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REMARKS ON THE DIRICHLET AND STATE-CONSTRAINT PROBLEMS FOR QUASILINEAR PARABOLIC EQUATIONS

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Abstract. We prove two different types of comparison results between semicontinuous viscosity sub- and supersolutions of the generalized Dirichlet problem (in the sense of viscosity solutions theory) for quasilinear parabolic equations: the first one is an extension of the Strong Comparison Result obtained previously by the second author for annular domains, to domains with a more complicated geometry. The key point in the proof is a localization argument based on a "strong maximum principle" type property. The second type of comparison result concerns a mixed Dirichlet-State-constraints problems for quasilinear parabolic equations in annular domains without rotational symmetry; in this case, we do not obtain a Strong Comparison Result but a weaker one on the envelopes of the discontinuous solutions. As a consequence of these results and the Perron's method we obtain the existence and the uniqueness of either a continuous or a discontinuous solution.

1. Introduction

In this article we are interested in the generalized Dirichlet problem (in the sense of viscosity solutions theory) for fully nonlinear, second-order parabolic partial differential equations, with a special emphasis on quasilinear equations: our aim is to obtain comparison results between, possibly discontinuous, sub- and supersolutions of these problems, including the case of state-constraints boundary conditions.

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Despite our results cover more general cases, we have mainly in mind application to geometric type equations such as the equation of motion by mean curvature for graphs

$$\frac{\partial u}{\partial t} - \Delta u + \frac{\langle D^2uDu, Du\rangle}{1+|Du|^2} = 0 \quad \text{in } \Omega \times (0,\infty), \tag{1.1}$$

where Ω is, say, a smooth bounded domain of $\mathbb{R}^N$ and $\langle\cdot,\cdot\rangle$ stands for the usual scalar product in $\mathbb{R}^N$, or the more singular equation of motion by mean curvature for general hypersurfaces appearing in the so-called "level-sets approach"

$$\frac{\partial u}{\partial t} - \Delta u + \frac{\langle D^2uDu, Du\rangle}{|Du|^2} = 0 \quad \text{in } \Omega \times (0,\infty). \tag{1.2}$$

In order to be more specific but also to simplify the presentation, we consider the case of stationary fully nonlinear elliptic equations of the form

$$F(x,u,Du,D^2u) = 0 \text{ in } \Omega, \tag{1.3}$$

with the boundary condition

$$u = \varphi \text{ on } \partial\Omega, \tag{1.4}$$

where φ is a continuous function, F is at least a real-valued, locally bounded function in $\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^N\times\mathcal{S}(N)$, which is continuous in $\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^N\setminus\{0\}\times\mathcal{S}(N)$, $\mathcal{S}(N)$ being the set of real symmetric $N\times N$ matrices. The solution u is a scalar function, Du, D^2u represent respectively the gradient and the Hessian matrix of u. As the example (1.2) shows, the operator may actually present a singularity at $Du=0$.

The (degenerate) ellipticity of the equation is expressed by the following monotonicity property satisfied by F

$$F(x,r,p,X) \le F(x,r,p,Y) \text{ whenever } X \ge Y,$$

for any $x\in\overline{\Omega}$, $r\in\mathbb{R}$, $p\in\mathbb{R}^N\setminus\{0\}$ and $X,Y\in\mathcal{S}(N)$ where " $\ge$ " stands for the usual partial ordering on symmetric matrices. This basic condition in viscosity solutions theory is assumed to hold throughout the paper.

It is well-known that this kind of Dirichlet problem cannot be solved in general for any boundary data, even in the more classical case of quasilinear equations, unless the boundary of the domain satisfies suitable geometric-curvature conditions. For an extensive presentation of the results on the solvability of the Dirichlet problem, we refer the reader to the books of Gilbarg and Trudinger [19] and Ladyzhenskaya and Ural'tseva [25] for the stationary case and to the book of Ladyzhenskaya, Solonikov and Ural'tseva [26] in the case of evolution equations. In Barles, Rouy and Souganidis [9], a simple, well-known example of non-solvability is given for the minimal surface equation.

Viscosity solutions theory provides a natural relaxation of the usual definition of boundary conditions which allows losses of boundary conditions. Of course, such losses of boundary conditions may occur as a consequence of degeneracies in the ellipticity of the equation or at least when the equation is not uniformly elliptic as in the case of (1.1) above.

The formulation of the generalized Dirichlet boundary condition (1.4) in the viscosity sense reads

$$\min(F(x,u,Du,D^2u), u-\varphi) \leq 0 \text{ on } \partial\Omega, \tag{1.5}$$

and

$$\max(F(x,u,Du,D^2u), u-\varphi) \geq 0 \text{ on } \partial\Omega. \tag{1.6}$$

Roughly speaking, these relaxed conditions mean that the equations has to hold up to the boundary, when the boundary condition is not assumed in the classical sense. The key argument to justifies them is that they appear naturally when one passes to the limit in the vanishing viscosity method using typically the half-relaxed limits method.

For a detailed presentation of the theory of viscosity solutions and of the boundary conditions in the viscosity sense, we refer to the "Users'guide" of Crandall, Ishii and Lions [13] and the book of Fleming and Soner [17] while the books of Bardi and Capuzzo Dolcetta [1] and Barles [2] provide an introduction to the theory in the case of first-order equations.

As it is usual in viscosity solutions theory, the key results to be obtained are comparison results. Ideally Strong Comparison Result (SCR in short), i.e., comparison results between discontinuous viscosity sub and supersolutions, because they provide as a by-product the existence and uniqueness of viscosity solutions but also the convergence of any reasonable approximations, even in the case when boundary layers appear.

In the case of generalized Neumann boundary conditions, rather general and natural SCR exist, for a large class of equations and boundary conditions, and the theory can be considered as being rather complete (See H. Ishii [22], G. Barles [3]). The situation is completely different for the generalized Dirichlet problem which is not so well understood: for example, until very recently there was no results at all for quasilinear equations.

The list of the SCR for the generalized Dirichlet problem is rather short: the first ones were proved for first-order equations by H. Ishii [21] and G. Barles and B. Perthame [2, 7]: it is worth pointing out that, in this framework, the results are rather general and natural. Extensions to second-order semilinear equations were obtained by G. Barles and J. Burdeau [5] using a reformulation with Neumann boundary conditions. The case of Hamilton-Jacobi-Bellman equations was first considered by M. Katsoulakis [24] and then, with different assumptions, by G. Barles and E. Rouy [8]: both works use in an essential way the control formulation.

As we already mention it above, no SCR have been obtained till recently for quasilinear equations (when there are actually losses of boundary conditions) because, in particular, of the gradient dependence in the equation which leads to various technical difficulties: in all the above mentioned SCR, a key property on the nonlinearity F is its Lipschitz continuity in Du (and D^2u), and this is clearly wrong in the quasilinear case. In particular, this Lipschitz continuity allows to use local arguments around a maximum point and this localization is not possible anymore for quasilinear equations; this is a key difficulty in the quasilinear case.

F. Da Lio [15] first broke this difficulty but unfortunately under rather restrictive conditions on the domain Ω and the nonlinearity F: Ω is assumed to be an annular domain and F has to satisfy some invariance properties with respect to rotation (a property satisfied anyway by our two examples above). The assumption on the domain is certainly the most restrictive and the aim of this work is to weaken it.

We are going to do it in two different ways: first by assuming that $\Omega = \Omega^e \setminus \cup_{i=1}^n \overline{B}_i$, where Ω^e is an open subset of $\mathbb{R}^N$ and for all i, $B_i = B(x_i, r_i)$[1], with $\overline{B}_i \subset \Omega^e$ and $\overline{B}_i \cap \overline{B}_j = \emptyset$ if $i \neq j$. In this case, if either Ω^e is a ball or if there is no loss of boundary condition on $\partial\Omega^e$ (i.e., if the Dirichlet boundary condition holds in the classical sense on $\partial\Omega^e$) and under suitable invariance conditions on F, we obtain a SCR for the generalized Dirichlet problem. The proof is based on some kind of "localization" argument which allows to use F. Da Lio's results or ideas in a thin annular domain around one of the B_i's or $\partial\Omega^e$ when Ω^e is a ball. It is worth noticing that our "localization" argument can be used in general domains, even if here we use it only in a particular situation.

The second case is when $\Omega = \Omega^e \setminus \Omega^i$, where $\Omega^i \subset\subset \Omega^e$ is a strictly star-shaped domain. In this framework we consider the case when we have a state-constraint boundary condition on $\partial\Omega^i$ and again no loss of boundary condition on $\partial\Omega^e$; we need also suitable conditions on F. We recall that state-constraint boundary conditions were first studied by H.M. Soner [28] and then by I. Capuzzo Dolcetta and P.L. Lions [12] in the context of first-order Hamilton-Jacobi Equations. State-constraint boundary conditions can

[1]For $x \in \mathbb{R}^N$ and $r > 0$, $B(x, r)$ denotes the ball with center x and radius r

be seen formally as the case when $\varphi \equiv +\infty$ and when we have actually a loss of boundary condition since we look for bounded solutions. In almost all the references we gave above for the generalized Dirichlet problem, comparison results for the state-constraint case are also provided since, in fact, these problems are closely related. It is worth pointing out anyway that the state-constraint case contains specific difficulties and we come back on this point later on.

Unfortunately, we were not able to obtain a SCR in this setting but only a weaker comparison result, namely, if u and v are respectively viscosity sub and supersolution of the generalized Dirichlet problem, then

$$u_* \leq v \quad \text{and} \quad u \leq v^* \quad \text{in } \Omega\,,$$

where, for any locally bounded function z, the notations z_* and z^* stands respectively for the lower semicontinuous and uppersemicontinuous envelope of z. Of course, this result provides the uniqueness of a discontinuous solution.

One of the main applications we have in mind for the above comparison results concerns the so-called "level-sets approach". We recall that the level-sets approach provides a weak notion for the evolution of an hypersurface in $\mathbb{R}^N$ with a prescribed normal velocity which may depend on the position of the hypersurface, on time, on its normal directions and curvature tensors. It was first introduced for numerical computations by Osher and Sethian in [27] and then rigorously settled up by Evans and Spruck in [16] for motion by mean curvature and by Chen, Giga, Goto in [14] for general motions. The basic underlying idea is to represent the evolving hypersurface, denoted by $(\Gamma_t)_t$, as the zero-level set of a function u which is solution of a suitable pde. For example, the equation (1.2) is related to motion by mean curvature.

We refer the reader to the above given references for the $\mathbb{R}^N$–case. When set in a bounded domain, the level-set equation has to be associated with boundary conditions which correspond to various constraints one wants to impose on the Γ_t's. The case when the Γ_t have to satisfy angle boundary conditions, is related to Neumann boundary conditions and the level-sets approach is provided for this case in Y. Giga and M.-H. Sato [18] and in G. Barles [3]. The attempt of "fixing" the positions of the Γ_t's on $\partial\Omega$ or imposing some velocity on the boundary seems intuitively related to Dirichlet boundary conditions, although a clear geometrical interpretation is still missing because it faces the possible loss of boundary conditions.

The first result concerning the study of fronts propagation by level-set approach in bounded domains with Dirichlet boundary condition was obtained by Sternberg and Ziemer in [29]: the authors focus their attention to the mean curvature equation and consider a set of assumptions implying that the Dirichlet boundary condition is satisfied in the classical sense. The only result in the case when loss of boundary conditions may occur are provided in F. Da Lio [15].

The level-sets approach, which consists, roughly speaking, in defining Γ_t through

$$\Gamma_t := \{x \in \mathbb{R}^N : \ u(x,t) = 0\} \ ,$$

requires two key results: on one hand, one needs clearly the existence and uniqueness of u; on the other hand, the geometric motion of the Γ_t is related to the fact that the 0–level set of u does not depend really on the choice of the initial and boundary data but only on their 0–level-sets and on their signs which play the role of orienting these level-sets. Comparison results play clearly a key role when performing the above program.

Following F. Da Lio [15], one can easily extend the level-sets approach to the first case we presented above by using the SCR. Therefore, we are going to describe this extension only in the case of state-constraint boundary conditions and when we have only a weak comparison result. A new point here is that the weak comparison result does not allow us to prove the existence of a continuous solution. As a consequence, we have to reformulate the level-set approach in a slightly different way.

In fact, in $\mathbb{R}^N$, the level-sets approach can be seen as providing a triplet of mutually disjoint subsets, namely (Γ_t, D_t^+, D_t^-) given by

$$\Gamma_t = \{x : u(x,t) = 0\} \ , \ D_t^+ = \{x : u(x,t) > 0\} \ , \ D_t^- = \{x : u(x,t) < 0\},$$

and the key point is that this triplet is uniquely determined by the initial triplet (Γ_0, D_0^+, D_0^-), independently of its representation through the initial data u_0. We are in our framework in a completely analogous situation, the triplet being given this time by (Γ_t, E_t^+, E_t^-) defined by

$$E_t := \{x \in \overline{\Omega} : \ u_*(x,t) > 0\} \ , \ E_t^- := \{x \in \overline{\Omega} : \ u^*(x,t) < 0\} \ ,$$

and $\Gamma_t = (E_t^+)^c \cap (E_t^-)^c$. As in the classical level-set approach, we show that this triplet depends only on F, the initial triplet (Γ_0, E_0^+, E_0^-) and the boundary triplet $(\Gamma_t^b, E_t^{+,b}, E_t^{-,b})$ defined by

$$E_t^{+,b} = \{x \in \partial\Omega^e : \varphi(x,t) > 0\} \ , \ E_t^{+,b} = \{x \in \partial\Omega^e : \varphi(x,t) < 0\},$$

and $\Gamma_t^b = \{x \in \partial\Omega^e : \varphi(x,t) = 0\}$ but not on their representation through u_0 and φ.

This analogy shows that, despite our approach seems weaker, we do not think it is in reality. Even if the classical-continuous approach provides us with a continuous function u, it does not imply more regularity for the triplet (Γ_t, D_t^+, D_t^-). In particular, one face the *no-empty interior difficulty*, i.e., the fact that $\bigcup_{t>0} \Gamma_t \times \{t\}$. This no-empty interior difficulty is a problem in the applications, in particular when studying the asymptotics of reaction-diffusion equations or particle systems since the asymptotic behaviors of the solutions of these equations are difficult to analyze on this set (See, for example, G. Barles, H.M. Soner and P.E. Souganidis [10], G. Barles and P.E. Souganidis [11] and G. Barles and F. Da Lio [6]). We refer to [10] for a discussion no-empty interior difficulty.

This article is organized as follows: in Section 2, we state and prove the SCR for the first case described above. The third section is devoted to the study of the second case. Both cases are treated in the parabolic case but it is rather easy to obtain related results in the stationary case and we leave these extensions to the reader. In preparation to the level-set approach described in Section 5, we provide in the fourth section an existence result of discontinuous solutions in the state-constraint case: in general, such result is rather easy to obtain using the Perron's method of H. Ishii [20] but here the state-constraint boundary creates a difficulty since one needs conditions on the nonlinearity to build the needed supersolution.

2. Some extensions of strong comparison result for generalized Dirichlet problem

In this section, we consider the following initial-boundary value problem

$$\begin{cases} u_t + F(x,t,u,Du,D^2u) = 0 & \text{in } \Omega \times (0,T], \\ u(x,t) = \varphi(x,t) & \text{in } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \overline{\Omega}, \end{cases} \qquad (BVP)$$

where Ω is a smooth domain of $\mathbb{R}^N$ and F is a locally bounded function in $\overline{\Omega} \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}(N)$ which is continuous in $\overline{\Omega} \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \setminus \{0\} \times \mathcal{S}(N)$. The boundary condition φ and the initial condition u_0 are real-valued, continuous functions defined respectively on $\partial\Omega \times [0,T]$ and on $\overline{\Omega}$; they satisfy in addition the following compatibility condition

$$\varphi(x,0) = u_0(x) \text{ in } \partial\Omega. \qquad (2.1)$$

We are going to use the following assumptions on F.

(H1) *For any $R > 0$, there exists $\gamma_R \in \mathbb{R}$ such that, for all $x \in \overline{\Omega}$, $t \in [0,T]$, $-R \le v \le u \le R$, $p \in \mathbb{R}^N$ and $M \in \mathcal{S}(N)$*

$$F(x,t,u,p,M) - F(x,t,v,p,M) \ge \gamma_R(u-v) .$$

(H2) *For all $R, K > 0$, there exists a function $m_{R,K} : \mathbb{R}^+ \to \mathbb{R}$ such that $m_{R,K}(t) \to 0$ when $t \to 0$ and, for all $\eta > 0$ and $\varepsilon > 0$*

$$F(y,t,u,q,Y) - F(x,t,u,p,X) \leq m_{R,K}(\eta + \varepsilon(1+|p|+|q|))$$

for all $x, y \in \overline{\Omega}$, $t \in [0,T]$, $|u| \leq R$, $p, q \in \mathbb{R}^N$ and for all matrices $X, Y \in \mathcal{S}(N)$ satisfying the following properties

$$-\frac{K\eta}{\varepsilon^2} Id \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \frac{K\eta}{\varepsilon^2} \begin{pmatrix} Id & -Id \\ -Id & Id \end{pmatrix} + K\eta Id, \tag{2.2}$$

$$|p-q| \leq K\varepsilon(|p|+|q|) \tag{2.3}$$

$$|x-y| \leq K\varepsilon. \tag{2.4}$$

(H3) For all $x \in \overline{\Omega}$, $t \in [0,T]$ and $u \in \mathbb{R}$

$$F^*(x,t,u,0,0) = F_*(x,t,u,0,0)\ .$$

For any $O \subseteq \mathbb{R}^N \times [0,T]$ we denote by $BUSC(O)$ the set of bounded, upper semicontinuous functions in O and by $BLSC(O)$ the set of bounded, lower semicontinuous functions in O. In the sequel we use also the following notations: $Q_T := \Omega \times (0,T]$, $\partial_l Q_T := \partial\Omega \times (0,T)$, $\partial Q_T := \partial_l Q_T \cup \overline{\Omega} \times \{0\}$. We will say below that a function is a strict subsolution (resp. strict supersolution) of (BVP) if it is a subsolution of (BVP) with F replaced by $F + \eta$ with $\eta > 0$ (resp. supersolution of (BVP) with F replaced by $F - \eta$ with $\eta > 0$).

Our first result which is valid for any smooth domain is:

Theorem 2.1. (No interior positive maximum) *Assume that Ω has a $W^{3,\infty}$-boundary, that F satisfies* **(H1)**-**(H3)** *and that u_0, φ are continuous functions satisfying* (2.1). *Let $u \in BUSC(\overline{Q}_T)$, $v \in BLSC(\overline{Q}_T)$ be respectively viscosity sub- and supersolution of (BVP) and denote by $R := \max(||u||_\infty, ||v||_\infty)$. If $M := \max_{\overline{Q}_T}(u-v)$ is achieved at some point of Q_T and if either $\gamma_R > 0$ or if $\gamma_R \geq 0$ with either u or v being a strict sub or supersolution of (BVP), then $M \leq 0$. In particular, if $M > 0$ and $K \subset\subset Q_T$, then $\max_K(u-v) < M$.*

As we will see below in the application, this result will be used as some kind of localization argument for the comparison proof and this is a useful tool for quasilinear type pdes for which localization is one of the key problems.

A priori the assumptions on γ_R seems rather restrictive but it is worth mentioning that, as it is classical, this result is used in general after some changes on the sub and supersolutions, like $\exp(-Kt)u, \exp(-Kt)v$ for some $K > 0$ large enough and/or $u - \eta t$, $v + \eta t$ for some $\eta > 0$, and not directly on the sub and supersolutions at hands.

We do not know if these assumptions are optimal but the following argument shows where the difficulty is: if one considers the singular parabolic pdes arising in the so-called "level-sets approach" like (1.2) in the introduction, by results of Barles, Soner and Souganidis [10], if u, v are sub and supersolutions of such pdes, then $\chi_1 = \mathbb{1}_{\{u\geq 0\}}$ and $\chi_2 = \mathbb{1}_{\{v>0\}}$ are still sub and supersolutions of the same pde. And for such sub and supersolutions taking only two values (0 and 1), either $\chi_1 \leq \chi_2$ in Q_T or the maximum of $\chi_1 - \chi_2$ is achieved in Q_T.

This shows that Theorem 2.1 would imply, in a very simple way, a comparison result for the Dirichlet problem associated to geometric pdes. Moreover, through the arguments of Barles, Biton and Ley [4] which allows to transform solutions of (non-singular) quasilinear pdes into solutions of geometric pdes, this would also provide a rather general comparison result for the Dirichlet problem associated to quasilinear pdes.

Unfortunately, we do not know how to completely justify this argument: indeed, for geometric pdes, if it is true that $\gamma_R \geq 0$ for any $R > 0$, it is far less easy to obtain some strict inequalities for u or v, keeping the property of having an interior maximum point. This is why we came up with a so complicated and apparently restrictive formulation on our assumptions.

Now we turn to the proof of Theorem 2.1.

Proof. The proof relies on an argument given, for the stationary case, in Barles, Rouy and Souganidis [9]. Despite the extension to the time-dependent case is rather easy, we reproduce it for the sake of completeness and for the reader's convenience.

To prove Theorem 2.1, we assume by contradiction that $M = \max_{\overline{Q}_T}(u - v) > 0$ and that it is achieved at some point $(x_0, t_0) \in Q_T$.

In the arguments below, we are going to use the regularity of the boundary: we denote by d a smooth function agreeing in a neighborhood $\mathcal{W}$ of $\partial\Omega$ with the signed distance function to $\partial\Omega$ which is positive in Ω and negative in $\mathbb{R}^N \setminus \overline{\Omega}$ and we denote by $n(x) := -Dd(x)$ for all $x \in \mathcal{W}$. If $x \in \partial\Omega$, $n(x)$ is just the unit outward normal to $\partial\Omega$ at x.

We define $\mathcal{M} = \{(x,t) \in \overline{Q}_T :\ u(x,t) - v(x,t)) = M\}$ and we introduce the sets

$$\Gamma_u = \{(x,t) \in \partial_l Q_T \cap \mathcal{M} :\ u(x,t) \leq \varphi(x,t)\},$$
$$\Gamma_v = \{(x,t) \in \partial_l Q_T \cap \mathcal{M} :\ v(x,t) \geq \varphi(x,t)\}.$$

Since $M > 0$, Γ_u, Γ_v are closed, disjoint subsets of $\partial_l Q_T$ and therefore there exists a smooth function $\psi \colon \mathbb{R}^N \to \mathbb{R}$ with compact support in $\mathcal{W}$ such that $\psi = 1$ in a neighborhood of Γ_u and $\psi = -1$ in a neighborhood of Γ_v.

We define a smooth function $\chi\colon \mathbb{R}^N \times [0,T] \to \mathbb{R}^N$ by setting

$$\chi(x,t) = \begin{cases} \psi(x,t)n(x) & \text{if } (x,t) \in \mathcal{W} \times [0,T] \ , \\ 0 & \text{otherwise.} \end{cases} \tag{2.5}$$

We may assume without loss of generality that $(x_0,t_0) \notin \mathcal{W}$ and therefore $\psi(x_0,t_0) = 0$. For all $\varepsilon > 0$, we define the auxiliary function $\Phi\colon \overline{\Omega} \times \overline{\Omega} \times [0,T] \to \mathbb{R}$ by

$$\Phi(z,w,t) = u(z,t) - v(w,t) - \left|\frac{z-w}{\varepsilon} + \chi(\frac{z+w}{2},t)\right|^4.$$

Let $(z_\varepsilon, w_\varepsilon, t_\varepsilon)$ be a global maximum of Φ in $\overline{\Omega} \times \overline{\Omega} \times [0,T]$.

The key point in the proof is the equality $M = \Phi(x_0,x_0,t_0)$ which allows to prove all the needed estimates to conclude the comparison proof. Indeed, it first leads to

$$M = \Phi(x_0,x_0,t_0) \le \Phi(z_\varepsilon,w_\varepsilon,t_\varepsilon) \le M + o_\varepsilon(1) \ ,$$

the last inequality being a consequence of the upper semicontinuity of u and the lower semicontinuity of v. And then, by standard arguments one can prove that (up to subsequences)

$$z_\varepsilon,\ w_\varepsilon \to \tilde{x},\ t_\varepsilon \to \tilde{t} \text{ as } \varepsilon \to 0. \tag{2.6}$$

$$\left|\frac{z_\varepsilon - w_\varepsilon}{\varepsilon} + \chi(\frac{z_\varepsilon + w_\varepsilon}{2}, t_\varepsilon)\right|^4 = o_\varepsilon(1), \tag{2.7}$$

$$u(z_\varepsilon,t_\varepsilon) - v(w_\varepsilon,t_\varepsilon) \to u(\tilde{x},\tilde{t}) - v(\tilde{x},\tilde{t}) = M,\ \text{as } \varepsilon \to 0, \tag{2.8}$$

and the upper and lower semicontinuity of u and v imply that

$$u(z_\varepsilon,t_\varepsilon) \to u(\tilde{x},\tilde{t}),\ v(w_\varepsilon,t_\varepsilon) \to v(\tilde{x},\tilde{t}),\ \text{as } \varepsilon \to 0.$$

We claim that for $\varepsilon > 0$ small enough the viscosity inequalities associated to the equation $F = 0$ hold for u and v. This is obviously the case when $\tilde{x} \in \Omega$ or $(\tilde{x},\tilde{t}) \notin \Gamma_u \cap \Gamma_v$. If $(\tilde{x},\tilde{t}) \in \Gamma_u$, then $v(w_\varepsilon,t_\varepsilon) < \varphi(w_\varepsilon,t_\varepsilon)$ for ε small enough. On the other hand we recall that $\psi(x,t) = 1$ in a neighborhood of Γ_u, thus if ε is small enough, we deduce from (2.7) and the regularity of the boundary

$$z_\varepsilon = w_\varepsilon - \varepsilon n(w_\varepsilon) + o(\varepsilon), \tag{2.9}$$

which implies again by the smoothness of the domain that $z_\varepsilon \in \Omega$. Hence, we have proved the claim. The proof for the case $(\tilde{x},\tilde{t}) \in \Gamma_v$ is similar. Let $\Psi(x,y,t) = |x + \chi(y,t)|^4$ and $\zeta(z,w,t) = \Psi(\frac{z-w}{\varepsilon}, \frac{z+w}{2}, t)$.

Since F may be singular at $p = 0$, we have to consider separately the two cases $D\zeta(z_\varepsilon,w_\varepsilon,t_\varepsilon) \neq 0$ and $D\zeta(z_\varepsilon,w_\varepsilon,t_\varepsilon) = 0$. Assume first there exists a subsequence of $(z_\varepsilon,w_\varepsilon,t_\varepsilon)$ (which we continue to denote by $(z_\varepsilon,w_\varepsilon,t_\varepsilon)$) such that $D\zeta(z_\varepsilon, w_\varepsilon, t_\varepsilon) \neq 0$. We know that for $\varepsilon > 0$ small enough and for all $\alpha > 0$, there exists $(a,p,X) \in \overline{\mathcal{P}}^{2,+}u(z_\varepsilon, t_\varepsilon)$, $(b,q,Y) \in \overline{\mathcal{P}}^{2,-}v(w_\varepsilon, t_\varepsilon)$ such that

$$a - b = \frac{\partial \zeta}{\partial t}(z_\varepsilon, w_\varepsilon, t_\varepsilon) \ ,$$
$$p = D_z\zeta(z_\varepsilon, w_\varepsilon, t_\varepsilon) = \frac{1}{\varepsilon}D_x\Psi + \frac{1}{2}D_y\Psi \ ,$$
$$q = -D_w\zeta(z_\varepsilon, w_\varepsilon, t_\varepsilon) = \frac{1}{\varepsilon}D_x\Psi - \frac{1}{2}D_y\Psi \ ,$$

and, if $A_\varepsilon = D^2\zeta(z_\varepsilon, w_\varepsilon, t_\varepsilon)$

$$-(\frac{1}{\alpha} + ||A_\varepsilon||)Id \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq (Id + \alpha A_\varepsilon)A_\varepsilon \ . \tag{2.10}$$

Moreover, the viscosity inequalities for u and v read

$$a + F_*(x_\varepsilon, t_\varepsilon, u(x_\varepsilon, t_\varepsilon), p, X) \leq 0, \tag{2.11}$$
$$b + F^*(y_\varepsilon, t_\varepsilon, v(y_\varepsilon, t_\varepsilon), q, Y) \geq 0. \tag{2.12}$$

From (2.10) it follows that $X \leq Y$, thus subtracting (2.12) to (2.11) we can write the result in the form

$$\frac{\partial \zeta}{\partial t}(z_\varepsilon, w_\varepsilon, t_\varepsilon) + F_*(x_\varepsilon, t_\varepsilon, u(x_\varepsilon, t_\varepsilon), p, X) - F_*(x_\varepsilon, t_\varepsilon, v(y_\varepsilon, t_\varepsilon), p, X)$$
$$\leq F^*(y_\varepsilon, t_\varepsilon, v(y_\varepsilon, t_\varepsilon), q, Y) - F_*(x_\varepsilon, t_\varepsilon, v(y_\varepsilon, t_\varepsilon), p, X).$$

If we assume $\gamma_R > 0$, by arguing as in [15], we deduce from this inequality $\gamma_R M \leq o(1)$, as $\varepsilon \to 0$. Hence, if ε is small enough, we get a contradiction. If $\gamma_R \geq 0$ and if u or v is a strict sub or supersolution, we are led this time to an inequality of the form $\eta \leq o(1)$, which leads again to a contradiction. This concludes the proof of Theorem 2.1. □

Now we turn to the main application of this result. To do so, we assume that the domain Ω satisfies the following condition

(H4) $\begin{cases} \Omega = \Omega^e \setminus \overline{\Omega^i} \ \text{ where } \Omega^e \subseteq \mathbb{R}^N \ \text{ is a smooth bounded domain,} \\ \Omega^i = \cup_{j=1}^n \overline{B}_j, \ B_j = B(x_j, r_j) \subset\subset \Omega^e \text{ and } \overline{B}_j \cap \overline{B}_k = \emptyset \ \text{ if } j \neq k. \end{cases}$

In Ω^e, Ω^i, the "e" is for "exterior" and the "i" for "interior".

For the nonlinearity F, we use, in a neighborhood of each ball B_i, analogous conditions to the ones used in Section 4 of [15]; more precisely, F is of the form

$$F(x,t,r,p,X) = \overline{F}(x,t,r,p,X) - f(x,t), \tag{2.13}$$

with $f \in C(\overline{\Omega} \times [0,T])$ and $\overline{F}$ satisfying **(H1)**, **(H2)**, **(H3)**. Moreover, it verifies the following structure assumptions

(H5) For any $j = 1, \cdots, n$, there exists $\delta_j > 0$ small enough such that $B(x_j, r_j + \delta_j) \subset \Omega^e$, $B(x_j, r_j + \delta_j) \cap B_k = \emptyset$ if $k \neq j$ and such that

$$\overline{F}(x_j + O^t h, t, r, O^t p, O^t XO) = \overline{F}(x_j + h, y, r, p, X)$$

for every $|h| \leq \delta_j$, $t \in [0, T]$, $r \in \mathbb{R}$, $p \in \mathbb{R}^N \setminus \{0\}$, $X \in \mathcal{S}(N)$ and $O \in \mathcal{O}(N)$.

Next we introduce the following subsets of $\partial_l Q_T$: Σ^p_- is the set of points $(x, t) \in \partial_l Q_T$ such that, for all $R > 0$, one has either

$$\liminf_{\substack{(y,s)\to(x,t)\\ \alpha\downarrow 0}} \Big\{ \inf_{-R\leq r\leq R} \Big[\frac{o_\alpha(1)}{\alpha} + F(y, s, r, \frac{Dd(y) + o_\alpha(1)}{\alpha}, \\ -\frac{1}{\alpha^2} Dd(y) \otimes Dd(y) + \frac{o_\alpha(1)}{\alpha^2})\Big]\Big\} > 0 \tag{2.14}$$

or

$$\liminf_{\substack{(y,s)\to(x,t)\\ \alpha\downarrow 0}} \Big\{ \inf_{-R\leq r\leq R} \Big[\frac{o_\alpha(1)}{\alpha} + F(y, s, r, \frac{Dd(y) + o_\alpha(1)}{\alpha}, \\ \frac{1}{\alpha} D^2 d(y) + \frac{o_\alpha(1)}{\alpha})\Big]\Big\} > 0. \tag{2.15}$$

while Σ^p_+ is the set of points $(x, t) \in \partial_l Q_T$ such that for all $R > 0$ one has either

$$\limsup_{\substack{(y,s)\to(x,t)\\ \alpha\downarrow 0}} \Big\{ \sup_{-R\leq r\leq R} \Big[\frac{o_\alpha(1)}{\alpha} + F(y, s, r, \frac{-Dd(y)) + o_\alpha(1)}{\alpha}, \\ \frac{1}{\alpha^2} Dd(y) \otimes Dd(y) + \frac{o_\alpha(1)}{\alpha^2})\Big]\Big\} < 0 \tag{2.16}$$

or

$$\limsup_{\substack{(y,s)\to(x,t)\\ \alpha\downarrow 0}} \Big\{ \sup_{-R\leq r\leq R} \Big[\frac{o_\alpha(1)}{\alpha} + F(y, s, r, \frac{-Dd(y) + o_\alpha(1)}{\alpha}, \\ -\frac{1}{\alpha} D^2 d(y) + \frac{o_\alpha(1)}{\alpha})\Big]\Big\} < 0, \tag{2.17}$$

where α is a positive parameter and $o_\alpha(1) \to 0$ as $\alpha \downarrow 0$. Finally, we set

$$\Sigma^p = \partial_l Q_T \setminus (\Sigma^p_- \cup \Sigma^p_+).$$

It is worth remarking that if F satisfies (2.13), then Σ^p_- and Σ^p_+ are the same for F and $\overline{F}$.

We will use the following assumption on Σ^p

(H5) For all $(x,t) \in \Sigma^p$, $r,a \in \mathbb{R}$, $p \in \mathbb{R}^N$ and $M \in S(N)$, there exist $\lambda_1, \mu_1 \in \mathbb{R}$ and λ_2, $\mu_2 \in \mathbb{R}$ such that

$$a + F_*(x,t,r,p+\lambda_1 Dd(x), M + \lambda_1 D^2 d(x) + \mu_1 Dd(x) \otimes Dd(x)) > 0$$

$$a + F^*(x,t,r,p+\lambda_2 Dd(x), M + \lambda_2 D^2 d(x) + \mu_2 Dd(x) \otimes Dd(x)) < 0.$$

We add the following condition on the exterior boundary
(H6) Either Ω^e is a ball or $\partial\Omega^e \times [0,T] \subset \Sigma^p_+ \cap \Sigma^p_-$.

We observe that because the symmetry of each ∂B_j and the invariance property **(H5)** of $\overline{F}$, the sets $\Sigma^p_-, \Sigma^p_+, \Sigma^p$ are either empty or unions of connected components of $\partial\Omega$.

Before giving the main result of this section we premise some comments on the sets $\Sigma^p_\pm$ and on the second part of condition **(H6)**. In Section 4 of [15], it is proved that there cannot be loss of boundary conditions respectively of the sub and supersolutions of (BVP), namely for any $(x,t) \in \Sigma^p_-$ (resp. Σ^p_+) and any subsolutions u (resp. supersolutions v) we have $u(x,t) \le \varphi(x,t)$ and $v(x,t) \ge \varphi(x,t)$. Therefore, the condition **(H6)** says that either Ω^e is a ball or the boundary data on $\partial\Omega^e \times [0,T]$ is assumed in a classical sense.

Our main SCR is the following:

Theorem 2.2. (Strong Comparison Result) *Assume that F is of the form* (2.13) *and that assumptions* **(H1)**-**(H6)** *hold. Let $u \in BUSC(\overline{Q}_T)$ and $v \in BLSC(\overline{Q}_T)$ be respectively viscosity sub- and supersolution of* (BVP). *Then $u \le v$ in Q_T. Moreover, if we define $\tilde{u}$ and $\tilde{v}$ on $\overline{Q}_T$ by setting*

$$\tilde{u}(x,t) = \begin{cases} \limsup\limits_{\substack{(y,s)\to(x,t)\\ (y,s)\in Q_T}} u(y,s) & \text{in } \partial_l Q_T \setminus \Sigma^p_+, \\ u(x,t) & \text{otherwise} \end{cases} \tag{2.18}$$

$$\tilde{v}(x,t) = \begin{cases} \liminf\limits_{\substack{(y,s)\to(x,t)\\ (y,s)\in Q_T}} v(y,s) & \text{in } \partial_l Q_T \setminus \Sigma^p_-, \\ v(x,t) & \text{otherwise,} \end{cases} \tag{2.19}$$

then $\tilde{u}$, $\tilde{v}$ are still respectively an usc subsolution and lsc supersolution of (BVP) *and $\tilde{u} \le \tilde{v}$ in $\overline{Q}_T$.*

Before providing the proof of Theorem 2.2, we describe the applications in the quasilinear case, namely,

$$u_t - \mathrm{Tr}\left(A(x,t,Du)D^2u\right) + H(x,t,u,Du) = 0 \quad \text{in } Q_T\,, \tag{2.20}$$

where A and H are continuous functions with values in the space of $N \times N$-matrices and $\mathbb{R}$ respectively, with A satisfying, for all $x \in \overline{\Omega}$, $t \in [0,T]$,

$p, q \in \mathbb{R}^N$, the ellipticity condition

$$\langle A(x,t,p)q,q\rangle \geq 0 \; . \tag{2.21}$$

Our result applies to (2.20) if typically A is independent of x, bounded, symmetric (an unnecessary but simplifying condition), and of the form $A = \sigma\sigma^T$, with σ satisfying for some constant C and for all $t \in [0,T]$, $p, q \in \mathbb{R}^N$

$$|\sigma(t,p) - \sigma(t,q)| \leq \frac{C}{|p| + |q|} \; , \tag{2.22}$$

and if H is of the form $H(x,t,u,p) = \overline{H}(t,u,|p|) - f(x,t)$ with $f \in C(\overline{Q}_T)$ and with $\overline{H}$ satisfying, for some constant $C > 0$ and for any $t \in [0,T]$, $u \in \mathbb{R}^N$ and $p, q \in \mathbb{R}^N$, and

$$D_u\overline{H}(t,u,p) \geq 1/C > 0 \; , \tag{2.23}$$

and

$$|\overline{H}(t,u,|p|) - \overline{H}(t,u,|q|)| \leq C|p-q| \; . \tag{2.24}$$

Of course the main restriction comes from the invariance properties around each ball because of which it is almost necessary to have x-independent nonlinearities (in fact we could have a more form on the nonlinearities above since the invariance property has to hold only in a neighborhood of each ball). It is worth noticing anyway that our result applies to equations (1.1) and (1.2).

Proof of Theorem 2.2. The proof basically consists in using Theorem 2.1 in order to localize in an annular domain around one of the balls B_i or $\partial\Omega^e$, and then in using, in this annular domain, the arguments of the proof of Theorem 3.1 in [15]. Thus, we just outline the main differences.

Under the current assumptions $\tilde{u}$ and $\tilde{v}$ are still viscosity sub and supersolution of (BVP), (see e.g. Corollary 4.1 in [15]). We want to show that $\tilde{u} \leq \tilde{v}$ in $\overline{Q}_T$.

We first use a change of variable by setting

$$\overline{u}(x,t) := \exp(-Kt)\tilde{u} \quad \text{and} \quad \overline{v}(x,t) := \exp(-Kt)\tilde{v} \; .$$

By classical arguments, $\overline{u}$, $\overline{v}$ are still respectively sub and supersolutions of a transformed equation whose nonlinearity still satisfies **(H1)**-**(H3)** with $\gamma_R > 0$ (for the suitable R) if K is chosen large enough.

We argue by contradiction assuming that $M := \max_{\overline{Q}_T}(\overline{u} - \overline{v}) > 0$. Since $\overline{u}$ is usc and $\overline{v}$ is lsc, this maximum is achieved at some point $(x_0, t_0) \in \overline{Q}_T$.

By Theorem 2.1 and since we know that we have $\overline{u}(x,0) \leq u_0(x) \leq \overline{v}(x,0)$ on $\overline{\Omega}$, necessarily $(x_0, t_0) \in \partial_l Q_T$.

If $(x_0, t_0) \in B_j$ for some j, we can argue in the annular domain $B(x_j, r_j + \delta_j) \setminus \overline{B}(x_j, r_j)$: indeed, again by Theorem 2.1, we know that $\overline{u} - \overline{v} < M$ on $\partial B(x_j, r_j + \delta_j)$ and we can readily apply the arguments of [15] in this domain; this lead to the desired contradiction.

If the first part of **(H6)** holds, $\Omega^e = B(z, R)$ for some $z \in \mathbb{R}^N$ and $R > 0$ and if $(x_0, t_0) \in \partial\Omega^e$, we can argue as above in the annular domain $B(z, R) \setminus \overline{B}(z, R - h)$ for $h > 0$ small enough. And again this leads to a contradiction.

Of course, if the second part of **(H6)** holds, we have no loss of boundary condition on $\partial\Omega^e$; this means that (x_0, t_0) cannot be on $\partial\Omega^e$. Therefore, we are in one of the cases which we already examine above and the proof is complete. □

Remark 2.1. We note that the proofs of Theorem 2.1 and Theorem 2.2 can be easily adapted (being even simpler) to the case of state-constraint problems of the type (DSCP) below and to the case of stationary equations. We leave these adaptations to the reader in order to keep this article with a reasonable length.

3. A comparison result in general annular domains

In this section we consider domains of the form $\Omega = \Omega^e \setminus \overline{\Omega}^i$, where $\Omega^i \subset\subset \Omega^e$ are smooth, bounded open subsets of $\mathbb{R}^N$, and we are interested in proving a comparison result between discontinuous viscosity sub and supersolutions to the following generalized Dirichlet-State Constraint problem

$$\begin{cases} u_t + F(x, t, u, Du, D^2u) = 0 & \text{in } \Omega \times (0, T), \\ u(x, t) = \varphi(x, t) & \text{in } \partial\Omega^e \times (0, T), \\ u_t + F(x, t, u, Du, D^2u) \geq 0 & \text{in } \partial\Omega^i \times (0, T), \\ u(x, 0) = u_0(x) & \text{in } \Omega \times \{0\}, \end{cases} \tag{$DSCP$}$$

where $T > 0$. As above, F is a locally bounded real-valued function defined in $\overline{\Omega} \times [0, T] \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}(N)$, the boundary conditions φ and the initial condition u_0 are continuous respectively on $\partial\Omega^1 \times [0, T]$ and on $\overline{\Omega}$ and satisfy the following compatibility condition

$$\varphi(x, 0) = u_0(x) \text{ on } \partial\Omega^e. \tag{3.1}$$

We recall that a function $u \in USC(\overline{Q}_T)$ is a viscosity subsolution of (DSCP) if satisfies

$$\begin{cases} u_t + F(x, t, u, Du, D^2u) \leq 0 & \text{in } Q_T, \\ \min(u_t + F(x, t, u, Du, D^2u), u(x, t) - \varphi(x, t)) \leq 0 & \text{in } \partial\Omega^e \times (0, T), \\ u(x, 0) \leq u_0(x) & \text{in } \Omega \times \{0\}, \end{cases}$$

and a function $v \in LSC(\overline{Q}_T)$ is a viscosity supersolution of (DSCP) if satisfies

$$\begin{cases} v_t + F(x,t,v,Dv,D^2v) \geq 0 & \text{in } Q_T, \\ \max(v_t + F(x,t,v,Dv,D^2v), v(x,t) - \varphi(x,t)) \geq 0 & \text{in } \partial\Omega^e \times (0,T), \\ v_t + F(x,Dv,D^2v) \geq 0 & \text{in } \partial\Omega^i \times (0,T), \\ v(x,0) \geq u_0(x) & \text{in } \Omega \times \{0\}, \end{cases}$$

The basic assumptions of this section for the domain Ω is the following:
(H7) $\Omega = \Omega^e \setminus \overline{\Omega^i}$, where $\Omega^i \subset\subset \Omega^e$ are bounded open subsets of $\mathbb{R}^N$, Ω^i being strictly starshaped with respect to the origin, i.e.,

$$\begin{cases} \exists \lambda_0 > 1,\ \exists \gamma > 0,\ \forall \lambda \in (1,\lambda_0)\ \text{dist}(x, \partial\overline{\Omega^i}) \geq \gamma(\lambda - 1), \ \text{ if } x \in \lambda\overline{\Omega^i} \\ \text{and } \text{dist}(x, \partial\overline{\Omega^i}) \geq \gamma(1 - \lambda^{-1}), \ \text{ if } x \in \lambda^{-1}\overline{\Omega^i}. \end{cases} \tag{3.2}$$

The key assumption on the equation is:
(H8) For any $\lambda > 1$ close to 1 and $R > 0$, there exists $\delta(\lambda), \mu(\lambda)$ converging to 1 as $\lambda \to 1$ such that F satisfies

$$\delta(\lambda)\mu(\lambda)F(\lambda x, \mu(\lambda)t, \frac{r}{\delta(\lambda)}, \frac{p}{\delta(\lambda)\lambda}, \frac{X}{\delta(\lambda)\lambda^2}) \geq F(x,t,r,p,X) + o_\lambda(1) \text{ as } \lambda \to 1, \tag{3.3}$$

uniformly for all $(x,t) \in \overline{\Omega} \times [0,T]$, $|r| \leq R$, $p \in \mathbb{R}^N \setminus \{0\}$, $X \in S(N)$.

Theorem 3.1. *Assume* **(H1)**-**(H3)** *and* **(H7)**-**(H8)**. *Let $u \in BUSC(\overline{Q}_T)$, $v \in BLSC(\overline{Q}_T)$ be respectively viscosity sub- and supersolution of (DSCP). Suppose that $u(x,t) \leq v(x,t)$ in $\partial\Omega^e \times [0,T]$, then $u_*(x,t) \leq v(x,t)$ and $u(x,t) \leq v^*(x,t)$ in Q_T. Moreover, if we define $\tilde{u}$ on $\overline{Q}_T$ by setting*

$$\tilde{u}(x,t) = \begin{cases} \limsup_{\substack{(y,s)\to(x,t) \\ (y,s)\in Q_T}} u(y,s) & \text{in } \partial\Omega^i \times [0,T), \\ u(x,t) & \text{otherwise}, \end{cases} \tag{3.4}$$

then $\tilde{u}$ is still a usc subsolution of (DSCP) and $(\tilde{u})_* \leq v$ and $\tilde{u} \leq v^*$ in $\overline{Q}_T$.

Before providing the proof of Theorem 3.1, we comment the rather technical assumption **(H8)**. To do so, we consider the case of quasilinear equation of the form

$$u_t - \text{Tr}[A(x,t,Du)D^2u] + H(x,t,u,Du) = 0\ , \tag{3.5}$$

where A is locally bounded in $\overline{\Omega} \times [0,\infty) \times \mathbb{R}^N$ and continuous in $\overline{\Omega} \times [0,\infty) \times \mathbb{R}^N$ and H is a continuous function on $\overline{\Omega} \times [0,\infty) \times \mathbb{R} \times \mathbb{R}^N$.

Here the natural choice for $\delta(\lambda)$ and $\mu(\lambda)$ is

$$\delta(\lambda) = \lambda^{-1} \quad \text{and} \quad \mu(\lambda) = \lambda^2\ .$$

Indeed as we will see it in the proof of Theorem 3.1, it corresponds to a change $u_\lambda(x,t) := \delta(\lambda)u(\lambda x, \mu(\lambda)t)$ with a parabolic scaling and is such that $Du_\lambda(x,t) = Du(\lambda x, \mu(\lambda)t)$.

The following restrictions are then natural on A and H: $A = A(\frac{x}{|x|}, p)$ and $H(x,t,p,r) = f(x,t,r) + h(p)$ with $h \geq 0$, $f(x,t,r)$ being continuous with respect to p and (x,t,r).

We need the following:

Lemma 3.1. *$u(x,0) \leq u_0(x)$ for all $x \in \Omega \cup \partial\Omega^e$ and $v(x,0) \geq u_0(x)$ for all $x \in \overline{\Omega}$.*

We refer the reader to [15] or [9] for the proof of Lemma 3.1 in the case when $x \in \Omega \cup \partial\Omega^e$. On $\partial\Omega^i$, because of the state-constraint boundary condition which violates the compatibility condition (2.1), we do not know how to prove this property for the subsolution at this level of generality.

Proof of Theorem 3.1. The proof is inspired from I. Capuzzo Dolcetta and P.L. Lions [12] and H. Ishii and P.L. Lions [23].

We first remark that, since there is no boundary condition for the subsolution on $\partial\Omega^i$, it is clear that $\tilde{u}$ is still a viscosity subsolution of (DSCP).

From now on, for the sake of simplicity of notations, we drop the "$\tilde{}$" on $\tilde{u}$, hence we consider u being a subsolution of (DSCP) which satisfies

$$u(x,t) = \limsup_{\substack{(y,s)\to(x,t)\\(y,s)\in Q_T}} u(y,s) \text{ for all } (x,t) \in \partial\Omega^i \times [0,T), \tag{3.6}$$

For $\lambda > 1$, close to 1, we set $\Omega_\lambda = \lambda^{-1}\Omega$ and we introduce the function

$$u_\lambda(x,t) := \delta(\lambda)u(\lambda x, \mu(\lambda)t),\ x \in \overline{\Omega}_\lambda \times [0, T_\lambda].$$

where the functions $\delta(\lambda), \mu(\lambda)$ are given by **(H3)** and $T_\lambda := \mu(\lambda)^{-1}T$. Because of assumption **(H3)**, u_λ is a subsolution of

$$(u_\lambda)_t + F(x,t,u_\lambda,Du_\lambda,D^2u_\lambda) \leq o_\lambda(1)\ .$$

We are going to compare the functions u_λ and v despite they are defined on slightly different domains and to do so, we observe that we may assume without loss of generality that the constant γ_R in **(H5)** is strictly positive; otherwise we make the change of variable $r \mapsto e^{-Kt}r$ with $K > 0$ large enough and compare the functions $e^{-Kt}u_\lambda, e^{-Kt}v$.

For fixed $\lambda > 1$ and for all $\varepsilon > 0$, we consider the function $\Phi_\varepsilon \colon \overline{\Omega}_\lambda \times \overline{\Omega} \times [0, T_\lambda] \to \mathbb{R}$, defined by

$$\Phi_\varepsilon(x,y,t) = u_\lambda(x,t) - v(y,t) - \frac{|x-y|^4}{\varepsilon^4}.$$

Let $(x_\varepsilon, y_\varepsilon, t_\varepsilon) \in \overline{\Omega}_\lambda \times \overline{\Omega} \times [0, T_\lambda]$ be a maximum point of Φ_ε. By standard arguments we obtain, up to a subsequence, that

$$t_\varepsilon \to t_\lambda \in [0, T_\lambda],\ |x_\varepsilon - y_\varepsilon| = o(\varepsilon),\ x_\varepsilon, y_\varepsilon \to x_\lambda \in \overline{\Omega}_\lambda \cap \overline{\Omega} \text{ as } \varepsilon \to 0$$

$$\lim_{\varepsilon \to 0} [u_\lambda(x_\varepsilon, t_\varepsilon) - v(y_\varepsilon, t_\varepsilon)] = u_\lambda(x_\lambda, t_\lambda) - v(x_\lambda, t_\lambda) = \max_{(\overline{\Omega}_\lambda \cap \overline{\Omega}) \times [0, T_\lambda]} [u_\lambda(x, t) - v(x, t)].$$

As $\lambda \to 1$, extracting if necessary a subsequence, we may assume that $x_\lambda \to \hat{x} \in \overline{\Omega}$ and the following two cases may occur: either $\hat{x} \in \partial\Omega^e$ or $\hat{x} \notin \partial\Omega^e$.

In the first case, we use the fact that there is no loss of boundary condition on $\partial\Omega^e \times [0, T]$ which means that $u(\hat{x}, \hat{t}) \le \varphi(\hat{x}, \hat{t}) \le v(\hat{x}, \hat{t})$. By the upper semicontinuity of u and the lower semicontinuity of v, this implies

$$u_\lambda(x_\lambda, t_\lambda) - v(x_\lambda, t_\lambda) \le o_\lambda(1) \quad \text{as } \lambda \to 1. \tag{3.7}$$

for λ close enough to 1.

We claim that (3.7) implies that $u_*(x, t) \le v(x, t)$ for all $(x, t) \in \overline{\Omega} \times [0, T]$. In fact, given $(x, t) \in \overline{\Omega} \times [0, T]$, then either $(x, t) \in \partial\Omega^e \times [0, T]$ and we are done or for λ close to 1 we have $(x, t) \in \overline{\Omega}_\lambda \times [0, T]$. Since (x_λ, t_λ) is a maximum of $u_\lambda - v$, from (3.7) we get

$$u_\lambda(x, t) \le v(x, t) + o_\lambda(1) \quad \text{as } \lambda \to 1 \tag{3.8}$$

thus,

$$u_*(x, t) \le v(x, t) \tag{3.9}$$

and the claim is proved.

If, on the contrary, $\hat{x} \notin \partial\Omega^e$, then x_λ cannot be on $\lambda^{-1}\partial\Omega^e$ for λ close enough to 1 and there exists a subsequence of $(x_\lambda)_\lambda$ (that we continue to denote by $(x_\lambda)_\lambda$ such that $x_\lambda \in \Omega_\lambda \cap \overline{\Omega}$. We notice that since $|x_\varepsilon - y_\varepsilon| = o(\varepsilon)$, as $\varepsilon \to 0$, and Ω satisfies **(H7)**, we have $x_\varepsilon, y_\varepsilon \in \Omega_\lambda \cap \overline{\Omega}$, for ε small enough.

If $t_\varepsilon = 0$ for all ε, then by Lemma 3.1, we have

$$u_\lambda(x_\varepsilon, 0) \le \delta(\lambda) u_0(\lambda x_\varepsilon) \text{ and } v(y_\varepsilon, 0) \ge u_0(y_\varepsilon).$$

Thus,

$$u_\lambda(x_\varepsilon, 0) - v(y_\varepsilon, 0) \le \delta(\lambda) u_0(\lambda x_\varepsilon) - u_0(y_\varepsilon),$$

and by passing to the limit as $\varepsilon \to 0$, using the continuity of u_0, we get (3.7) and we conclude as before.

If there exists a subsequence of t_ε such that $t_\varepsilon > 0$, then the viscosity inequality holds for both u_λ and v since we know that $x_\varepsilon \in \Omega_\lambda$.

Setting $R = \max(||u||_\infty, ||v||_\infty)$, standard arguments leads to the inequality

$$\gamma_R(u_\lambda(x, t) - v(y, t)) \le o_\lambda(1) \text{ as } \lambda \to 1,$$

and we conclude as before.

In order to prove that $u(x,t) \leq v^*(x,t)$ for all $(x,t) \in \overline{\Omega} \times [0,T]$, we remark that because of the upper semicontinuity of u, it is enough to do it for $x \in \Omega$. We use (3.8): changing x into $\lambda^{-1}x$ and t into $\mu(\lambda)^{-1}t$, for λ close to 1, it yields

$$u(x,t) \leq \delta^{-1}(\lambda)v(\lambda^{-1}x, \mu(\lambda)^{-1}t) + o_\lambda(1)\ ,$$

and by similar arguments as above, letting λ go to 1, we obtain the desired inequality, completing the proof of Theorem 3.1. □

4. An existence result for the state-constraint case

In this section, we provide an existence result of possibly discontinuous solutions for (DSCP) using Perron's method and Theorem 3.1. An analogous –and even stronger– existence result of a continuous solution for (BVP) can be obtained along the lines of F. Da Lio [15]; therefore we skip it and concentrate on the difficulty connected to the state-constraint boundary condition.

It is known that the solvability of (DSCP) depends on various factors such as the value of the boundary data (see e.g. the example in [9]), the structural properties of the operator and the geometry of the domains, in particular the sign of its principal curvatures (see e.g. Section 2 in [15]). Thus it seems reasonable that in order to get the existence of a solution of (DSCP), one need some kind of compatibility conditions involving the structure of the operator and the curvature of the domain.

Now we give some sufficient conditions implying the existence of a solution of (DSCP). In order to simplify the exposure, we suppose that F satisfies **(H1)** with $\gamma_R \geq 0$ for all $R > 0$.

We assume that

(H9) There exists $C \in \mathbb{R}$ and $\nu > 0$ such that, if $\lambda,\ \mu > 0$ satisfy $\mu \leq \nu\lambda(1+\lambda^2)$, then

$$F(x,t,0,-\lambda Dd, -\lambda D^2 d + \mu Dd \otimes Dd) \geq C\ ,$$

for all (x,t) in a neighborhood $\mathcal{W}$ of $\partial\Omega^i \times [0,T]$.

Our result is:

Corollary 4.1. *Assume that the function F satisfies the hypotheses of Theorem 3.1 with $\gamma_R \geq 0$ for any $R > 0$ in* **(H1)**, **(H9)** *holds and $\partial\Omega^e \times [0,T] \subset \Sigma_+^p \cap \Sigma_-^p$. Then for any $\varphi \in C(\partial\Omega^e \times [0,T])$ and $u_0 \in C(\overline{\Omega})$ satisfying (3.1), there exists a unique discontinuous viscosity solution of $(DSCP)$.*

Proof. We use the Perron's method introduced for viscosity solutions by H. Ishii [20] with the version up to the boundary of F. Da Lio [15].

Let d be a smooth extension of the distance function to Ω^i which is constant outside a neighborhood of Ω^i. Under the assumption **(H9)**, one can verify that, for well-chosen constants $M, C > 0$ large enough then $\overline{u}(x,t) = M + Ct - d^\alpha(x)$ with $0 < \alpha < 1$ is a supersolution of (DSCP). Indeed, on $\partial\Omega^i \times (0,T)$, since $\frac{\partial \overline{u}}{\partial n}(x) = +\infty$, we have $J^{2,-}\overline{u}(x,t) = \emptyset$, where $J^{2,-}\overline{u}(x,t)$ is the second-order subjet of $\overline{u}$ at (x,t) relative to $\overline{\Omega} \times [0,T]$, and therefore the state-constraint boundary conditions is automatically satisfied.

It is worth pointing out that for $M > 0$ large enough, $\overline{u} \geq 0$ and since we assume that $\gamma_R \geq 0$ for any $R > 0$, to have 0 instead of r in **(H9)** is enough. On other hand, if $M, C > 0$ are large enough, then the function $\underline{u}(x,t) = -Ct - M$ is a subsolution of $(DSCP)$.

The Perron's method provides us with a (possibly discontinuous) solution u of (DSCP) such that $\overline{u} \leq u \leq \underline{u}$ in $\Omega \times (0,T)$. Moreover, the condition "$\partial\Omega^e \subseteq \Sigma_+^p \cap \Sigma_-^p$." implies that there is no loss of boundary condition on $\partial\Omega^e$ and therefore, any subsolution v and any supersolution w of (DSCP) satisfies

$$v \leq \varphi \leq w \quad \text{on } \partial\Omega^e \times [0,T] \,.$$

The first consequence of this inequality is that $u_* = u^* = \varphi$ on $\partial\Omega^e$ and therefore u is continuous at points of $\partial\Omega^e$. The second one is the uniqueness of the discontinuous solution u which follows from Theorem 3.1. □

We want to test the condition **(H9)** on quasilinear equations of the form

$$u_t - \frac{1}{2}\mathrm{Tr}[A(x,t,Du)D^2u] + H(x,t,u,Du) = 0 \text{ in } \Omega, \tag{4.1}$$

where A, H are locally bounded functions defined on $\overline{\Omega} \times \mathbb{R}^+ \times \mathbb{R}^N \setminus \{0\}$ and $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N$ respectively and with values in the set of nonnegative symmetric $N \times N$ matrices and $\mathbb{R}$. We suppose there exist two functions A_∞, H_∞ such that

$$\lim_{\eta \to +\infty} A(x,t,\eta p) = A_\infty(x,t,p) \tag{4.2}$$

and

$$\lim_{\eta \to +\infty} \frac{H(x,t,r,\eta p)}{\eta} = H_\infty(x,t,r,p). \tag{4.3}$$

These limits being locally uniform respectively on $\overline{\Omega} \times [0,T] \times \mathbb{R}^N \setminus \{0\}$ and $\overline{\Omega} \times [0,T] \times \mathbb{R} \times \mathbb{R}^N$. In this case, the condition in **(H9)** implies

$$\left\{ \begin{array}{c} \langle A_\infty(x,t,-Dd(x))Dd(x), Dd(x)\rangle = 0 \\ \text{and} \\ \frac{1}{2}\mathrm{Tr}[A_\infty(x,t,-Dd(x))D^2d(x)] + H_\infty(x,t,0,-Dd(x)) \geq 0, \end{array} \right. \tag{4.4}$$

Thus, we find the conditions which "allow" loss of boundary condition for the supersolution of $(DSCP)$, (see e.g. Section 2 in [15]).

On the other hand if we assume that $A(x,t,\lambda p) = O(\frac{1}{\lambda^2})$ as $\lambda \to \infty$, and

$$\frac{1}{2}\mathrm{Tr}[A_\infty(x,t,-Dd(x))D^2d(x)] + H_\infty(x,t,0,-Dd(x)) \geq \tilde{\nu} > 0 \tag{4.5}$$

for some $\tilde{\nu} > 0$ then **(H9)** is satisfied.

To be more concrete, we consider the case when

$$A(x,p) = Id - \frac{p \otimes p}{1+|p|^2} \text{ and } H(x,t,r,p) = f(x,t,r) + |p|.$$

In this case

$$A_\infty(x,p) = Id - \frac{p \otimes p}{|p|^2} \text{ and } H_\infty(x,t,r,p) = |p|$$

and the condition **(H9)** is satisfied if

$$\frac{1}{2}\Delta d(x) + 1 > 0 \quad \text{in } \partial\Omega^i.$$

5. Applications to the level-sets approach

In this section we provide an extension of the level-set approach for discontinuous solutions of (DSCP). As we mention it in the introduction, an analogous result can be obtained for (BVP) following exactly F .Da Lio [15] by using Theorem 2.2.

In this section, we first assume that the nonlinearity F satisfies the hypothesis of Theorem 3.1 and Corollary 4.1. In addition, we suppose that the following "geometrical property" holds

$$F(x,t,\lambda p, \lambda X + \nu p \otimes p) = \lambda F(x,t,p,X), \tag{5.1}$$

for all $\lambda > 0,\ \nu \in \mathbb{R}$ and for all $x \in \overline{\Omega},\ t \in [0,T],\ p \in \mathbb{R}^N \setminus \{0\},\ X \in S(N)$.

The main, and rather immediate, consequence of (5.1) is that the equation in (DSCP) is invariant with respect to nondecreasing changes $u \to \phi(u),\ \phi' \geq 0$. Moreover, (5.1) and the local boundedness of F imply

$$F_*(x,t,0,O) = F^*(x,t,0,O) = 0, \tag{5.2}$$

for all $x \in \overline{\Omega}$ and $t \in [0,T]$.

This geometrical property allows us to simplify the assumptions **(H9)** and "$\partial\Omega^e \times [0,T] \subset \Sigma_+^p \cap \Sigma_-^p$". To do so, we introduce

(H10) $\begin{cases} (i)\ F(x,t,-Dd,-D^2d) \geq 0 \text{ in a neighborhood of } \partial\Omega^i \times [0,T], \\ (ii)\ F(x,t,Dd,D^2d) > 0 \text{ on } \partial\Omega^e \times [0,T], \\ (iii)\ F(x,t,-Dd,-D^2d) < 0 \text{ on } \partial\Omega^e \times [0,T]. \end{cases}$

In the sequel, we use the following notations. We set $\partial_l Q_T^e := \partial\Omega^e \times [0,T]$, $\partial Q_T^e := (\Omega \times \{0\}) \cup \partial_l Q_T^e$ and given $\varphi \colon \partial\Omega^e \times [0,T] \to \mathbb{R}$ and $u_0 \colon \overline{\Omega} \to \mathbb{R}$, we denote by $\chi[u_0,\varphi]$ the real-valued, continuous function defined on ∂Q_T^e by

$$\chi[u_0,\varphi](x,t) = \begin{cases} u_0(x) & \text{if } (x,t) \in \overline{\Omega} \times \{0\}, \\ \varphi(x,t) & \text{if } (x,t) \in \partial_l Q_T^e. \end{cases}$$

A consequence of the comparison result (Theorem 3.1) and the existence of a discontinuous solution of (DSCP) (Corollary 4.1) is the following result which asserts the well-posedness of $(DSCP)$.

Theorem 5.1. *Suppose that F satisfies the hypotheses of Theorem* 3.1, (5.1) *and* **(H10)** *and that $u_0 \in C(\overline{\Omega})$, $\varphi \in C(\partial_l Q_T^e)$ satisfies* (3.1). *Then there exists a unique* (*discontinuous*) *solution $U[u_0,\varphi]$ of (DSCP).*

Moreover, if $u_0^1,\ u_0^2 \in C(\overline{\Omega})$, $\varphi_1, \varphi_2 \in C(\partial_l Q_T^e)$ satisfies (3.1) *and*

$$\chi[u_0^1,\varphi_1](x,t) \le \chi[u_0^2,\varphi_2](x,t) \quad \text{on } \partial Q_T^e\ ,$$

then $U_*[u_0^1,\varphi_1](x,t) \le U_*[u_0^2,\varphi_2](x,t)$ and $U^*[u_0^1,\varphi_1](x,t) \le U^*[u_0^2,\varphi_2](x,t)$ in $\overline{Q}_T$.

As we mention it already above, the proof of this result is straightforward by just remarking that, because of (5.1), on one hand, **(H9)** is equivalent to the condition $F(x,t,-Dd,-D^2d) \ge 0$ in a neighborhood of $\partial\Omega^i \times [0,T]$. On the other hand, the conditions (ii) and (iii) in **(H10)** implies respectively that $\partial\Omega^e \times [0,T] \subset \Sigma_-^p$ and $\partial\Omega^e \times [0,T] \subset \Sigma_+^p$.

Next we show that if u is a discontinuous solution of (DSCP), then for any $t \in [0,T]$, the regions

$$E_t^+ := \{x \in \overline{\Omega} :\ u_*(x,t) > 0\}, \quad E_t^- := \{x \in \overline{\Omega} :\ u^*(x,t) < 0\}\ ,$$

$$\Gamma_t = (E_t^+)^c \cap (E_t^-)^c\ ,$$

depend only on F and on the initial-boundary data, namely

$$E_0^+ := \{x \in \overline{\Omega} :\ u_0(x) > 0\}, \quad E_0^- := \{x \in \overline{\Omega} :\ u_0(x) < 0\}\ ,$$

$$\Gamma_0 = \{x \in \overline{\Omega} :\ u_0(x) = 0\}\ ,$$

and

$$E_t^{+,b} := \{(x,t) \in \partial\Omega^e :\ \varphi(x,t) > 0\}, \quad E_t^{+,b} := \{(x,t) \in \partial\Omega^e :\ \varphi(x,t) < 0\}\ ,$$

$$\Gamma_t^b = \{(x,t) \in \partial\Omega^e :\ \varphi(x,t) = 0\}\ ,$$

but not on the choice of their representation through u_0 and φ.

Theorem 5.2. *Under the assumptions of Theorem* 5.1, *let* $U[u_0^1,\varphi_1]$ *and* $U[u_0^2,\varphi_2]$ *be two bounded viscosity solutions of (DSCP) with* $u_0^1, u_0^2 \in C(\overline{\Omega})$, $\varphi_1,\varphi_2 \in C(\partial_l Q_T^e)$ *satisfying* (3.1). *Assume in addition that*

$$\{(x,t)\in\partial Q_T^e : \chi[u_0^1,\varphi_1)](x,t)=0\} = \{(x,t)\in\partial Q_T^e : \chi[u_0^2,\varphi_2](x,t)=0\},$$
$$\{(x,t)\in\partial Q_T^e : \chi[u_0^1,\varphi_1](x,t)>0\} = \{(x,t)\in\partial Q_T^e : \chi[u_0^2,\varphi_2](x,t)>0\},$$
$$\{(x,t)\in\partial Q_T^e : \chi[u_0^1,\varphi_1](x,t)<0\} = \{(x,t)\in\partial Q_T^e : \chi[u_0^2,\varphi_2](x,t)<0\}.$$

Then, for all $t\in[0,T]$, *we have*

$$E_t := \{x\in\overline{\Omega} : U_*[u_0^1,\varphi_1](x,t)>0\} = \{x\in\overline{\Omega} : U_*[u_0^2,\varphi_2](x,t)(x,t)>0\},$$
$$I_t := \{x\in\overline{\Omega} :\ U^*[u_0^1,\varphi_1](x,t)<0\} = \{x\in\overline{\Omega} : U^*[u_0^2,\varphi_2](x,t)<0\},$$
$$\Gamma_t(U[u_0^1,\varphi_1]) = \Gamma_t(U[u_0^2,\varphi_2]).$$

Proof. We follow readily [14] by considering the functions Φ,Ψ given by

$$\Phi(\eta) = \inf\{\chi[u_0^2,\varphi_2](x,t) :\ \chi[u_0^1,\varphi_1](x,t)\geq\eta,\ (x,t)\in\partial Q_T^e\} \tag{5.3}$$
$$\Psi(\eta) = \sup\{\chi[u_0^2,\varphi_2](x,t) :\ \chi[u_0^1,\varphi_1](x,t)\leq\eta,\ (x,t)\in\partial Q_T^e\}. \tag{5.4}$$

By the same arguments in [15] one can see that $\Phi,\ \Psi$ are nondecreasing, respectively lower and upper semicontinuous functions such that

$$\Phi(\chi[u_0^1,\varphi_1](x,t)) \leq \chi[u_0^2,\varphi_2](x,t) \leq \Psi(\chi[u_0^1,\varphi_1](x,t)). \tag{5.5}$$

Moreover, the functions $\Phi,\ \Psi$ are continuous at $\eta=0$ and $\Phi(0)=\Psi(0)=0$. Standard regularization procedure yields the existence of two sequences $\Phi_k,\Psi_k\in C^1(\mathbb{R})$ such that for all k, Φ_k,Ψ_k are nondecreasing functions and

$$\Phi = \sup_k \Phi_k,\ \ \Psi = \inf_k \Psi_k. \tag{5.6}$$

Condition (5.1) implies that, for $i=1,2$ and for all k, $\Phi_k(U[u_0^i,\varphi_i])$ and $\Psi_k(U[u_0^i,\varphi_i])$ are still solution of (DSCP) and since (5.5) holds for all k, Theorem 5.1 yields

$$(\Phi_k(U[u_0^1,\varphi_1](x,t)))_* \leq U_*[u_0^2,\varphi_2](x,t) \leq (\Psi_k(U[u_0^1,\varphi_1](x,t)))_* \quad \text{in } \overline{Q}_T\ ,$$

and

$$(\Phi_k(U[u_0^1,\varphi_1](x,t)))^* \leq U^*[u_0^2,\varphi_2](x,t) \leq (\Psi_k(U[u_0^1,\varphi_1](x,t)))^* \quad \text{in } \overline{Q}_T\ .$$

Since $\Phi_k,\ \Psi_k$ are continuous and nondecreasing, this implies

$$\Phi_k(U_*[u_0^1,\varphi_1](x,t)) \leq U_*[u_0^2,\varphi_2](x,t) \leq \Psi_k(U_*[u_0^1,\varphi_1](x,t)) \quad \text{in } \overline{Q}_T\ ,$$

and

$$\Phi_k(U^*[u_0^1,\varphi_1](x,t)) \leq U^*[u_0^2,\varphi_2](x,t) \leq \Psi_k(U^*[u_0^1,\varphi_1](x,t)) \quad \text{in } \overline{Q}_T\ ,$$

therefore, by (5.6), taking either the sup or inf in k, we obtain

$$\Phi(U_*[u_0^1,\varphi_1](x,t)) \le U_*[u_0^2,\varphi_2](x,t) \quad \text{in } \overline{Q}_T,$$
$$U^*[u_0^2,\varphi_2](x,t) \le \Psi(U^*[u_0^1,\varphi_1](x,t)) \quad \text{in } \overline{Q}_T.$$

In order to conclude, we remark that $\Phi(\eta)>0$ if $\eta>0$ and $\Psi(\eta)<0$ if $\eta<0$. Indeed fixed $\eta>0$, a priori we have $\Phi(\eta)\ge 0$. Suppose by contradiction that $\Phi(\eta)=0$, then $0=\Phi(\eta)=\chi[u_0^2,\varphi_2](x_\eta,t_\eta)$ and $\chi[u_0^1,\varphi_1](x_\eta,t_\eta)\ge\eta>0$, for some $(x_\eta,t_\eta)\in\partial Q_T^e$, but this contradicts the hypotheses. In a similar way it can be proved that if $\eta<0$, then $\Psi(\eta)<0$ and the proof is complete. □

To conclude this section, we examine the case of the mean curvature equation (1.2). Clearly, this equation satisfies the hypotheses of Theorem 3.1 with the natural parabolic scaling $\delta(\lambda)=\lambda^{-1}$ and $\mu(\lambda)=\lambda^2$, and (5.1). For checking **(H10)**, we compute

$$F(x,t,-Dd,-D^2d)=\Delta d(x)\ ,\ F(x,t,Dd,D^2d)=-\Delta d(x)\ ,$$

and it is therefore clear that **(H10)** is satisfied if $\Delta d(x)\ge 0$ in a neighborhood of $\partial\Omega^i$ and $\Delta d(x)<0$ on $\partial\Omega^e$. The first condition on $\partial\Omega^i$ is satisfied, for example, if Ω^i is a smooth convex subset of $\mathbb{R}^N$ (an assumption which is in agreement with the star-shaped hypotheses on Ω^i), while the second is, of course, the classical assumption for having no loss of boundary condition on $\partial\Omega^e$.

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