V)$ is never $-1$ and that $FS(V) = 1$ if and only if $\chi(g)^2 = 1$ for every $g \in G$. 

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4. Let $G = D_4$ the dihedral group of order 8 and let $V$ be the 2-dimensional irreducible representation of $G$. Does there exist a $G$-invariant symmetric bilinear form on $V$? (Hint: use the previous exercise and count the number of involutions). Same question for $Q_8$.

You can find more exercises in Serre’s book and in Etingof et al lecture notes.