Motivate every answer

1. We will construct the character table of $S_4$, the symmetric group on 4 letters. Recall that conjugacy classes in $S_4$ are in bijection with partitions of 4.

(a) Prove that $S_4$ has 5 irreducible representations, of dimension 1, 1, 2, 3, 3 respectively. Let us label their characters by $\chi_0, \chi_1, \chi_2, \chi_3$ and $\chi_4$, respectively.

(b) Consider the permutation representation $\rho: S_4 \to GL_4(\mathbb{C})$ given, on the vectors of the canonical basis by $\rho(\sigma)(e_i) = e_{\sigma(i)}$ for every $\sigma \in S_4$. Compute its character $\chi$ and show that $\chi$ is the sum of the trivial representation $\chi_0$ and an irreducible representation $\chi_3$ of dimension 3.

(c) Compute $\chi_1$ (hint: divide the permutations into even and odd ones).

(d) Deduce from (c) that $\chi_2((12)) = \chi_2((1234)) = 0$.

(e) Deduce all values of $\chi_4$ from (c).

(f) Write down the complete character table of $S_4$.

(g) Describe the kernel of all irreducible representations of $S_4$.

(h) Show that the Frobenius-Schur indicator of every irreducible representations is equal to 1.

2. Let $p$ be a prime number, and let $G$ be a group of order $p^3$.

(a) Show that all complex irreducible representations of $G$ have dimension 1 or $p$.

(b) Show that if $G$ is not abelian, then $|[G,G]| = p$.

(c) Show that the number of conjugacy classes of $G$ is either $p^3$ or $p^2 + p - 1$. 