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Local $C^{0,\alpha}$ estimates for viscosity solutions of Neumann-type boundary value problems

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Abstract

In this article, we prove the local $C^{0,\alpha}$ regularity and provide $C^{0,\alpha}$ estimates for viscosity solutions of fully nonlinear, possibly degenerate, elliptic equations associated to linear or nonlinear Neumann type boundary conditions. The interest of these results comes from the fact that they are indeed regularity results (and not only a priori estimates), from the generality of the equations and boundary conditions we are able to handle and the possible degeneracy of the equations we are able to take into account in the case of linear boundary conditions.

Keywords: Regularity of solutions of partial differential equations; Hölder regularity; Elliptic equations; Linear and nonlinear boundary conditions; Viscosity solutions; Degenerate equations

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Contents

1. Introduction

In this article, we are interested in the local $C^{0,\alpha}$ regularity of viscosity solutions of nonlinear Neumann boundary value problems of the form

$$\begin{cases} F(x,u,Du,D^2u)=0 & \text{in } O, \\ G(x,u,Du)=0 & \text{on } \partial O, \end{cases} \tag{1}$$

where $O \subset \mathbb{R}^n$ is a smooth domain, F and G are, at least, real-valued continuous functions defined respectively on $\overline{O} \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n$ and $\partial O \times \mathbb{R} \times \mathbb{R}^n$, where $\mathcal{S}^n$ denotes the space of real, $n \times n$, symmetric matrices. The solution u is a scalar function and Du, D^2u denote respectively its gradient and Hessian matrix. More precise assumptions on O, F, G are given later on.

We recall that the boundary condition $G = 0$ is said to be a nonlinear Neumann boundary condition if the function G satisfies the following conditions:

(G1) For all $R > 0$, there exists $\mu_R > 0$ such that, for every $(x,u,p) \in \partial O \times [-R,R] \times \mathbb{R}^n$, and $\lambda > 0$, we have

$$G\big(x,u,p+\lambda n(x)\big) - G(x,u,p) \geqslant \mu_R \lambda, \tag{2}$$

where $n(x)$ denotes the unit outward normal vector to ∂O at $x \in \partial O$.

(G2) For all $R > 0$ there is a constant $K_R > 0$ such that, for all $x, y \in \partial O$, $p, q \in \mathbb{R}^n$, $u, v \in [-R,R]$, we have

$$\big|G(x,u,p) - G(y,v,q)\big| \leqslant K_R\big[\big(1+|p|+|q|\big)|x-y| + |p-q| + |u-v|\big]. \tag{3}$$

The main examples of boundary conditions we have in mind are the following: first, linear type boundary conditions like oblique derivative boundary conditions, in which G is given by

$$G(x,u,p) = \big\langle p, \gamma(x)\big\rangle + g(x), \tag{4}$$

where $\gamma : \partial O \to \mathbb{R}^n$ is a bounded, Lipschitz continuous vector field such that

$$\big\langle \gamma(x), n(x) \big\rangle \geqslant \beta > 0 \quad \text{for all } x \in \partial O,$$

and $g \in C^{0,\beta}(\partial O)$ for some $0 < \beta < 1$. Here and below, "$\langle p, q \rangle$" denotes the usual scalar product of the vectors p and q of $\mathbb{R}^n$.

Next, nonlinear boundary conditions: the first example is capillarity type boundary conditions for which G is given by

$$G(x, u, p) = \big\langle p, n(x) \big\rangle - \theta(x)\sqrt{1 + |p|^2}, \tag{5}$$

where $\theta : \partial O \to \mathbb{R}^n$ is a Lipschitz scalar function such that $|\theta(x)| < 1$ for every $x \in \partial O$. A second example is the boundary condition arising in the optimal control of processes with reflection when there is control on the reflection, namely

$$G(x, u, p) = \sup_{\alpha \in A} \big\{ \big\langle \gamma_\alpha(x), p \big\rangle + c_\alpha(x) u - g_\alpha(x) \big\}, \tag{6}$$

where A is a compact metric space, $\gamma_\alpha : \partial O \to \mathbb{R}^n$ are Lipschitz continuous vector fields such that $\langle \gamma_\alpha(x), n(x) \rangle \geqslant \beta > 0$ for all $x \in \partial O$, and $c_\alpha, g_\alpha : \partial O \to \mathbb{R}$ are Lipschitz continuous, scalar functions.

We are going to show that, under suitable assumptions on O, F, G, any continuous viscosity solution of (1) is in $C^{0,\alpha}_{\text{loc}}(\overline{O})$ for some $0 < \alpha < 1$, with an estimate on the local $C^{0,\alpha}$-norm of u. These results are indeed regularity results (and not only a priori estimates); this is their first main advantage. But they are also valid, in the case of linear boundary conditions, for possibly degenerate equations, a second original feature. The counterpart is that the regularity properties we have to impose on O, F, and G, are stronger than in the case of a priori estimates where the solution u is assumed to be either in $C^2(O) \cap C^1(\overline{O})$ or at least in $W^{2,n}(O) \cap C^1(\overline{O})$.

The classical a priori estimates for this type of problems are in fact proved for linear equations and extended to fully nonlinear equations by a simple linearization procedure described in Lieberman and Trudinger [14]; this linearization requires the regularity of the solution. For linear equations, the classical results are obtained under rather weak assumptions on the coefficients of the operators in the equation and in the boundary condition. To the best of our knowledge, the first results in this direction are the ones of Nadirashvili [18,19] for linear, uniformly elliptic equations with L^∞ coefficients associated to oblique derivative boundary condition of the form

$$\big\langle Du, \gamma(x) \big\rangle + a(x)u(x) + g(x) = 0 \quad \text{in } \partial O. \tag{7}$$

He first proves them with a continuous direction of reflection γ and for Lipschitz domains, and then for a direction of reflection in L^∞ for C^2 domains. Results in this direction were also obtained in the 80s by Lieberman [12] by different methods. Recent improvements on the regularity of the coefficients of the equation (which can be assumed to be L^p for p large enough or in L^n), were obtained by Kenig and Nadirashvili [11] and Lieberman [13].

The case of fully nonlinear equations was first considered in Lions and Trudinger [17] who show the existence of a smooth classical solution in $C^2(O) \cap C^{1,1}(\overline{O})$ for Hamilton–Jacobi–Bellman equations with smooth coefficients and directions of reflection. As mentioned above, in Lieberman and Trudinger [14], the case of fully nonlinear equations is considered in a more systematic way but most of the results are obtained by a linearization procedure and are based on results for linear equations; it is worth pointing out that, in general, the passage from a priori estimates to regularity results requires the existence of smooth enough solutions for a sequence of approximate problems (and even uniqueness for the problem itself), and is often only valid for convex or concave equations.

Our approach is based on classical viscosity solutions (not L^p viscosity solutions): for a detailed presentation of the theory of viscosity solutions and of the boundary conditions in the viscosity sense, we refer the reader to the "User's guide" of Crandall et al. [7] and the book of Fleming and Soner [8], while the books of Bardi and Capuzzo Dolcetta [1] and Barles [2] provide an introduction to the theory in the case of first-order equations.

Clearly, this approach requires more regularity properties for the operator G which has to be locally Lipschitz with respect to its variables, for the domain O which has to be assumed to be C^2 and for the equation (F has to be continuous). Its advantage is that, in the case of oblique boundary conditions of the form (7), we require only $F(x,u,p,M)$ to be nondegenerate (in a sense precised below) in one direction which depends, near the boundary, only on p and γ. Whereas in the case of more general nonlinear boundary conditions we require F to be uniformly elliptic. To prove such results, we use systematically an idea introduced in Ishii and Lions [10] which has already been used to obtain interior, local regularity (or global regularity) in [3] and Barles and Souganidis [6].

In this paper, for technical reasons, we treat separately the "linear case," i.e., typically the case of oblique derivative boundary condition where the operator G is linear with respect to u and p and the "nonlinear" case where G is not linear. A surprising fact in the linear case—and maybe our result is not optimal in this direction—is that the assumptions on F, and in particular the ellipticity one, depends on γ. We were unable to remove this dependence.

In the case of Neumann boundary condition, i.e., when $\gamma(x) = n(x)$, our "strong ellipticity condition" can be written formally as

$$\frac{\partial F}{\partial M}(x,u,p,M) \leqslant -\lambda \hat{p} \otimes \hat{p} \quad \text{for almost every } (x,u,p,M), \tag{8}$$

where $\lambda > 0$ and where, here and below, the notation $\hat{p}$ stands for $p/|p|$. This condition is the natural requirement for the interior $C^{0,\alpha}$ regularity to hold and it allows to extend the results up to the boundary.

Next, if the direction of reflection γ is C^2, then a classical property which is used in Lions [15] (see also Lions and Sznitman [16]) is the existence of a C^2 function $A(x)$, taking values in the set of nonnegative symmetric matrices and such that $A(x)\gamma(x) = n(x)$ for any $x \in \partial O$. In this case, we have to require that the above "strong ellipticity condition" is valid but replacing in (8) $\hat{p}$ by $\widehat{A^{-1}(x)p}$. Since A is not unique, this assumption is admittedly not completely satisfactory.

Finally, if $\gamma(x) \neq n(x)$ is just Lipschitz continuous, then we have again to assume "strong ellipticity condition" of F in the $\widehat{A^{-1}(x)p}$ direction where again $A(x)\gamma(x) = n(x)$ but, here, $A(x)$ is just Lipschitz continuous and this creates technical difficulties.

We mention that, both in the linear and nonlinear case, we prove the regularity result by assuming that G does not depend on u. Indeed, one can always reduce to this case by a suitable change of variable that we show later on.

The proofs of these results rely on the constructions of suitable test-functions inspired by the test-functions built for proving uniqueness results: in the case of Neumann or regular oblique derivatives problems, the corresponding uniqueness results were proved by Lions [15] (see also [7]) and in the case of nonlinear Neumann-type boundary condition in [4]. It is worth pointing out anyway that the construction in the case of Lipschitz continuous γ's, which is the difficult case, takes a completely different form here.

It is worth mentioning also the results of Ishii [9] proved, in the case of nonlinear Neumann boundary conditions, under weaker assumptions on O but stronger assumptions on the boundary condition than in [4]; our approach requires more regularity of the boundary and therefore we do no use the test-function built in [9].

This paper is organized as follows: in Section 2, we state our regularity results both in the linear (Section 2.1) and in the nonlinear case (Section 2.2) and we provide the main proofs. Such proofs rely on the constructions of a suitable test-functions which are different in the linear and nonlinear case: these constructions are given in Section 3. It is worth pointing out anyway that, despite most of the arguments are common in these two cases, the conclusion is a little bit different because of the particular "ellipticity conditions" used in these two cases.

2. The local $C^{0,\alpha}$ estimates

In this section, we state and prove the local $C^{0,\alpha}$ regularity of the solutions of the problem (1) both in the linear and nonlinear cases. As pointed out in the Introduction, these two cases requires slightly different assumptions. We first introduce the assumptions which are common of both cases. First, for the domain O, we require

(H1) (*Regularity of the boundary*) O is a domain with a C^2-boundary.

This assumption on O implies the existence of an $\mathbb{R}^n$-neighborhood $\mathcal{V}$ of ∂O such that the signed distance function d which is positive in O and negative in O^c is in $C^2(\mathcal{V})$. We still denote by d a C^2-extension of the signed distance function to $\mathbb{R}^n$ which agrees with d in $\mathcal{V}$ and we use below the notation $n(x) = -Dd(x)$ even if x is not on the boundary.

The "strong ellipticity" conditions on F are different in the linear and nonlinear cases but the following natural growth condition on F is, on the contrary, the same:

(H2) (*Growth condition on F*) For any $R > 0$, there exist positive constants C_1^R, C_2^R, C_3^R and functions ω_1^R, ω_2^R, $\varpi^R : \mathbb{R}^+ \to \mathbb{R}$ such that $\omega_1^R(0+) = 0$, $\omega_2^R(r) = O(r)$ as $r \to +\infty$, $\varpi^R(t) \to 0$ as $t \to +\infty$, and for any $x, y \in \overline{O}$, $-R \leqslant u, v \leqslant R$, $p, q \in \mathbb{R}^n$, $M \in \mathcal{S}^n$ and $K > 0$,

$$
\begin{aligned}
&F(x,u,p,M)-F(y,v,q,M+K\mathrm{Id})\\
&\quad\leqslant \omega_1^R\big(|x-y|\big(1+|p|+|q|\big)+\varpi^R\big(|p|\wedge|q|\big)|p-q|\big)\|M\|\\
&\qquad+\omega_2^R(K)+C_1^R+C_2^R\big(|p|^2+|q|^2\big)+C_3^R|x-y|\big(|p|^3+|q|^3\big),
\end{aligned}
$$

where $|p|\wedge|q|=\min(|p|,|q|)$.

In the sequel, K always denotes a positive constant which may vary from line to line, depends only on the data of the problem and is, in particular, independent on the small parameters we are going to introduce.

2.1. *The case of linear boundary conditions*

In this subsection we examine the case when G is linear with respect to p, namely it is of the form (7).

The main additional assumptions on F and G are the following.

(H3a) (*Oblique-derivative boundary condition and ellipticity*) There exists a Lipschitz continuous function $A:\overline{O}\to\mathcal{S}^n$ with $A\geqslant c_0\mathrm{Id}$, for some $c_0>0$ such that $A(x)\gamma(x)=n(x)$ for every $x\in\partial O$, and for any $R>0$, there exist $L_R,\lambda_R 0$ such that, for all $x\in\overline{O}$, $|u|\leqslant R$, $|p|>L_R$ and $M,N\in\mathcal{S}^n$ with $N\geqslant 0$, we have

$$
\begin{aligned}
&F(x,u,p,M+N)-F(x,u,p,M)\\
&\quad\leqslant -\lambda_R\big\langle N\widehat{A^{-1}(x)p},\widehat{A^{-1}(x)p}\big\rangle+o(1)\|N\|,
\end{aligned}
\tag{9}
$$

where $o(1)$ denotes a function of the real variable $|p|$ which converges to 0 as $|p|$ tends to infinity.

Before stating the assumption on the boundary condition, we want to point out that the existence of such A is really an assumption in a neighborhood of ∂O, then, under suitable assumptions on F, A can be extended to $\overline{O}$ and even to $\mathbb{R}^n$.

For the boundary condition, we require

(H4) (*Regularity of the boundary condition*) The functions γ and a in (7) are Lipschitz continuous on ∂O, $\langle\gamma(x),n(x)\rangle\geqslant\beta>0$ for any $x\in\partial O$ and g is in $C^{0,\beta}_{\mathrm{loc}}(\partial O)$ for some $0<\beta\leqslant 1$.

Our result is

Theorem 2.1. *Assume* (H1)–(H2)–(H3a)–(H4). *Then every continuous viscosity solution u of* (1) *with G given by* (7) *is in $C^{0,\alpha}_{\mathrm{loc}}(\overline{O})$ for any $0<\alpha<1$ if $\beta=1$ and with $\alpha=\beta$ if $\beta<1$. Moreover, the $C^{0,\alpha}_{\mathrm{loc}}$-norms of u depend only on O, F, γ, a, g through the constants and functions appearing in* (H2)–(H3a)*, the local $C^{0,1}$-norm of γ and a, the local $C^{0,\beta}$-*

norm of g and the local C^2-norm of the distance function of the boundary including the modulus of continuity of D^2d.

Despite we make a point here to have a rather unified result as we do it for the proof, Theorem 2.1 contains clearly three cases which are rather different from the technical point of view.

(1) The homogeneous Neumann boundary condition $\frac{\partial u}{\partial n} = 0$ or more generally $\frac{\partial u}{\partial n} + g(x) = 0$ when g is a C^2-function, is the simplest case. Of course, one can take A as being the identity matrix and the construction of the test-function does not require the heavy regularization procedures we use in Section 3.

It is worth pointing out here that, in the construction of the test-function, the two terms $\frac{\partial u}{\partial n}$ and g are treated in fact separately. To prove a result with g being either Hölder or Lipschitz continuous requires the rather sophisticated regularization argument of case (3) and is therefore of a different level of difficulty.

(2) The case of "regular" oblique derivative boundary condition does not differ so much technically from the first case. The assumption which says that $A(x)\gamma(x) = n(x)$ where, for any $x \in \partial O$, $A(x)$ is a nonnegative symmetric matrix with a C^2 dependence in x, implies that γ is a C^1 function of x.

(3) The case when γ is only Lipschitz continuous and when g is only Hölder or Lipschitz continuous, is technically very different as it is for the comparison results (cf. [4]). Here the only (known) way to treat this case is through a nontrivial regularization argument which we are going to use also.

Of course, in the proof below, we emphasize more the (difficult) third case: the proofs are far easier in the two first ones.

Assumption (H2) is a classical hypothesis in such type of regularity result: it is the same as the one which appears for the interior regularity (cf. Ishii and Lions [10], Barles [3]). It is worth pointing out, anyway, that the treatment of the oblique derivative boundary condition does not lead to a stronger assumption.

Concerning (H3a), we recall that, for the interior regularity, just the strong ellipticity "in the gradient direction" (cf. (8)) is needed as in the case of homogeneous boundary condition. Unfortunately, in the case of oblique derivatives boundary conditions, this natural assumption does not seem to be enough or, at least, it has to be reformulated in a far less natural way. Of course, all these conditions hold when a classical uniform ellipticity property holds, like (H3b) below.

One of the main examples we have in mind is the case of standard quasilinear equations

$$-\operatorname{Tr}\big[b(x, Du)D^2u\big] + H(x, u, Du) = 0 \quad \text{in } O, \tag{10}$$

where b is a $n \times n$ matrix and H a continuous function. In this case, the assumptions (H2) and (H3a) are easily checkable.

(H3a) is equivalent to: there exists $\lambda > 0$ such that, for any $x \in \overline{O}$, $p \in \mathbb{R}^n$,

$$b(x,p) \geqslant \lambda \widehat{A^{-1}(x)p} \otimes \widehat{A^{-1}(x)p} - o(1)\mathrm{Id},$$

where, as in (H3a), $o(1)$ is a function of $|p|$ which converges to 0 as $|p| \to +\infty$.

This assumption may seem restrictive, in particular the fact that the constant λ does not depend on x and p; but, for general equations, if b satisfies the above statement with a strictly positive λ depending on x and p, one can divide the equation (i.e., b and H) by $\lambda(x,p)$ and the above property becomes true with $\lambda = 1$.

We conclude these remarks about (H3a) by emphasizing the role of the "$o(1)$" term and, for the sake of simplicity, we assume that $A \equiv \mathrm{Id}$. Without this term, (H3a) would be essentially reduced to

$$b(x,p) \geqslant \lambda \hat{p} \otimes \hat{p},$$

for any $x \in \overline{O}$ and $p \in \mathbb{R}^n - \{0\}$, while, with this term, (H3a) is satisfied if

$$b(x,p) \geqslant \lambda \widehat{q(x,p)} \otimes \widehat{q(x,p)},$$

where q is a continuous function such that $|p|^{-1}(q(x,p) - p) \to 0$ as $p \to \infty$, uniformly with respect to $x \in \overline{O}$. This condition is not only a more general assumption on b but it is also far easier to check it. Of course, a similar remark can be made for general A's.

Now we turn to (H2). It is satisfied when

(i) b is a bounded, continuous function of x and p and there exists a modulus of continuity $\omega_1 : \mathbb{R}^+ \to \mathbb{R}^+$ and a function $\varpi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\varpi(t) \to 0$ as $t \to +\infty$ and

$$\left|b(x,p) - b(y,q)\right| \leqslant \omega_1\big(|x-y|\big(1 + |p| + |q|\big) + \varpi\big(|p| \wedge |q|\big)|p-q|\big).$$

Moreover, the uniform bound on b provides a ω_2 with a linear growth.

(ii) The function H satisfies: for any $R > 0$, there exist positive constants C_1^R, C_2^R, C_3^R such that, for any $x, y \in \overline{O}$, $-R \leqslant u, v \leqslant R$ and $p, q \in \mathbb{R}^n$,

$$H(x,u,p) - H(y,v,q) \leqslant C_1^R + C_2^R\big(|p|^2 + |q|^2\big) + C_3^R|x-y|\big(|p|^3 + |q|^3\big).$$

As we already mentioned it above, this assumption is classical (see Ishii and Lions [10], Barles [3]).

Proof of Theorem 2.1. We are going to do the proof in two steps: the first one consists in proving the result when $a \equiv 0$ and contains the main arguments. In the second one, we show how to handle the "$a(x)u$" term. According to the proof of the comparison result, it is clear that to take in account such a u-term in regularity result is not immediate and we are going to do it in a very indirect way, by adding an extra variable.

Step 1. The $a \equiv 0$ case.

Since we are going to argue locally, we start with some notations. For every $x_0 \in \overline{O}$ and $r > 0$, $B_{\overline{O}}(x_0, r) := B(x_0, r) \cap \overline{O}$ denotes the open ball in the topology of $\overline{O}$ while $\overline{B}_{\overline{O}}(x_0, r) := \overline{B}(x_0, r) \cap \overline{O}$ denotes the closed ball in the topology of $\overline{O}$ and $\partial B_{\overline{O}}(x_0, r) := \partial B(x_0, r) \cap \overline{O}$. In a similar way we define $B_O(x_0, r)$, $\overline{B}_O(x_0, r)$, and $\partial B_O(x_0, r)$.

We assume that $g \in C^{0,\beta}_{\mathrm{loc}}(\partial O)$ for some $0 < \beta < 1$, the case $g \in C^{0,1}_{\mathrm{loc}}(\partial O)$ being treated in a similar way.

We are going to prove that, if we choose $\alpha = \beta$, for all $x_0 \in \overline{O}$, if r is small enough, then there exists a constant C depending on the different data of the problem, such that, for any $x \in B_{\overline{O}}(x_0, r)$, we have

$$u(x_0) - u(x) \leqslant C|x - x_0|^\alpha. \tag{11}$$

All together, these inequalities give the answer provided that we control the dependence of r and C in x_0, which will be the case.

The proof of this estimate is done in two steps whose arguments are the same: the first step consists in proving that the result holds for α small enough (depending on the local L^∞ norm of u and the data of the problem) and then that this property implies that the result holds for $\alpha = \beta$.

We provide the main arguments of the proof of (11) in the case when the boundary condition plays a role, i.e., when $B(x_0, r) \cap \partial O \neq \emptyset$; the other case is simpler and can be treated by the methods of [3]. In the common proof of the two steps, we therefore argue with some $\alpha \leqslant \beta$. The following lemma is the key stone of the proof.

Lemma 2.1. *Assume that $B(x_0, r) \cap \partial O \neq \emptyset$ and that u is a bounded, continuous solution of* (1) *in $B_{\overline{O}}(x_0, 3r)$ with the oblique derivative boundary condition on $B(x_0, 3r) \cap \partial O$. Under the assumptions of Theorem* 2.1 *on F, γ and g, then there exists a constant $C > 0$, depending on F, γ, g and $\|u\|_{L^\infty(\overline{B}_{\overline{O}}(x_0,3r))}$ such that for all $x \in B_{\overline{O}}(x_0, r)$ the estimate* (11) *holds.*

Proof. In order to prove (11), we consider the auxiliary function

$$\Phi_0(x, y) = u(x) - u(y) - \Theta_0(x, y),$$

where the function Θ_0 has the following form:

$$\Theta_0(x, y) = Ce^{-\tilde{K}(d(x)+d(y))}\big[\psi_0(x, y)\big]^{\alpha/2} + Le^{-\tilde{K}(d(x)+d(x_0))}\psi_0(x, x_0) + \chi_0(x, y),$$

where $\alpha \in (0, \beta]$ is a fixed constant, C, L, $\tilde{K}$ are some large constants to be chosen later on and where the continuous functions $\psi_0(x, y)$, $\chi_0(x, y)$ satisfy the properties listed in the lemma below. In order to point out the main dependences in these functions but also to simplify the rather technical estimates we have to make in the proofs, we introduce the following notations which are used in all the sequel:

$$X := \frac{x+y}{2}, \qquad Y := x - y, \qquad Z := d(x) - d(y), \qquad T := d(x) + d(y). \qquad \square$$

Lemma 2.2. *Under the assumptions of Theorem* 2.1*, for $\delta \geqslant 0$ small enough, there exist real-valued, continuous functions $\tilde{\psi}_\delta(X,Y,T)$, $\tilde{\chi}_\delta(X,Y,T,Z)$ defined respectively in $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ and in $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$, such that if we set*

$$\psi_\delta(x,y) := \tilde{\psi}_\delta\left(\frac{x+y}{2}, x-y, d(x)+d(y)\right),$$

$$\chi_\delta(x,y) := \tilde{\chi}_\delta\left(\frac{x+y}{2}, x-y, d(x)+d(y), d(x)-d(y)\right),$$

the following facts hold for some constant K depending only on the local $C^{0,1}$-norm of γ and the $C^{0,\beta}$-norm of g:

(i) *For any $X, Y \in \mathbb{R}^n$, $T \in \mathbb{R}$ and for $\delta \geqslant 0$ small enough,*

$$K^{-1}|Y|^2 \leqslant \tilde{\psi}_\delta(X,Y,T) \leqslant K|Y|^2 + K\delta, \tag{12}$$

$$-K|Z| \leqslant \tilde{\chi}_\delta(X,Y,T,Z) \leqslant K|Z| + K\delta^\alpha. \tag{13}$$

(ii) *When $\delta \to 0$, $\tilde{\psi}_\delta(X,Y,T) \to \tilde{\psi}_0(X,Y,T)$ and $\tilde{\chi}_\delta(X,Y,T,Z) \to \tilde{\chi}_0(X,Y,T,Z)$, uniformly on each compact subset of $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ and $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$, respectively.*

(iii) *When $\delta > 0$, the functions $\tilde{\psi}_\delta$, $\tilde{\chi}_\delta$ are C^2-functions and the following estimates hold for any X, Y, Z, T:*

$$\left\langle D^2_{YY}\tilde{\psi}_\delta(X,Y,T)Y, Y\right\rangle = 2\tilde{\psi}_\delta + O\left(|Y|^3\right) + O(\delta) \quad \text{as } Y \to 0,\ \delta \to 0,$$

$$\left\langle D_Y\tilde{\psi}_\delta(X,Y,T), Y\right\rangle = 2\tilde{\psi}_\delta + O\left(|Y|^3\right) + O(\delta) \quad \text{as } Y \to 0,\ \delta \to 0,$$

$$\left|D_X\tilde{\psi}_\delta(X,Y,T)\right| \leqslant K|Y|^2, \qquad \left|D_Y\tilde{\psi}_\delta(X,Y,T)\right| \leqslant K|Y|,$$

$$\left|D^2_{XX}\tilde{\psi}_\delta(X,Y,T)\right| \leqslant K|Y|, \qquad \left|D^2_{XY}\tilde{\psi}_\delta(X,Y,T)\right| \leqslant K|Y|,$$

$$\left|D^2_{YY}\tilde{\psi}_\delta(X,Y,T)\right| \leqslant K, \qquad \left|D_T\tilde{\psi}_\delta(X,Y,T)\right| \leqslant K\delta,$$

$$D^2_{TT}\tilde{\psi}_\delta = D^2_{TX}\tilde{\psi}_\delta = D^2_{TY}\tilde{\psi}_\delta = 0.$$

Moreover, if $\Lambda_\delta := (|Y|^2 + \delta^2)^{1/2}$

$$\left|D_X\tilde{\chi}_\delta(X,Y,T,Z)\right| \leqslant \Lambda_\delta^{\alpha-1}|Z|,$$

$$\left|D_Y\tilde{\chi}_\delta(X,Y,T,Z)\right| \leqslant \Lambda_\delta^{\alpha-1}|Z|,$$

$$\left|D_Z\tilde{\chi}_\delta(X,Y,T,Z)\right| \leqslant K, \qquad \left|D_T\tilde{\chi}_\delta(X,Y,T,Z)\right| \leqslant K\delta,$$

$$\left|D^2\tilde{\chi}_\delta(X,Y,T,Z)\right| \leqslant \Lambda_\delta^{\alpha-2}|Z|.$$

(iv) *There exists a constant $\tilde{K} > 0$ large enough* (*independent of C and L*) *such that, if we set*

$$\Theta_\delta(x,y) = Ce^{-\tilde{K}(d(x)+d(y))}\big[\psi_\delta(x,y)\big]^{\alpha/2} + Le^{-\tilde{K}(d(x)+d(x_0))}\psi_\delta(x,x_0) + \chi_\delta(x,y), \tag{14}$$

then, for $|x-y|$ small enough, we have

$$\big\langle D_x\Theta_\delta(x,y),\gamma(x)\big\rangle + g(x) > 0 \quad \text{if } x \in \partial O, \tag{15}$$

$$\big\langle -D_y\Theta_\delta(x,y),\gamma(y)\big\rangle + g(y) < 0 \quad \text{if } y \in \partial O. \tag{16}$$

The proof of the key Lemma 2.2 is postponed to Section 3.1. The first two properties in the point (iii) in Lemma 2.2 are going to play a central role in the proof.

We continue with

Proof of Lemma 2.1. We are going to show that, for a suitable choice of $L > 0$, chosen large enough in order to localize, then for $C > 0$ large enough, we have

$$M_{L,C} := \max_{\overline{B}_{\overline{O}}(x_0,r)\times\overline{B}_{\overline{O}}(x_0,r)} \Phi_0(x,y) \leqslant 0. \tag{17}$$

Indeed, if (17) holds then plugging $x = x_0$ and using the estimates (12) and (13), we get

$$u(x_0) - u(y) \leqslant K|x_0 - y|^\alpha,$$

for some constant $K > 0$ depending on α, r, $\|u\|_{L^\infty(\overline{B}_{\overline{O}}(x_0,r))}$, F through (H2)–(H3a), the uniform Lipschitz norm of γ and the uniform Hölder norm of g in $\overline{B}_{\overline{O}}(x_0,r)$ and independent of x_0 (at least if x_0 remains in a compact subset of $\overline{O}$), which is the desired regularity result.

To prove (17), we first choose $L, C > 0$ large enough in order to have $\Phi_0(x,y) \leqslant 0$ for x or y on $\partial B(x_0,r) \cap \overline{O}$. This is possible since u is locally bounded on $\overline{O}$ and since the conditions (12) and (13) imply that

$$C_1|x-y|^2 \leqslant \psi_0(x,y) \leqslant C_2|x-y|^2, \tag{18}$$

$$-C_3|x-y| \leqslant \chi_0(x,y) \leqslant C_4|x-y|. \tag{19}$$

Of course, L and C depends on r.

From now on, we fix such an L and we argue by contradiction assuming that, for all $C > 0$, $M_{L,C} > 0$. Since Φ_0 is a continuous function, the maximum is achieved at some $(\bar{x},\bar{y}) \in \overline{B}_{\overline{O}}(x_0,r) \times \overline{B}_{\overline{O}}(x_0,r)$ and we observe that, by the choice of L, C, we may even assume that $\bar{x} \in B_{\overline{O}}(x_0, 3r/4)$ and $\bar{y} \in B_{\overline{O}}(x_0,r)$. Here we have dropped the dependence of $\bar{x},\bar{y}$ on C for simplicity of notations.

Two quantities are going to play a key role in the proof:

$$Q_1 := C|\bar{x}-\bar{y}|^\alpha, \qquad Q_2 := L|\bar{x}-x_0|^2$$

(again we have dropped the dependence of Q_1, Q_2 in C for the sake of simplicity of notations). The reason for that is the following: by using only the local boundedness of u, we are only able to show that Q_1, Q_2 are uniformly bounded when C becomes very large while if we use the local modulus of continuity of u, we can show that $Q_1, Q_2 \to 0$ as $C \to +\infty$. The idea of the proof can therefore be described in the following way: we first show that u is locally in $C^{0,\alpha}$ for α small enough with suitable estimates depending only on the local L^∞ norm of u and on the data, and this is done by using only the uniform boundedness of Q_1, Q_2. Then this first step provides us with a local modulus of continuity for u and we obtain the full result using this time that $Q_1, Q_2 \to 0$ as $C \to +\infty$.

As we just mention it, from the fact that $\Phi_0(\bar{x}, \bar{y}) > 0$, using classical arguments, Q_1, Q_2 are bounded and, more precisely the following estimates hold, in which $\overline{K}$ denotes the constant $(e^{2\tilde{K}r}(2\|u\|_\infty + \|\chi_0\|_\infty))^{1/\alpha}$

$$Q_1 \leqslant \overline{K}^\alpha \quad \text{or equivalently} \quad |\bar{x} - \bar{y}| \leqslant \overline{K} C^{-1/\alpha},$$
$$L|\bar{x} - x_0|^2 \leqslant \overline{K}^\alpha. \tag{20}$$

From the first estimates (20) it follows, in particular, that $|\bar{x} - \bar{y}| \to 0$ as $C \to +\infty$. These estimates hold true for any maximum point of the function Φ_0 in $\overline{B}_{\overline{O}}(x_0, r) \times \overline{B}_{\overline{O}}(x_0, r)$.

The function Θ_0 is not (a priori) a smooth function and therefore we cannot use directly viscosity solutions arguments; this is why we have to consider the functions ψ_δ and χ_δ defined in Lemma 2.2. Since $\psi_\delta \to \psi_0$ and $\chi_\delta \to \chi_0$ as $\delta \to 0$ locally uniformly in $\overline{O} \times \overline{O}$, for all $L, C > 0$, there is $\delta_{C,L} > 0$ such that for $0 < \delta \leqslant \delta_{C,L}$, we have

$$\max_{\overline{B}_{\overline{O}}(x_0,r) \times \overline{B}_{\overline{O}}(x_0,r)} \big(u(x) - u(y) - \Theta_\delta(x, y)\big) > 0. \tag{21}$$

Let (x^δ, y^δ) be the maximum point of the function $(x, y) \mapsto u(x) - u(y) - \Theta_\delta(x, y)$ in $\overline{B}_{\overline{O}}(x_0, r) \times \overline{B}_{\overline{O}}(x_0, r)$. Standard arguments show that, up to subsequence, (x^δ, y^δ) converges to a maximum point $(\bar{x}, \bar{y})$ of Φ_0 as $\delta \to 0$. Moreover, we may suppose that $x^\delta - y^\delta \neq 0$. Indeed if for all $\delta > 0$ we have $x^\delta - y^\delta = 0$, then $\bar{x} - \bar{y} = 0$ as well. But in this case we would have $\Phi_0(\bar{x}, \bar{y}) \leqslant 0$ which is a contradiction. Hence, we can assume without loss of generality that $x^\delta - y^\delta$ remains bounded away from 0. For simplicity of notations we now drop the dependence of (x^δ, y^δ) on δ as we already dropped it on C. For C large enough, we have $x, y \in B_{\overline{O}}(x_0, r)$. Moreover, from Lemma 2.2, it follows that

$$\big\langle D_x \Theta_\delta(x, y), \gamma(x)\big\rangle + g(x) > 0 \quad \text{if } x \in \partial O, \tag{22}$$
$$\big\langle -D_y \Theta_\delta(x, y), \gamma(x)\big\rangle + g(y) < 0 \quad \text{if } y \in \partial O. \tag{23}$$

Thus the viscosity inequalities associated to the equation $F = 0$ hold for $u(x)$ and $u(y)$ whenever x, y lie.

By the arguments of User's guide [7], for all $\varepsilon > 0$, there exist $(p, B_1) \in \overline{J}^{2,+}u(x)$, $(q, B_2) \in \overline{J}^{2,-}u(y)$ such that

$$p = D_x\Theta_\delta(x,y), \qquad q = -D_y\Theta_\delta(x,y),$$
$$-\left(\varepsilon^{-1} + \left\|D^2\Theta_\delta(x,y)\right\|\right)\mathrm{Id} \leqslant \begin{pmatrix} B_1 & 0 \\ 0 & -B_2 \end{pmatrix} \leqslant D^2\Theta_\delta(x,y) + \varepsilon\left(D^2\Theta_\delta(x,y)\right)^2, \tag{24}$$

and

$$F\left(x,u(x),p,B_1\right) \leqslant 0, \qquad F\left(y,u(y),q,B_2\right) \geqslant 0. \tag{25}$$

We choose below $\varepsilon = \rho\|D^2\Theta_\delta(x,y)\|^{-1}$ for ρ small enough but fixed. Its size is determined in the proof below. Next we need the following lemma whose proof is postponed at the end of this subsection. We recall that Y denotes $x - y$ and $\hat{Y} = Y/|Y|$.

Lemma 2.3. *If ρ is small enough and if B_1, B_2 satisfy* (24) *then, for $|Y|$ small enough* (*i.e., for C large enough*), *there is $K > 0$ such that*

$$\left\langle (B_1 - B_2)\hat{Y}, \hat{Y}\right\rangle \leqslant -CK^{-1}\alpha(1-\alpha)\frac{(\tilde{\psi}_\delta)^{\alpha/2}}{|Y|^2} + O(\delta)|Y|^{\alpha-4} \tag{26}$$
$$+ CK|Y|^{\alpha-1} + K, \tag{27}$$

as $\delta \to 0$. Moreover, $B_1 - B_1 \leqslant \tilde{K}_1(Y,\delta)\mathrm{Id}$ with $\tilde{K}_1(Y,\delta)$ given by

$$\tilde{K}_1(Y,\delta) = K\left(C\delta|Y|^{\alpha-2} + C|Y|^{\alpha-1} + 1\right).$$

Now we estimate $|p-q|$, $|p|$, $|q|$ and $\|B_1\|$, $\|B_2\|$. For some $K > 0$, we have

$$|p|, |q| \geqslant CK^{-1}\alpha|Y|^{\alpha-1} - O(\delta)|Y|^{\alpha-3} - K,$$
$$|p|, |q| \leqslant CK\alpha|Y|^{\alpha-1} + O(\delta)|Y|^{\alpha-1} + CK|Y|^\alpha + K,$$
$$|p-q| \leqslant KC\alpha|Y|^\alpha + O\left(|x-x_0|\right) + o_\delta(1), \quad \text{as } |Y| \to 0,\ \delta \to 0,$$
$$\|B_1\|, \|B_2\| \leqslant K\left(1 + \frac{1}{O(\rho)}\right)\left[C\alpha|Y|^{\alpha-2} + O(\delta)|Y|^{\alpha-4} + 1\right].$$

At this point, it is worth noticing that we are going to let first δ tends to 0 for fixed C and we recall that, since we assume that $M_{L,C} > 0$, Y does not converge to 0 when δ tends to 0 for fixed C. The first consequence of this fact is that the term $O(\delta)|Y|^{\alpha-3}$ is playing no role in the lower estimate of $|p|, |q|$ since we can choose δ as small as necessary and, by the above estimates, we have $|p|, |q| \to +\infty$ as $C \to +\infty$.

We are going to use (H2)–(H3a) with $R = \|u\|_{L^\infty(B_{\overline{O}}(x_0,r))}$. We drop the dependence in R in the coefficients and modulus which appear in these assumptions. We subtract the two inequalities (25) and write the difference in the following way

$$F\left(x,u(x),p,B_1\right) - F\left(x,u(x),p,B_2 + \tilde{K}_1(Y,\delta)\mathrm{Id}\right)$$
$$\leqslant F\left(y,u(y),q,B_2\right) - F\left(x,u(x),p,B_2 + \tilde{K}_1(Y,\delta)\mathrm{Id}\right), \tag{28}$$

and, using the fact that $B_1 - B_2 \leqslant \tilde{K}_1(Y,\delta)\mathrm{Id}$, we apply (H3a) to the left-hand side and (H2) to the right-hand side of (28). Recalling also that $|p|, |q| \to +\infty$ as $C \to +\infty$, this yields

$$\lambda \operatorname{Tr}\big[\big(B_2 - B_1 + \tilde{K}_1(Y,\delta)\mathrm{Id}\big)\big(\widehat{A^{-1}(x)p} \otimes \widehat{A^{-1}(x)p}\big)\big] + o(1)\big\|B_2 - B_1 + \tilde{K}_1(Y,\delta)\mathrm{Id}\big\|$$
$$\leqslant \omega_1\big(|x-y|\big(1+|p|+|q|\big) + \varpi\big(|p|\wedge|q|\big)|p-q|\big)\|B_2\| + \omega_2\big(\tilde{K}_1(Y,\delta)\big) + C_1$$
$$+ C_2\big(|p|^2+|q|^2\big) + C_3|x-y|\big(|p|^3+|q|^3\big). \tag{29}$$

Now we use the following result which is a consequence of the construction of the test-function and whose proof is given at the end of Section 3.1.

Lemma 2.4. *We have*

$$\widehat{A^{-1}(x)p} = \hat{Y} + o_Y(1) + o_\delta(1) \quad \textit{as } |Y| \to 0,\ \delta \to 0.$$

We want to point out that, in the above lemma, $|Y| \to 0$ is in fact equivalent to C going to infinity.

We come back to (29): by Lemma 2.4 and recalling also that $|p|, |q| \to +\infty$ as $C \to +\infty$, we get

$$\operatorname{Tr}\big[\big(B_2 - B_1 + \tilde{K}_1(Y,\delta)\mathrm{Id}\big)\big(\widehat{A^{-1}(x)p} \otimes \widehat{A^{-1}(x)p}\big)\big]$$
$$\geqslant \big\langle (B_2 - B_1)\hat{Y}, \hat{Y}\big\rangle + \tilde{K}_1(Y,\delta) - \|B_2 - B_1\|\big(o_Y(1) + o_\delta(1)\big).$$

Moreover, by using the estimates on B_1, B_2 we have

$$\|B_2 - B_1\| \leqslant K\left[1 + \frac{1}{O(\rho)}\right]\big[C\alpha|Y|^{\alpha-2} + O(\delta)|Y|^{\alpha-4} + 1\big]. \tag{30}$$

As we already pointed out above, we are going to let first δ tends to 0 for fixed C and, since we assume that $M_{L,C} > 0$, Y does not converge to 0 when δ tends to 0 for fixed C. Hence in the estimates below, we are going to replace the terms which converge to 0 as $\delta \to 0$ by $o_\delta(1)$. On the other hand, as we already mention it above, C going to infinity is equivalent to Y going to 0 and when C is going to infinity, p, q are also going to infinity; we can therefore incorporate the $o(1)$-term coming from (H3a) in the $o_Y(1)$-term.

Therefore, by combining (30), the estimates on $\|B_2\|$, $|p|$, $|q|$ and $|p-q|$, and Lemma 2.4 we are lead to

$$\operatorname{Tr}\big[\big(B_2 - B_1 + \tilde{K}_1(Y,\delta)\mathrm{Id}\big)\big(\widehat{A^{-1}(x)p} \otimes \widehat{A^{-1}(x)p}\big)\big]$$
$$\geqslant CK^{-1}\alpha(1-\alpha)|Y|^{\alpha-2} + o_\delta(1) - K - \big(CK\alpha|Y|^{\alpha-2} + o_\delta(1) + K\big)\big(o_Y(1) + o_\delta(1)\big).$$

On the other hand, for the right-hand side of (29), we first look at the ω_1-term. By tedious but straightforward computations, we have

$$
\begin{aligned}
&|x-y|\big(1+|p|+|q|\big)+\varpi\big(|p|\wedge|q|\big)|p-q| \\
&\quad = K\alpha Q_1 + K\varpi\big(|p|\wedge|q|\big)Q_2^{1/2} + o_Y(1)+o_\delta(1),
\end{aligned}
$$

since $O(|x-x_0|)$ is like $Q_2^{1/2}$. This estimate is emphasizing the role of Q_1, Q_2 and the necessity of having the ϖ-term.

The complete estimate of the right-hand side of (29) is

$$
\begin{aligned}
&K\omega_1\big(K\alpha Q_1 + K\varpi\big(|p|\wedge|q|\big)Q_2^{1/2} + o_Y(1)+o_\delta(1)\big)C\alpha|Y|^{\alpha-2} \\
&\quad + KC^2\alpha^2|Y|^{2\alpha-2} + C^3\alpha^3|Y|^{3\alpha-2} + K + o_Y(1)+o_\delta(1),
\end{aligned}
$$

where we (partially) use the fact that $Q_1 = C|Y|^\alpha$ is bounded for C large enough.

By dividing all the above inequalities by the (very large) term $C\alpha|Y|^{\alpha-2}$, we obtain the following (almost) final estimate:

$$
\begin{aligned}
\lambda K^{-1} \leqslant{}& K\omega_1\big(K\alpha Q_1 + K\varpi\big(|p|\wedge|q|\big)Q_2^{1/2} + o_Y(1)+o_\delta(1)\big) + K\alpha Q_1 + K\alpha^2 Q_1^2 \\
&+ o_Y(1)+o_\delta(1).
\end{aligned}
$$

And by using the fact that $|p|, |q| \to +\infty$ as C tends to $+\infty$, this yields

$$
\begin{aligned}
\lambda K^{-1} \leqslant{}& K\omega_1\big(K\alpha Q_1 + o_Y(1)Q_2^{1/2} + o_Y(1)+o_\delta(1)\big) \\
&+ K\alpha Q_1 + K\alpha^2 Q_1^2 + o_Y(1)+o_\delta(1). \qquad (31)
\end{aligned}
$$

Using this last estimate, the conclusions of the two steps we mention at the beginning of the proof follow rather easily.

On one hand, by using the uniform control on Q_1, Q_2, we can choose α small enough (depending only on the local L^∞-norm of u and the data) in order to have

$$
\lambda K^{-1} \geqslant \frac{3}{2}\big(K\omega_1(K\alpha Q_1) + K\alpha Q_1 + K\alpha^2 Q_1^2\big) > K\omega_1(K\alpha Q_1) + K\alpha Q_1 + K\alpha^2 Q_1^2.
$$

With this choice, it is clear that the above inequality cannot hold for δ small and C large enough (depending again only on the local L^∞-norm of u and the data) and the local $C^{0,\alpha}$ estimate is proved for small enough α.

On the other hand, repeating this proof for any $x_0 \in \overline{B}_{\overline{O}}(x_0, 2r)$, this $C^{0,\alpha}$ property provides us with a modulus of continuity in $B_{\overline{O}}(x_0, r)$ (which depends only on the L^∞-norm of u in $B_{\overline{O}}(x_0, 3r)$ and the data), and in the above estimate, for any $\alpha \leqslant \beta$, we can use the fact that $Q_1, Q_2 \to 0$ as $C \to +\infty$. Arguing as above, we obtain the $C^{0,\alpha}$ estimate for any $\alpha \leqslant \beta$. The proof of Lemma 2.1 (and of Step 1) is complete. □

Step 2. How to handle the "$a(x)u$" term.

We are going to introduce an extra variable to reduce this case to the previous one. More precisely, we consider the function $v:\overline{O}\times\mathbb{R}\to\mathbb{R}$ defined by

$$v(x,y)=\exp(ky)u(x),$$

where $k>0$ is a large constant to be chosen.

This new function is a solution of

$$-\frac{\partial^2 v}{\partial y^2}+\exp(ky)F\big(x,\exp(-ky)v,\exp(-ky)D_x v,\exp(-ky)D^2_{xx}v\big)+k^2v=0 \quad \text{in } O\times\mathbb{R},$$

and

$$\big\langle\gamma(x),D_x v\big\rangle+\frac{1}{k}a(x)D_y v+\exp(ky)g(x)=0 \quad \text{on } \partial O\times\mathbb{R}.$$

We first remark that since we are going to argue in a neighborhood of the point $(x_0,0)$, the exponential terms are not a problem to check the assumptions on either the equation or the boundary condition. The only difficulty comes from (H3a) since we have changed the boundary, γ and therefore A is not given anymore.

In order to define the new matrix A, denoted below by $\tilde{A}_k$ since it depends on k, we first set $\tilde{n}:=(n,0)$, the exterior unit normal vector to $\partial O\times\mathbb{R}$ and $\tilde{\gamma}:=(\gamma,k^{-1}a)$. In fact, because of the form of (H3a), it is more convenient to define $\tilde{A}_k^{-1}$ and we do it by setting

$$\tilde{A}_k^{-1}(x):=\begin{pmatrix} A^{-1}(x) & k^{-1}a(x)n(x)\\ k^{-1}a(x)n^T(x) & 1\end{pmatrix},$$

where n^T is the transpose of the column vector n.

An easy computation shows that $\tilde{A}_k^{-1}(x)\tilde{n}(x)=\tilde{\gamma}(x)$. Moreover, if $P=(p,p_{n+1})\in\mathbb{R}^{n+1}$, we have

$$\big\langle\tilde{A}_k^{-1}(x)P,P\big\rangle=\big\langle A^{-1}(x)p,p\big\rangle+2k^{-1}a(x)\big\langle n(x),p\big\rangle p_{n+1}+|p_{n+1}|^2,$$

and applying Cauchy–Schwartz inequality to the second term of the right-hand side, it is straightforward to show that, for k large enough, $\tilde{A}_k^{-1}(x)$ is still a definite positive matrix. Finally, we consider the operator $\tilde{F}:\overline{O}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^{n+1}\times\mathcal{S}^{n+1}\to\mathbb{R}$ defined by

$$\begin{aligned}&\tilde{F}(x,y,v,P,M)\\ &\quad=-M_{n+1,n+1}+\exp(ky)F\big(x,\exp(-ky)v,\exp(-ky)p,\exp(-ky)\tilde{M}\big)+k^2v,\end{aligned}$$

where $\tilde{M}$ is the $n\times n$ symmetric matrix obtained from M by removing the last column and row.

Next we claim that the operator $\tilde{F}$ satisfies (H3a) with $\tilde{A}_k^{-1}$ but by replacing the $o_P(1)$-term which tends to 0 as $|P| \to \infty$ by a $o_P(1) + o_k(1)$ where $o_k(1) \to 0$ as $k \to +\infty$.

Indeed, we first observe that, since the new solution v is Lipschitz continuous w.r.t. y and since y is a variable which corresponds to a tangent direction to the boundary, the property "$|P| \to +\infty$" is equivalent in fact to "$|p| \to +\infty$" because p_{n+1} remains bounded.

On the other hand, by using (H3a) for F, one can easily see that the checking of our property reduces to show that, for all $R > 0$, $x \in \overline{O}$, $P = (p, p_{n+1})$ and $\zeta = (\xi, \eta) \in \mathbb{R}^{n+1}$, we have

$$\lambda_R \langle A^{-1}(x)p, \xi \rangle^2 + |\eta|^2 \geqslant \tilde{\lambda}_R \langle \tilde{A}_k^{-1}(x)P, \zeta \rangle^2 - o_k(1),$$

for some constant $\tilde{\lambda}_R > 0$. Because of the particular form of $\tilde{A}_k^{-1}(x)$, this property is obvious for "$k = +\infty$" and, of course, this implies that it is also satisfied for k large within a $o_k(1)$-term.

We finally observe that the $C^{0,1}_{\mathrm{loc}}$-norm of $\tilde{A}_k^{-1}$ does not depend on k if we choose it large enough. In order to conclude, we just remark that the proof of the first step still works if the term $o_k(1)$ is small enough and the proof of Theorem 2.1 is complete. □

Remark 2.1. We remark that, under further regularity assumptions on O and the coefficients appearing in the boundary condition (7), it is possible to handle the $a(x)u$-term without adding an extra variable but by using another change of variable. More precisely, let us suppose that the following assumption holds:

(H5) There exists a C^2-function $\chi : \overline{O} \to \mathbb{R}$ such that $\frac{\partial \chi}{\partial \gamma} = a(x)$ on ∂O.

Then the function v defined by $v(x) = e^{\chi(x)} u(x)$, is a solution of a modified equation in O (but still satisfying (H2)–(H3a)) with the boundary condition

$$\frac{\partial v}{\partial \gamma} + e^{\chi(x)} g(x) = 0 \quad \text{on } \partial O,$$

and we can apply the proof of Step 1 of Theorem 2.1 to v.

Assumption (H5) holds for example in the following case: if O is a $C^{2,\beta}$ domain and γ, a are $C^{1,\beta}$-functions, then the existence of χ is given by Theorem 7.4 (p. 539) in Lieberman and Trudinger [14]. Indeed, to build χ, one can solve the Laplace equation in O together with the oblique derivative boundary condition.

Moreover, if O is a C^3 domain and γ, a are C^2, then one can just take

$$\chi(x) = \frac{a(x)d(x)}{\langle n(x), \gamma(x) \rangle}$$

in O where a, γ and n denote here suitable extensions of these functions to $\overline{O}$.

We end this section with the proof of Lemma 2.3.

Proof of Lemma 2.3. By the regularity properties of ψ_δ, χ_δ given in Lemma 2.2, it is tedious (but easy) to check that all the terms in $D^2\Theta_\delta(x,y)$ are estimated by $K+CK|Y|^{\alpha-1}$ except perhaps the ones coming from the derivation of the first term. More precisely, we have

$$D^2\Theta_\delta(x,y)=M_1+M_2+M_3+M_4,$$

where

$$M_1=Ce^{-\tilde{K}(d(x)+d(y))}\left[\frac{\alpha}{2}(\psi_\delta)^{\alpha/2-1}D^2\psi_\delta+\frac{\alpha}{2}\left(\frac{\alpha}{2}-1\right)(\psi_\delta)^{\alpha/2-2}D\psi_\delta\otimes D\psi_\delta\right],$$
$$M_2=C(\psi_\delta)^{\alpha/2}D^2\bigl(e^{-\tilde{K}(d(x)+d(y))}\bigr),$$
$$M_3=C\alpha(\psi_\delta)^{\alpha/2-1}\bigl(D\bigl(e^{-\tilde{K}(d(x)+d(y))}\bigr)\otimes D\psi_\delta\bigr),$$
$$M_4=LD^2\bigl(e^{-\tilde{K}d(x)}\psi_\delta(x,x_0)\bigr)+D^2\chi_\delta(x,y).$$

But we have also

$$D_x\psi_\delta(x,y)=\frac{D_X\tilde{\psi}_\delta}{2}+D_Y\tilde{\psi}_\delta+D_T\tilde{\psi}_\delta Dd(x),$$
$$D_y\psi_\delta(x,y)=\frac{D_X\tilde{\psi}_\delta}{2}+D_Y\tilde{\psi}_\delta+D_T\tilde{\psi}_\delta Dd(y),$$
$$D^2_{xx}\psi_\delta(x,y)=\frac{D^2_{XX}\tilde{\psi}_\delta}{4}+D^2_{YY}\tilde{\psi}_\delta+D^2_{XY}\tilde{\psi}_\delta+D_T\tilde{\psi}_\delta D^2d(x),$$
$$D^2_{yy}\psi_\delta(x,y)=\frac{D^2_{XX}\tilde{\psi}_\delta}{4}+D^2_{YY}\tilde{\psi}_\delta-D^2_{XY}\tilde{\psi}_\delta+D_T\tilde{\psi}_\delta D^2d(x),$$
$$D^2_{xy}\psi_\delta(x,y)=\frac{D^2_{XX}\tilde{\psi}_\delta}{4}-D^2_{YY}\tilde{\psi}_\delta$$

and, taking into account the properties given in Lemma 2.2, it can be readily checked that $\|M_2\|\leqslant CK|Y|^\alpha$, $\|M_3\|\leqslant CK\alpha(|Y|^{\alpha-1}+\delta|Y|^{\alpha-2})$ and $\|M_4\|\leqslant K$.

On the other hand, for all $\xi,\zeta\in\mathbb{R}^n$, we have

$$\begin{aligned}\bigl\langle M_1(\xi,\zeta),(\xi,\zeta)\bigr\rangle=Ce^{-\tilde{K}(d(x)+d(y))}\bigl[&\bigl\langle P_1(\xi-\zeta),(\xi-\zeta)\bigr\rangle+\bigl\langle P_2(\xi+\zeta),(\xi-\zeta)\bigr\rangle\\&+\bigl\langle P_3(\xi+\zeta),(\xi+\zeta)+\bigl\langle P_4(\xi,\zeta),(\xi,\zeta)\bigr\rangle\bigr],\end{aligned}\tag{32}$$

where

$$P_1=D^2_{YY}(\tilde{\psi}_\delta)^{\alpha/2}(X,Y,T),\qquad P_2=2D^2_{XY}(\tilde{\psi}_\delta)^{\alpha/2}(X,Y,T),$$
$$P_3=D^2_{XX}(\tilde{\psi}_\delta)^{\alpha/2}(X,Y,T),$$

and P_4 is the matrix involving all the terms $D^2_{XT}(\tilde\psi_\delta)^{\alpha/2}$, $D^2_{YT}(\tilde\psi_\delta)^{\alpha/2}$, $D^2_{TT}(\tilde\psi_\delta)^{\alpha/2}$ together with the Dd and D^2d derivatives. One can easily check that P_4 is estimated by $O(\delta)(\tilde\psi_\delta)^{(\alpha-4)/2}$. On the other hand, Lemma 2.2 implies

$$|P_1| = O\big((\tilde\psi_\delta)^{(\alpha-2)/2}\big), \qquad |P_2| = O\big((\tilde\psi_\delta)^{(\alpha-1)/2}\big), \qquad |P_3| = O\big((\tilde\psi_\delta)^{(\alpha-1)/2}\big).$$

Choosing $\xi = \zeta$ in (32), we first deduce that

$$M_1 \leqslant C\big(O\big((\tilde\psi_\delta)^{(\alpha-1)/2}\big) + O(\delta)\big((\tilde\psi_\delta)^{(\alpha-4)/2}\big)\big)\mathrm{Id}.$$

We next choose $\xi = -\zeta = \hat Y$. According to the two first properties in the point (iii) of Lemma 2.2 and taking in account the above estimate on P_4, we get

$$\begin{aligned}\big\langle M_1(\hat Y, -\hat Y), (\hat Y, -\hat Y)\big\rangle &\leqslant 4Ce^{-2\tilde K(d(x)+d(y))}\frac{\alpha}{2}\frac{(\tilde\psi_\delta)^{\alpha/2}}{|Y|^2} \\ &\quad\times\left[\big\langle D^2_{YY}\tilde\psi_\delta Y, Y\big\rangle + \left(\frac{\alpha}{2}-1\right)\frac{\langle D_Y\tilde\psi_\delta, Y\rangle^2}{\tilde\psi_\delta}\right] \\ &\quad + C\big\langle P_4(\hat Y, -\hat Y), (\hat Y, -\hat Y)\big\rangle \\ &\leqslant CK^{-1}\alpha(\alpha-1)\tilde\psi_\delta^{(\alpha-2)/2} + K(\tilde\psi_\delta)^{\alpha/2} + O(\delta)\tilde\psi_\delta^{(\alpha-4)/2}.\end{aligned}$$

By combining the above estimates, we obtain

$$\langle B_1 - B_2\hat Y, \hat Y\rangle \leqslant CK^{-1}\alpha(\alpha-1)|Y|^{\alpha-2} + CO(\delta)|Y|^{\alpha-4} + CK|Y|^{\alpha-1} + K. \tag{33}$$

And the final upper estimate on $B_1 - B_2$ follows from the estimates on M_i for $i = 1, 2, 3, 4$. □

2.2. *The case of nonlinear boundary conditions*

In this subsection, we consider the case of nonlinear boundary conditions of the form

$$G(x, u, Du) = 0 \quad \text{in } \partial O, \tag{34}$$

where $G : \partial O \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a continuous function, satisfying the conditions (G1) and (G2).

In this section, we use the following assumptions on F and G:

(H3b) (*Uniform ellipticity*) For any $R > 0$, there is $\lambda_R > 0$ such that, for all $x \in \overline{O}$, $-R \leqslant u \leqslant R$, $p \in \mathbb{R}^n$ and $M, N \in \mathcal{S}^n$ such that $M \leqslant N$, we have

$$F(x, u, p, M) - F(x, u, p, N) \geqslant \lambda_R \operatorname{Tr}(N - M).$$

(G3) For all $R > 0$ and $M > 0$ there is $K_{R,M} > 0$ such that

$$\left| \left\langle \frac{\partial G}{\partial p}(x, u, p), p \right\rangle - G(x, u, p) \right| \leqslant K_{R,M}, \tag{35}$$

for all $x \in \partial O$ and for all $p \in \mathbb{R}^n$, $|p| \geqslant M$, $|u| \leqslant R$.

(G4) There is a function $G_\infty : \partial O \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ such that

$$\frac{1}{\lambda} G(x, u, \lambda p) \to G_\infty(x, u, p) \quad \text{as } \lambda \to \infty, \tag{36}$$

locally uniformly in (x, u, p).

Before providing our result, we want to point out that the G_∞ appearing in (G4) is homogeneous of degree 1 and satisfies (G1) and (G2).

Our result is

Theorem 2.2. *Assume* (H1)–(H2)–(H3b) *and* (G1)–(G4). *Then every bounded continuous solution u of* (1) *is in $C^{0,\alpha}_{\mathrm{loc}}(\overline{O})$ for any $0 < \alpha < 1$. Moreover, the $C^{0,\alpha}_{\mathrm{loc}}$-norms of u depend only on O, F, G, through the constants and functions appearing in* (H2)–(H3b)*, and in* (G1)–(G4)*, the local C^2-norm of the distance function of the boundary including the modulus of continuity of $D^2 d$.*

Proof. We are going to do the proof in three steps: in the first one we prove the result in the case when G is independent of u and homogeneous of degree 1 in p, then in the second step, we remove the homogeneity restriction and finally, in Step 3, we use the method of the second step of the proof of Theorem 2.1 to deal with the dependence in u.

Step 1. The case when G is independent of u and homogeneous of degree 1 in p.

The proof is similar to the one of Theorem 2.1 and we just outline the main differences. Again we treat only the case when the boundary plays a role.

Since the boundary ∂O is C^2, by making a suitable change of variables, we can assume without loss of generality that the boundary is flat and more precisely that $O \cap B(x_0, 3r) \subset \{x_n 0\}$ and $\partial O \cap B(x_0, 3r) \subset \{x_n = 0\}$. It is worth noticing that the assumptions made on F and G are preserved by such a change. In order to keep simple notations, we still denote by F, G the functions arising in the equation and in the boundary condition in the domain with flat boundary.

We have to prove the following lemma which is the key stone of the proof.

Lemma 2.5. *Assume that $B(x_0, r) \cap \{x_n > 0\} \neq \emptyset$ and that u is a continuous solution of* (1) *in $B(x_0, 3r) \cap \{x_n > 0\}$ with nonlinear boundary condition $G = 0$ on $B(x_0, 3r) \cap \{x_n = 0\}$. Under the assumptions of Theorem* 2.2 *on F, G, there exists a constant $C > 0$ depending on $F, G, \|u\|_{L^\infty(B_{\overline{O}}(x_0,3r))}$ such that, for any $x \in B(x_0, r) \cap \{x_n \geqslant 0\}$, the estimate* (11) *holds.*

Proof. In order to prove (11), we consider the auxiliary function

$$\Phi_0(x,y) = u(x) - u(y) - \Theta_0(x,y),$$

where the function Θ_0 has in this case the following form:

$$\Theta_0(x,y) = Ce^{-\tilde{K}(x_n+y_n)}\big[\psi_0(x,y)\big]^{\alpha/2} + L\phi_0(x,x_0),$$

for some large constants C, L, $\tilde{K}$ to be chosen later on and where the continuous functions $\psi_0(x,y)$, $\phi_0(x,y)$ satisfy the properties listed in the following lemma in which we use the notations

$$X := \frac{x+y}{2}, \qquad Y := x - y.$$

We want also to point out that the parameters δ and η we introduce in this lemma play completely different roles in the proof, the role of δ being far more important than the role of η which is a small but fixed parameter; this is why we choose to drop the dependence in η of the functions ψ_δ, $\tilde{\psi}_\delta$ below.

Lemma 2.6. *Under the assumptions of Theorem* 2.2, *there is a function $\phi_0 \in C^2(\mathbb{R}^{2n})$ and, for $\delta \geqslant 0$ and $\eta > 0$ small enough, there exists a real-valued, continuous function $\tilde{\psi}_\delta(X,Y)$ defined in $\mathbb{R}^n \times \mathbb{R}^n$ such that if we set*

$$\psi_\delta(x,y) := \tilde{\psi}_\delta\left(\frac{x+y}{2}, x-y\right)$$

the following facts hold:

(i) *There exists a constant $K > 0$ such that, for $\delta \geqslant 0$ small enough,*

$$K^{-1}|Y|^2 \leqslant \tilde{\psi}_\delta(X,Y) \leqslant K|Y|^2 + K\delta, \tag{37}$$

for any $Y = (Y_1, \ldots, Y_n) \in \mathbb{R}^n$.

(ii) *When $\delta \to 0$, $\tilde{\psi}_\delta(X,Y) \to \tilde{\psi}_0(X,Y)$ uniformly on each compact subset of $\mathbb{R}^n \times \mathbb{R}^n$.*

(iii) *When $\delta > 0$, the function $\tilde{\psi}_\delta$ is C^2 and the following estimates are valid for some constant $K > 0$:*

$$\big|D_X\tilde{\psi}_\delta(X,Y)\big| \leqslant K|Y|^2 + K\delta, \qquad \big|D_Y\tilde{\psi}_\delta(X,Y)\big| \leqslant K|Y|,$$
$$\big\langle D_Y\tilde{\psi}_\delta(X,Y), Y\big\rangle = 2\tilde{\psi}_\delta(X,Y) + O(\eta)O\big(|Y|^2\big) + O(\delta),$$
$$\big\langle D^2_{YY}\tilde{\psi}_\delta(X,Y)Y, Y\big\rangle = 2\tilde{\psi}_\delta(X,Y) + O(\eta)O\big(|Y|^2\big) + O(\delta)$$

as $Y \to 0$, $\eta \to 0$ and $\delta \to 0$.

$$\left|D^2_{XX}\tilde{\psi}_\delta(X,Y)\right| \leqslant \frac{K}{\eta}|Y|\left(\delta^2+|Y|^2\right)^{1/2}, \qquad \left|D^2_{XY}\tilde{\psi}_\delta(X,Y)\right| \leqslant K|Y|,$$
$$\left|D^2_{YY}\tilde{\psi}_\delta(X,Y)\right| \leqslant K.$$

(iv)
$$K^{-1}|x-y|^4 \leqslant \phi_0(x,y) \leqslant K|x-y|^4. \tag{38}$$

(v) *There exists* $\tilde{K}>0$ *large enough* (*independent of* C *and* L) *such that, if we set*

$$\Theta_\delta(x,y) = Ce^{-\tilde{K}(x_n+y_n)}\left[\psi_\delta(x,y)\right]^{\alpha/2} + L\phi_0(x,x_0), \tag{39}$$

then, for $|x-y|$ *small enough, we have*

$$G\big(x, D_x\Theta_\delta(x,y)\big) > 0 \quad \textit{if } x_n = 0, \tag{40}$$
$$G\big(y, -D_y\Theta_\delta(x,y)\big) < 0 \quad \textit{if } y_n = 0. \tag{41}$$

The proof of the key Lemma 2.6 is postponed to Section 3.2. To prove (17), we first choose $L, C > 0$ large enough in order to have $\Phi_0(x,y) \leqslant 0$ for x or y on $\partial B(x_0,r)\cap\overline{\mathcal{O}}$. This is possible since because of the conditions (37) and (38). Of course, L, C depends on r.

As in the proof of Theorem 2.1, we fix such an L and we argue by contradiction assuming that, for all $\alpha\in(0,1)$ and $C>0$, $M_{L,C}>0$. Since Φ_0 is a continuous function, the maximum is achieved at some $(\bar{x},\bar{y}) \in \overline{B}_{\overline{\mathcal{O}}}(x_0,r)\times\overline{B}_{\overline{\mathcal{O}}}(x_0,r)$ and we observe that, by the choice of L, C, we may even assume that $\bar{x}\in B_{\overline{\mathcal{O}}}(x_0,3r/4)$ and $\bar{y}\in B_{\overline{\mathcal{O}}}(x_0,r)$. Here we have dropped the dependence of $\bar{x},\bar{y}$ on C for simplicity of notations.

We use here $Q_1 := C|\bar{x}-\bar{y}|^\alpha$ and $Q_2 := L|\bar{x}-x_0|^4$. From the fact that $\Phi_0(\bar{x},\bar{y})>0$, using classical arguments, the following estimates follow, in which $\overline{K}$ is the constant $(2e^{2\tilde{K}r}\|u\|_\infty)^{1/\alpha}$:

$$Q_1 \leqslant \overline{K}^\alpha \quad \text{or equivalently} \quad |\bar{x}-\bar{y}| \leqslant \overline{K}C^{-1/\alpha},$$
$$Q_2 \leqslant \overline{K}^\alpha. \tag{42}$$

These estimates hold true for any maximum point of the function Φ_0 in $\overline{B}_{\overline{\mathcal{O}}}(x_0,r)\times\overline{B}_{\overline{\mathcal{O}}}(x_0,r)$.

Since the function Φ_0 is not (a priori) a smooth function, we have to consider the functions ψ_δ defined in Lemma 2.6. The following property holds: for all $L, C>0$, there is $\delta_{C,L}>0$ such that for $0<\delta\leqslant\delta_{C,L}$ and $\tilde{K}$ large, we have

$$\max_{\overline{B}_{\overline{\mathcal{O}}}(x_0,r)\times\overline{B}_{\overline{\mathcal{O}}}(x_0,r)} \big(u(x)-u(y)-\Theta_\delta(x,y)\big) > 0. \tag{43}$$

Let (x^δ,y^δ) be the maximum point of $u(x)-u(y)-\Theta_\delta(x,y)$ in $\overline{B}_{\overline{\mathcal{O}}}(x_0,r)\times\overline{B}_{\overline{\mathcal{O}}}(x_0,r)$. Standard arguments show that, up to subsequence, (x^δ,y^δ) converges to a maximum point $(\bar{x},\bar{y})$ of Φ_0 as $\delta\to0$. Again we may suppose that $x^\delta-y^\delta\neq0$. For simplicity of notations we drop the dependence of (x^δ,y^δ) on δ as we already dropped it on C.

For C large enough, we have $x, y \in B_{\overline{O}}(x_0, r)$. Moreover, from Lemma 2.6, it follows that

$$G\big(x, D_x\Theta_\delta(x,y)\big) > 0 \quad \text{if } x_n = 0, \tag{44}$$

$$G\big(y, -D_y\Theta_\delta(x,y)\big) < 0 \quad \text{if } y_n = 0. \tag{45}$$

Thus the viscosity inequalities associated to the equation $F = 0$ hold for $u(x)$ and $u(y)$ whenever x, y lie.

By the arguments of User's guide [7], for all $\varepsilon > 0$, there exist $(p, B_1) \in \overline{J}^{2,+}u(x)$, $(q, B_2) \in \overline{J}^{2,-}u(y)$ such that

$$p = D_x\Theta_\delta(x,y), \qquad q = -D_y\Theta_\delta(x,y),$$
$$-\big(\varepsilon^{-1} + \big\|D^2\Theta_\delta(x,y)\big\|\big)\mathrm{Id} \leqslant \begin{pmatrix} B_1 & 0 \\ 0 & -B_2 \end{pmatrix} \leqslant D^2\Theta_\delta(x,y) + \varepsilon\big(D^2\Theta_\delta(x,y)\big)^2, \tag{46}$$

and

$$F\big(x, u(x), p, B_1\big) \leqslant 0, \qquad F\big(y, u(y), q, B_2\big) \geqslant 0. \tag{47}$$

We choose below $\varepsilon = \rho\|D^2\Theta_\delta(x,y)\|^{-1}$ for ρ small enough but fixed. Its size is determined in the proofs below. In order to have estimates on B_1 and B_2, we set $\overline{\psi}(X,Y) = e^{-\tilde{K}X_n}(\tilde{\psi}_\delta(X,Y))^{\alpha/2}$, with the correspondence given above between X, Y and x, y.

By the regularity properties of ψ_δ, ϕ_0 given in Lemma 2.6, it is tedious (but easy) to check that all the terms in $D^2\Theta_\delta(x,y)$ are bounded except perhaps the ones from $D^2\overline{\psi}(X,Y)$. Therefore, the inequality (46) can be rewritten as: for all $\xi, \zeta \in \mathbb{R}^n$,

$$\begin{aligned}\langle B_1\xi,\xi\rangle - \langle B_2\zeta,\zeta\rangle \leqslant \big(1+O(\rho)\big)\big[&\big\langle P_1(\xi-\zeta),(\xi-\zeta)\big\rangle + \big\langle P_2(\xi+\zeta),(\xi-\zeta)\big\rangle \\ &+ \big\langle P_3(\xi+\zeta),(\xi+\zeta)\big\rangle + K\big(|\xi|^2+|\zeta|^2\big)\big],\end{aligned} \tag{48}$$

for some constant K and where

$$P_1 = D^2_{YY}\overline{\psi}(X,Y), \qquad P_2 = D^2_{XY}\overline{\psi}(X,Y), \qquad P_3 = D^2_{XX}\overline{\psi}(X,Y).$$

Moreover, as we remark above, we can assume that $Y = x - y$ remains bounded away from 0 and this implies (after again tedious computations) that P_3 is bounded as well.

Choosing $\xi = \zeta$ in the above inequality, we first deduce that

$$B_1 - B_2 \leqslant \tilde{K}_2(Y,\delta,\eta)\mathrm{Id}$$

with

$$\tilde{K}_2(Y,\delta,\eta) = K\big[C\big(\alpha\eta^{-1}|Y|^{\alpha-1}\big(\delta^2+|Y|^2\big)^{1/2} + \delta|Y|^{\alpha-4}\big) + 1\big].$$

We next choose in (48), $\xi = -\zeta = \hat{Y}$. By using the properties on the first and second derivatives of $\tilde{\psi}_\delta$ and ϕ_0 proved in Section 3.2, we get, for some $K > 0$,

$$\begin{aligned}
&\big\langle (B_1 - B_2)\hat{Y}, \hat{Y}\big\rangle \\
&\quad\leqslant KCe^{-2\tilde{K}X_n}\frac{\alpha}{2}\frac{(\tilde{\psi}_\delta)^{\alpha/2-1}}{|Y|^2}\left[\big\langle D^2_{YY}(\tilde{\psi}_\delta)Y, Y\big\rangle + \left(\frac{\alpha}{2} - 1\right)\frac{\langle D_Y\tilde{\psi}_\delta, Y\rangle^2}{\tilde{\psi}_\delta}\right] + K \\
&\quad\leqslant -KC\alpha(1-\alpha)\frac{(\tilde{\psi}_\delta)^{\alpha/2}}{|Y|^2} + C\alpha O(\eta)|Y|^{\alpha-2} + O(\delta)|Y|^{\alpha-6} + K.
\end{aligned}$$

If $(e_i)_{1\leqslant i\leqslant n-1}$ are $(n-1)$ vectors such that $(e_1, \dots, e_{n-1}, \hat{Y})$ is an orthonormal basis of $\mathbb{R}^n$, we know that

$$\operatorname{Tr}(B_1 - B_2) = \sum_{i=1}^{n-1}\big\langle (B_1 - B_2)e_i, e_i\big\rangle + \big\langle (B_1 - B_2)\hat{Y}, \hat{Y}\big\rangle,$$

and by combining the above estimates, we deduce

$$\begin{aligned}
\operatorname{Tr}(B_1 - B_2) \leqslant{}& -CK\alpha(1-\alpha)\frac{(\tilde{\psi}_\delta)^{\alpha/2}}{|Y|^2} + O(\delta)|Y|^{\alpha-6} + C\alpha O(\eta)|Y|^{\alpha-2} \\
&+ \tilde{K}_2(Y, \delta, \eta).
\end{aligned}\tag{49}$$

Finally, for $|Y|$ small enough (i.e., for C large enough), we get

$$\begin{aligned}
\operatorname{Tr}(B_1 - B_2) \leqslant{}& -CK\alpha(1-\alpha)|Y|^{\alpha-2} + O(\delta)|Y|^{\alpha-6} + C\alpha O(\eta)|Y|^{\alpha-2} \\
&+ \tilde{K}_2(Y, \delta, \eta).
\end{aligned}\tag{50}$$

Now by using the estimates on the first and second derivatives on Θ_δ shown in Section 3.2, we get, for some $K > 0$,

$$\begin{aligned}
|p|, |q| &\geqslant C\alpha\big(K^{-1} - K\eta\big)|Y|^{\alpha-1} + O(\delta)|Y|^{\alpha-3} - K, \\
|p|, |q| &\leqslant C\alpha K|Y|^{\alpha-1} + O(\delta)|Y|^{\alpha-1} + K, \\
|p - q| &\leqslant C\alpha K|Y|^{\alpha} + O(\delta) + O\big(|x - x_0|^3\big), \\
\|B_1\|, \|B_2\| &\leqslant K\left(1 + \frac{1}{O(\rho)}\right)\big(1 + C\alpha|Y|^{\alpha-2} + O(\delta)\big).
\end{aligned}$$

As in the proof of Lemma 2.1, we notice that we are going to let first δ tends to 0 for fixed C and we recall that, since we assume that $M_{L,C} > 0$, Y does not converge to 0 when δ tends to 0 for fixed C. The first consequence of this fact is again that the term $O(\delta)|Y|^{\alpha-3}$ is playing no role in the lower estimate of $|p|, |q|$ since we can choose δ as small as necessary. The new point in the above estimate is the η-term: we choose it sufficiently small in order to have, say, $K^{-1} - K\eta \geqslant \frac{1}{2}K^{-1}$. With this choice, by the above estimates, we have $|p|, |q| \to +\infty$ as $C \to +\infty$.

We are going to use (H2)–(H3b) with $R = \|u\|_{L^\infty(B_{\overline{O}}(x_0,r))}$. We drop the dependence in R in the coefficients and modulus which appear in these assumptions. We subtract the two inequalities (47) and write the difference in the following way:

$$\begin{aligned} &F\big(x, u(x), p, B_1\big) - F\big(x, u(x), p, B_2 + \tilde{K}_2(Y, \delta, \eta)\mathrm{Id}\big) \\ &\quad \leqslant F\big(y, u(y), q, B_2\big) - F\big(x, u(x), p, B_2 + \tilde{K}_2(Y, \delta, \eta)\mathrm{Id}\big), \end{aligned} \tag{51}$$

and, using the fact that $B_1 - B_2 \leqslant \tilde{K}_2(Y, \delta, \eta)\mathrm{Id}$, we apply (H3b) to the left-hand side and (H2) to the right-hand side of (51). This yields

$$\begin{aligned} \lambda \operatorname{Tr}\big(B_2 - B_1 + \tilde{K}_2(Y, \delta, \eta)\mathrm{Id}\big) \leqslant\; &\omega_1\big(|x-y|\big(1 + |p| + |q|\big) + \varpi\big(|p| \wedge |q|\big)|p-q|\big)\|B\| \\ &+ \omega_2\big(\tilde{K}_2(Y, \delta, \eta)\big) + C_1 \\ &+ C_2\big(|p|^2 + |q|^2\big) + C_3|x-y|\big(|p|^3 + |q|^3\big). \end{aligned} \tag{52}$$

The estimates on the two sides of (52) are done in the same way as in the proof of Theorem 2.1. The only difference is a term of the form $C\alpha O(\eta)|Y|^{\alpha-2}$ in the left-hand side of the estimate.

Taking into account this additional term, we are lead to an analogous estimate to (31) with a right-hand side of the form $\lambda K^{-1} + O(\eta)$ instead of λK^{-1}. We conclude in the same way by choosing first η small enough. □

Step 2. The case when G is independent of u but with a general dependence in p.

As for the treatment of the dependence in u, we are going to introduce a new variable. More precisely, we introduce the function $v : \overline{O} \times \mathbb{R} \to \mathbb{R}$ defined by

$$v(x, y) := u(x) - y.$$

This new function is formally a solution of

$$-\frac{\partial^2 v}{\partial y^2} + F\big(x, v + y, D_x v, D_{xx}^2 v\big) = 0 \quad \text{in } O \times \mathbb{R}$$

and

$$-D_y v G\left(x, -\frac{D_x v}{D_y v}\right) = 0 \quad \text{in } \partial O \times \mathbb{R}.$$

In fact, in order to justify this, one has just to be a little bit more precise about the definition of the boundary condition. We set, for $x \in \partial O$, $p_x \in \mathbb{R}^n$ and $p_y \in \mathbb{R}$,

$$\tilde{G}(x, p_x, p_y) := \begin{cases} -p_y G\big(x, -\frac{p_x}{p_y}\big) & \text{if } p_y < 0, \\ G_\infty(x, p_x) & \text{if } p_y \geqslant 0. \end{cases}$$

With this notation, the boundary condition for v becomes $\tilde{G}(x, D_x v, D_y v) = 0$ and, because in particular of the assumptions (G3)–(G4), it is rather easy to show that $\tilde{G}$, in addition to be homogeneous of degree 1 in (p_x, p_y), satisfies (G1)–(G2).

On the other hand, the assumptions on the equation can be checked easily and therefore the conclusion follows from Step 1.

Step 3. The general case.

In order to treat the dependence in u, as mentioned above, we use the method of the second step of the proof of Theorem 2.1. We are not going to give all the details since they are essentially the same. We just want to point out that in order to take care of the dependence in u and to be sure that the transformed boundary condition actually satisfies (G3)–(G4), one has first to introduce the function G_R defined by $R > 0$ large enough and for $x \in \partial O$, $u \in \mathbb{R}$ and $p \in \mathbb{R}^n$, by

$$G_R(x, u, p) := \begin{cases} G(x, u, p) & \text{if } |u| \leqslant R, \\ G(x, -R, p) & \text{if } u \leqslant -R, \\ G(x, R, p) & \text{if } u \geqslant R. \end{cases}$$

Clearly, if R is large enough, u is still a solution of the Neumann problem with G_R and this transformation prevents difficulties with the behavior of G in u for $|u|$ large.

And the proof of Theorem 2.2 is complete. □

Remark 2.2. We want to point out that, in (H2), the ϖ-term is needed only because we want to obtain local estimates: this is clear in the proof since, in (31), this ϖ-term is used to take care of Q_2 which comes from the localization term. Therefore, in the case of global estimates, the same result holds with $\varpi \equiv 1$. This remark can be used either on bounded domains or in unbounded domains where, if the equation and the boundary condition satisfy suitable uniform properties, L can be taken as small as we want (a "mild" localization) and the same effect occurs.

3. The construction of the test-functions

In this section, we provide the proofs of Lemmas 2.2 and 2.6. In particular we show how we construct the functions Θ_0 and Θ_δ which are used in the proofs of Lemmas 2.1 and 2.5. We will consider separately as in Section 1 the case of linear and nonlinear boundary conditions.

3.1. The test-function for linear boundary conditions

In this subsection we consider the case

$$\frac{\partial u}{\partial \gamma} + g(x) = 0 \quad \text{on } \partial O,$$

where $\gamma : \partial O \to \mathbb{R}^n$ is a locally Lipschitz continuous vector field such that $\langle \gamma(x), n(x) \rangle \geqslant \nu > 0$ for any $x \in \partial O$, and $g : \partial O \to \mathbb{R}^n$ is either a locally Lipschitz continuous or a locally Hölder continuous scalar function.

According to assumption (H3a), there exists a function $A(\cdot) \in C^{0,1}_{\text{loc}}(\partial O, \mathcal{S}^n)$ such that, for any $x \in \partial O$, $n(x) = A(x)\gamma(x)$ and $A(x) \geqslant c_0 \mathrm{Id}$, for some constant $c_0 > 0$ and such that (9) holds. Of course, this last property is the most important information in (H3a), the existence of such A without the connection with the ellipticity of the equation, being easy to show.

We may assume without loss of generality, that $\langle \gamma(x), n(x) \rangle = 1$ for any $x \in \partial O$, otherwise we change γ in $\frac{\gamma(x)}{\langle \gamma(x), n(x) \rangle}$, g in $\frac{g(x)}{\langle \gamma(x), n(x) \rangle}$ and $A(x)$ in $\langle \gamma(x), n(x) \rangle A(x)$; these transformations do not change the properties of γ and g.

Proof of Lemma 2.2. As in the proof of the comparison result for this kind of problems (cf. Barles [3]), we are going to use regularizations of A and g. To do so, it is convenient to introduce the following lemma whose proof is classical and therefore left to the reader.

Lemma 3.1. *Assume that $\rho \in D(\mathbb{R}^n)$, $\rho \geqslant 0$, $\operatorname{supp}(\rho) \subset B(0,1)$ and $\int_{\mathbb{R}^n} \rho(y)\,dy = 1$. If $f \in C^{0,\beta}(\mathbb{R}^n)$ for some $0 < \beta \leqslant 1$, and f is bounded, then the function $\tilde{f} : \mathbb{R}^n \times [0,+\infty) \to \mathbb{R}$ defined by*

$$\tilde{f}(x,\varepsilon) := \int_{\mathbb{R}^n} f(z)\rho\left(\frac{x-z}{\varepsilon}\right)\frac{1}{\varepsilon^n}\,dz \quad \text{for } x \in \mathbb{R}^n,\ \varepsilon > 0,$$

$$\tilde{f}(x,0) = f(x) \quad \text{for } x \in \mathbb{R}^n,$$

is in $C^{0,\beta}(\mathbb{R}^n \times [0,+\infty))$. Moreover, the function $\tilde{f}$ is C^2 in $\mathbb{R}^n \times (0,+\infty)$ with

$$\left|D_x \tilde{f}(x,\varepsilon)\right|, \left|D_\varepsilon \tilde{f}(x,\varepsilon)\right| \leqslant K\varepsilon^{\beta-1},$$

$$\left|D^2_{xx} \tilde{f}(x,\varepsilon)\right|, \left|D^2_{x\varepsilon} \tilde{f}(x,\varepsilon)\right|, \left|D^2_{\varepsilon\varepsilon} \tilde{f}(x,\varepsilon)\right| \leqslant K\varepsilon^{\beta-2} \quad \text{in } \mathbb{R}^n \times (0,+\infty)$$

for some constant K depending only on ρ, the L^∞ and the Hölder norm of f.

Step 1. The functions A and g and their regularizations.

We first extend g and A to $\mathbb{R}^n$; we still denote by g and A these extensions. We may assume that these extensions are respectively in $C^{0,\beta}(\mathbb{R}^n)$ and $C^{0,1}(\mathbb{R}^n)$. For some function ρ satisfying the properties of Lemma 3.1 (which is chosen and fixed from now on), we consider the functions $\tilde{A}$ and $\tilde{g}$ associated to A and g as in this lemma. Finally, we introduce, for some $\delta \geqslant 0$, the following quantity which is defined for $\xi \in \mathbb{R}^n$ by

$$\Lambda_\delta(\xi) = \left(\delta^2 + |\xi|^2\right)^{1/2},$$

and we set

$$\tilde{A}_\delta(x,\xi) = \tilde{A}\big(x, \Lambda_\delta(\xi)\big), \qquad \tilde{g}_\delta(x,\xi) = \tilde{g}\big(x, \Lambda_\delta(\xi)\big).$$

According to Lemma 3.1, these functions are C^2 as long as $\delta > 0$.

We also observe that

$$\big|D_\xi \Lambda_\delta(\xi)\big| \leqslant 1, \qquad \big|D^2_{\xi\xi} \Lambda_\delta(\xi)\big| \leqslant K \Lambda_\delta^{-1}.$$

Step 2. Construction of the functions ψ_0, ψ_δ and their main properties.

For $\delta \geqslant 0$, we introduce the following function, for $X, Y \in \mathbb{R}^n$ and $T > 0$:

$$\tilde{\psi}_\delta(X,Y,T) = \big\langle \tilde{A}_\delta(X,Y)Y, Y\big\rangle + K_1\delta(2M - T), \tag{53}$$

where $K_1 > 0$ is a constant to be chosen later and M is chosen so that $2M - T$ remains bounded. Moreover, we set, for x and y in a suitable neighborhood of x_0,

$$\psi_\delta(x,y) := \tilde{\psi}_\delta\left(\frac{x+y}{2}, x-y, d(x)+d(y)\right).$$

We observe that, as $\delta \to 0$, $\tilde{\psi}_\delta$ and ψ_δ converge locally uniformly respectively to $\tilde{\psi}_0$ and ψ_0. Depending on the simplicity, we provide below result either on $\tilde{\psi}_\delta$ or ψ_δ, the translation from one to the other being straightforward. Most of the time we will use $\tilde{\psi}_\delta$.

In the sequel $K > 0$ denotes a constant which may vary from line to line but which depends only on the data of the problem and is independent of the small parameter δ.

Proposition 3.1. *We have, for any $X, Y \in \mathbb{R}^n$,*

$$K^{-1}|Y|^2 \leqslant \tilde{\psi}_0(X,Y) \leqslant K|Y|^2, \tag{54}$$

$$K^{-1}|Y|^2 \leqslant \tilde{\psi}_\delta(X,Y) \leqslant K|Y|^2 + K\delta. \tag{55}$$

The proposition is straightforward consequence of the fact that $A(x) \geqslant c_0\mathrm{Id}$ for all $x \in \mathbb{R}^n$. Next we examine the regularity properties and the estimates on $\tilde{\psi}_\delta$ and ψ_δ.

Proposition 3.2. *We have, for any $X, Y \in \mathbb{R}^n$, $T \in \mathbb{R}$,*

$$\big|D_Y\tilde{\psi}_\delta(X,Y,T)\big| \leqslant K|Y|, \qquad \big|D_X\tilde{\psi}_\delta(X,Y,T)\big| \leqslant K|Y|^2,$$
$$\big|D^2_{YY}\tilde{\psi}_\delta(X,Y,T)\big| \leqslant K, \qquad \big|D^2_{XX}\tilde{\psi}_\delta(X,Y,T)\big| \leqslant K|Y|,$$
$$\big|D^2_{XY}\tilde{\psi}_\delta(X,Y,T)\big| \leqslant K|Y|,$$
$$\big|D_T\tilde{\psi}_\delta(X,Y,T)\big| \leqslant K\delta, \qquad D^2_{TT}\tilde{\psi}_\delta = D^2_{TX}\tilde{\psi}_\delta = D^2_{TY}\tilde{\psi}_\delta = 0.$$

Moreover,

$$\begin{aligned}\big\langle D_Y\tilde\psi_\delta(X,Y,T),Y\big\rangle &= 2\tilde\psi_\delta + O\big(|Y|^3\big)+O(\delta),\\ \big\langle D^2_{YY}\tilde\psi_\delta Y,Y\big\rangle &= 2\tilde\psi_\delta + O\big(|Y|^3\big)+O(\delta),\end{aligned}$$

as $Y\to 0$ *and* $\delta\to 0$.

The proof of these estimates is tedious but straightforward: the main reason to provide Lemma 3.1 and to write $\tilde A_\delta$ with a dependence in x and $\Lambda_\delta(\xi)$ was to have a simple way to check these computations.

Proof of Proposition 3.2. We have

$$\begin{aligned} D_X\tilde\psi_\delta(X,Y,T) &= \langle D_X\tilde A_\delta Y,Y\rangle,\\ D_Y\tilde\psi_\delta(X,Y,T) &= \langle D_Y\tilde A_\delta Y,Y\rangle + 2\tilde A_\delta Y,\\ D^2_{XX}\tilde\psi_\delta(X,Y,T) &= \langle D_{XX}\tilde A_\delta Y,Y\rangle,\\ D^2_{YY}\tilde\psi_\delta(X,Y,T) &= \langle D_{YY}\tilde A_\delta Y,Y\rangle + 2\tilde A_\delta + 2D_Y\tilde A_\delta Y.\end{aligned}$$

We premise some useful estimates on the first and second derivatives of $\tilde A_\delta$. By using Lemma 3.1 and the estimates on the first and second derivatives of Λ_δ, we have

$$\begin{aligned} D_X\tilde A_\delta(X,Y) &= D_{\Lambda_\delta}\tilde A = O_Y(1),\\ D^2_{XX}\tilde A_\delta(X,Y) &= D^2_{\Lambda_\delta}\tilde A = \Lambda_\delta^{-1}O_Y(1),\\ D_Y\tilde A_\delta(X,Y) &= D_{\Lambda_\delta}\tilde A\frac{\partial\Lambda_\delta}{\partial Y} = O_Y(1),\\ D^2_{YY}\tilde A_\delta(X,Y) &= (D_{\Lambda_\delta}\tilde A)\frac{\partial^2\Lambda_\delta}{\partial^2 Y} + D^2_{\Lambda_\delta}\tilde A\left(\frac{\partial\Lambda_\delta}{\partial Y}\right)^2 = \Lambda_\delta^{-1}O_Y(1).\end{aligned}$$

Using the estimates on the first and second derivatives of properties of $\tilde A_\delta$, we obtain easily all the estimates on the derivatives of $\tilde\psi_\delta$ and the second part of Proposition 3.2. □

We turn to the properties of ψ_δ with respect to the boundary condition.

Proposition 3.3. *If* $|x-y|$ *is small enough and* K_1 *is large enough, then we have, for some* $K>0$,

$$\big\langle D_x\psi_\delta(x,y),\gamma(x)\big\rangle > -K|x-y|^2 \quad \textit{if } x\in\partial O, \tag{56}$$

$$\big\langle -D_y\psi_\delta(x,y),\gamma(y)\big\rangle < K|x-y|^2 \quad \textit{if } y\in\partial O. \tag{57}$$

Proof. We only check (56) the other case being similar.

By a direct computation, we have

$$D_x\psi_\delta(x,y) = 2\tilde{A}_\delta(x-y) + \left\langle\left(\frac{D_X\tilde{A}_\delta}{2} + D_Y\tilde{A}_\delta\right)Y, Y\right\rangle - K_1\delta Dd(x)$$
$$= 2\tilde{A}_\delta(x-y) + O\big(|x-y|^2\big) + K_1\delta n(x).$$

But we recall that

$$\|\tilde{A}_\delta - A\| \leqslant K\big(|x-y| + \delta\big)$$

and since $x \in \partial O$, by the regularity of the boundary, we have

$$\big\langle n(x), (x-y)\big\rangle = d(y) + O\big(|x-y|^2\big) \quad \text{as } |x-y| \to 0.$$

Thus if K_1 is large enough we have

$$\big\langle D_x\psi_\delta(x,y), \gamma(x)\big\rangle \geqslant \big\langle 2A(x-y), \gamma\big\rangle - K|x-y|\big(|x-y| + \delta\big) - K|x-y|^2 + K_1\delta$$
$$\geqslant \big\langle 2(x-y), n(x)\big\rangle - K\delta - K|x-y|^2 + K_1\delta$$
$$\geqslant -K|x-y|^2. \qquad \square$$

Step 3. Construction of the functions χ_0 and χ_δ.

In the same way as above, we set for $\delta \geqslant 0$,

$$\tilde{\chi}_\delta(X,Y,T,Z) = \tilde{g}_\delta(X,Y)Z + K_2\delta^\beta(2M - T), \tag{58}$$

where $K_2 > 0$ is a constant to be chosen later and $M > 0$ is chosen as above. We also set

$$\chi_\delta(x,y) := \tilde{\chi}_\delta\left(\frac{x+y}{2}, x-y, d(x)+d(y), d(x)-d(y)\right).$$

One can easily check that $\tilde{\chi}_\delta$ and χ_δ converges locally uniformly respectively to $\tilde{\chi}_0$ and χ_0 as $\delta \to 0$.

As in the previous step, we first consider the regularity properties of $\tilde{\chi}_\delta$.

Proposition 3.4. *For every $\delta > 0$ we have*

$$\big|D_X\tilde{\chi}_\delta(X,Y,T,Z)\big| \leqslant K\Lambda_\delta^{\beta-1}|Z|, \qquad \big|D_Y\tilde{\chi}_\delta(X,Y,T,Z)\big| \leqslant K\Lambda_\delta^{\beta-1}|Z|,$$
$$\big|D^2\tilde{\chi}_\delta(X,Y,T,Z)\big| \leqslant K\Lambda_\delta^{\beta-2}|Z|.$$

Proof. Again the tedious computations are simplified by the way we write down $\tilde{g}_\delta$. We first observe that

$$\begin{aligned} D_X \tilde{g}_\delta &= D_X \tilde{g} = \Lambda_\delta^{\beta-1} O_Y(1), \\ D^2_{XX} \tilde{g}_\delta &= D_{XX} \tilde{g} = \Lambda_\delta^{\beta-2} O_Y(1), \\ D_Y \tilde{g}_\delta &= D_{\Lambda_\delta} \tilde{g} \frac{\partial \Lambda_\delta}{\partial Y} = \Lambda_\delta^{\beta-1} O_Y(1), \\ D^2_{YY} \tilde{g}_\delta &= D^2_{\Lambda_\delta} \tilde{g} \left(\frac{\partial \Lambda_\delta}{\partial Y} \right)^2 + D_{\Lambda_\delta} \tilde{g} \frac{\partial^2 \Lambda_\delta}{\partial^2 Y} = \Lambda_\delta^{\beta-2} O_Y(1). \end{aligned}$$

Then, by the definition of $\tilde{\chi}_\delta$, we deduce

$$\begin{aligned} D_X \tilde{\chi}_\delta(X,Y,T,Z) &= D_X \tilde{g}_\delta Z \leqslant K \Lambda_\delta^{\beta-1} |Z|, \\ D_Y \tilde{\chi}_\delta(X,Y,T,Z) &= D_Y \tilde{g}_\delta Z \leqslant K \Lambda_\delta^{\beta-1} |Z|, \\ D^2_{XX} \tilde{\chi}_\delta(X,Y,T,Z) &= D^2_{XX} \tilde{g}_\delta Z \leqslant K \Lambda_\delta^{\beta-2} |Z|, \\ D^2_{YY} \tilde{\chi}_\delta(X,Y,T,Z) &= D^2_{YY} \tilde{g}_\delta Z \leqslant K \Lambda_\delta^{\beta-2} |Z|. \qquad \square \end{aligned}$$

Then we consider the boundary condition.

Proposition 3.5. *If $|x-y|$ is small enough and K_2 is large enough, then we have, for some $K>0$,*

$$\langle D_x \chi_\delta(x,y), \gamma(x) \rangle + g(x) > -K|x-y|^\beta + K\delta^\beta \quad \text{if } x \in \partial O, \tag{59}$$

$$\langle -D_y \chi_\delta(x,y), \gamma(y) \rangle + g(y) < K|x-y|^\beta - K\delta^\beta \quad \text{if } y \in \partial O. \tag{60}$$

Proof. Again we only check the first property (59). We first notice that, by the definition and properties of $\tilde{g}_\delta$,

$$\left| \tilde{g}_\delta \left(\frac{x+y}{2}, x-y \right) - g(x) \right| \leqslant K \big(|x-y|^\beta + \delta^\beta \big).$$

On the other hand, we have

$$D_x \chi_\delta(x,y) = \tilde{g}_\delta Dd(x) + \left(\frac{1}{2} D_X \tilde{g}_\delta + D_Y \tilde{g}_\delta \right) \big(d(x) - d(y) \big) - K_2 \delta^\beta Dd(x).$$

Therefore since $|x-y|$ is small, we obtain for K_2 large enough,

$$
\begin{aligned}
\langle D_x\chi_\delta(x,y),\gamma(x)\rangle + g(x) \geqslant{}& \left[g(x) - \tilde g_\delta\left(\frac{x+y}{2}, x-y\right)\right] \\
&+ \left\langle\left(\frac{1}{2}D_X\tilde g_\delta + D_Y\tilde g_\delta\right),\gamma(x)\right\rangle\big(d(x)-d(y)\big) + K_2\delta^\beta \\
\geqslant{}& -K|x-y|^\beta + K\delta^\beta.
\end{aligned}
$$

Thus we have shown (59). □

Now we can prove the following result.

Proposition 3.6. *If* $|x-y| \ll 1$*, then we have*

$$\langle D_x\Theta_\delta(x,y),\gamma(x)\rangle + g(x) > 0 \quad \text{if } x\in\partial O, \tag{61}$$

$$\langle -D_y\Theta_\delta(x,y),\gamma(y)\rangle + g(y) < 0 \quad \text{if } y\in\partial O. \tag{62}$$

Proof. Again we only check the first property (61). We first observe that, because of the properties of ψ_δ and since $d(x)=0$, we have

$$
\begin{aligned}
&\left\langle D_x\left(e^{-\tilde K(d(x)+d(y))}\big(\psi_\delta(x,y)\big)^{\alpha/2}\right),\gamma(x)\right\rangle \\
&\quad= \tilde K e^{-\tilde K d(y)}\tilde\psi_\delta(x,y) + \frac{\alpha}{2}e^{-\tilde K d(y)}\big(\psi_\delta(x,y)\big)^{\alpha/2-1}\langle D_x\psi_\delta(x,y),\gamma(x)\rangle \\
&\quad\geqslant \tilde K e^{-\tilde K d(y)}|x-y|^\alpha - K e^{-\tilde K d(y)}|x-y|^\alpha.
\end{aligned}
$$

Thus, if $\tilde K$ is sufficiently large, we obtain

$$\left\langle D_x\left[e^{-\tilde K(d(x)+d(y))}\big(\psi_\delta(x,y)\big)^{\alpha/2}\right],\gamma(x)\right\rangle > \overline{K}|x-y|^\alpha,$$

for some constant $\overline{K}$. Similarly one can show that

$$\left\langle D_x\left[e^{-\tilde K(d(x)+d(x_0))}\big(\psi_\delta(x,x_0)\big)\right],\gamma(x)\right\rangle > \overline{K}|x-x_0|^2.$$

Then if $|x-y| \ll 1$ and $C>0$ is large enough, we have

$$
\begin{aligned}
\langle D_x\Theta_\delta(x,y),\gamma(x)\rangle + g(x) ={}& \left\langle C D_x\left[e^{-\tilde K(d(x)+d(y))}\big(\psi_\delta(x,y)\big)^{\alpha/2})\right],\gamma(x)\right\rangle \\
&+ \left\langle L D_x\left[e^{-\tilde K d(x)}\psi_\delta(x,x_0)\right],\gamma(x)\right\rangle \\
&+ \langle D_x\chi_\delta(x,y),\gamma\rangle + g(x) \\
\geqslant{}& CKe^{-\tilde K(d(x)+d(y))}|x-y|^\alpha - K|x-y|^\beta + K\delta^\beta > 0. \qquad \square
\end{aligned}
$$

The proof of Lemma 2.2 is obtained by combining Propositions 3.2, 3.4 and 3.6. □

We conclude this section with the following lemma.

Lemma 3.2. *If* $P = A^{-1}(x)D_x\Theta_\delta(x,y)$, *we have*

$$\hat{P} = \hat{Y} + o_Y(1) + O(\delta) \quad \text{as } |Y| \to 0 \text{ and } \delta \to 0.$$

Proof. We just give a sketch of proof. We first observe that

$$\left\| A_\delta\left(\frac{x+y}{2}, x-y\right) - A(x)\right\| \leqslant K\big(|x-y| + \delta\big).$$

Therefore, by direct computations, we obtain

$$\begin{aligned} D_x\Theta_\delta(x,y) &= Ce^{-\tilde{K}(d(x)+d(y))}(\tilde{\psi}_\delta)^{\alpha/2-1}\big[AY + O\big(|Y|^2\big) + O(\delta)\big] \\ &= Ce^{-\tilde{K}(d(x)+d(y))}(\tilde{\psi}_\delta)^{\alpha/2-1}\big[AY + O\big(|Y|^2\big) + O(\delta)\big]. \end{aligned}$$

Thus we get

$$\frac{A^{-1}(x)D_x\Theta_\delta(x,y)}{|A^{-1}(x)D_x\Theta_\delta(x,y)|} = \frac{Y + O(|Y|^2) + O(\delta)}{|Y + O(|Y|^2) + O(\delta)|} = \hat{Y} + o_Y(1) + O(\delta)$$
$$\text{as } |Y| \to 0,\ \delta \to 0,$$

which gives the desired result. □

3.2. The test-function for nonlinear boundary conditions

We recall that we have to build this test-function in the case when the function G is independent of u and homogeneous of degree 1 with respect to p.

We first extend the function $G(x,p)$ to $\mathbb{R}^n \times \mathbb{R}^n$ and we may assume that all the properties of G are still satisfied in $\mathcal{V} \times \mathbb{R}^n$ where $\mathcal{V}$ is a neighborhood of ∂O. The properties of G imply that, for every $x \in \mathcal{V},\ p \in \mathbb{R}^n$ there exists a unique solution $t = \varphi(x,p)$ of the equation

$$G\big(x, p + tn(x)\big) = 0. \tag{63}$$

One can verify that φ is still homogeneous of degree 1 and satisfies (G2).

It is not restrictive to reduce to the case when the boundary is flat and more precisely $O = \{x_n > 0\}$ and $\partial O = \{x_n = 0\}$.

Proof of Lemma 2.6.

Step 1. The function φ and its regularization.

In order to regularize the function φ, we first extend it to $\mathbb{R}^n$ and we still denote by φ this extension. We may assume that this extension satisfies (G2).

We introduce, for $\delta, \eta > 0$, the following quantity which is defined for $\xi \in \mathbb{R}^n$ by

$$\Gamma_\delta(\xi) = \eta\left(\frac{\delta^2 + (\xi_n)^2}{\delta^2 + |\xi|^2}\right)^{1/2},$$

and we set, for $x, \xi \in \mathbb{R}^n$,

$$\tilde{\varphi}_\delta(x, \xi) = \tilde{\varphi}\big(x, \hat{\xi}, \Gamma_\delta(\xi)\big),$$

where $\tilde{\varphi}$ is defined as in Lemma 3.1 and $\hat{\xi} = \xi/|\xi|$.

We first observe that the following estimates, which are used extensively in the sequel, hold:

$$|D_\xi \Gamma_\delta| \leqslant \frac{K\eta}{(1 + |\xi|^2)^{1/2}}, \qquad |D_{\xi\xi} \Gamma_\delta| \leqslant \frac{K\eta}{(1 + |\xi|^2)^{1/2}(1 + |\xi_n|^2)^{1/2}}.$$

Step 2. Construction of the functions ψ_0, ψ_δ, and their main properties.

For $\delta \geqslant 0$, we introduce the following function, for $X, Y \in \mathbb{R}^n$ with as above $Y = (Y_1, \ldots, Y_n)$ and $X = (X_1, \ldots, X_n)$:

$$\tilde{\psi}_\delta(X, Y) = |Y|^2 - 2\tilde{\varphi}_\delta(X, Y)|Y|Y_n + 2A_1 Y_n^2 + K_1\delta(R - X_n), \tag{64}$$

with A_1, $K_1 > 0$ constants to be chosen later. The constant $R > 0$ has to be chosen in order that the term $R - X_n$ remains positive; this does not create any problem since we argue locally. Moreover, we set, for x and y in a suitable neighborhood of x_0,

$$\psi_\delta(x, y) := \tilde{\psi}_\delta\left(\frac{x + y}{2}, x - y\right).$$

We observe that, as $\delta \to 0$, $\tilde{\psi}_\delta$ and ψ_δ converge locally uniformly respectively to $\tilde{\psi}_0$ and ψ_0. As in previous subsection we provide below result either on $\tilde{\psi}_\delta$ or ψ_δ, the translation from one to the other being straightforward. Most of the time we will use $\tilde{\psi}_\delta$.

In the sequel $K > 0$ will denote a nonnegative constant which may vary from line to line but which depends only on the data of the problem and is independent of the small parameters δ and η.

Proposition 3.7. *If $A_1 > 0$ is large enough, we have, for any $X, Y \in \mathbb{R}^n$,*

$$K^{-1}|Y|^2 \leqslant \tilde{\psi}_0(X, Y) \leqslant K|Y|^2, \tag{65}$$

$$K^{-1}|Y|^2 \leqslant \tilde{\psi}_\delta(X, Y) \leqslant K|Y|^2 + K\delta. \tag{66}$$

We skip the proof of this proposition which is straightforward: it is based only on Cauchy–Schwartz's inequality to control the $\tilde{\varphi}_\delta$ or the $\tilde{\varphi}_0$ term and on the fact that $\tilde{\varphi}_\delta$ and $\tilde{\varphi}_0$ are bounded.

Next we examine the regularity properties and the estimates on $\tilde{\psi}_\delta$ and ψ_δ.

Proposition 3.8. *If the constant $\eta > 0$ is chosen small enough and A_1, K_1 large enough, then, for any δ small enough and for all $X, Y \in \mathbb{R}^n$, we have*

$$\left|D_X\tilde{\psi}_\delta(X,Y)\right| \leqslant K|Y|^2 + K\delta, \qquad \left|D_Y\tilde{\psi}_\delta(X,Y)\right| \leqslant K|Y|,$$
$$\left\langle D_Y\tilde{\psi}_\delta(X,Y), Y\right\rangle = 2\tilde{\psi}_\delta(X,Y) + O(\eta)O\left(|Y|^2\right) + O(\delta),$$
$$\left\langle D^2_{YY}\tilde{\psi}_\delta(X,Y)Y, Y\right\rangle = 2\tilde{\psi}_\delta(X,Y) + O(\eta)O\left(|Y|^2\right) + O(\delta),$$

as $Y \to 0$, $\eta \to 0$ and $\delta \to 0$,

$$\left|D^2_{XX}\tilde{\psi}_\delta(X,Y)\right| \leqslant \frac{K}{\eta}|Y|\left(\delta^2 + |Y|^2\right)^{1/2}, \qquad \left|D^2_{XY}\tilde{\psi}_\delta(X,Y)\right| \leqslant K|Y|,$$
$$\left|D^2_{YY}\tilde{\psi}_\delta(X,Y)\right| \leqslant K.$$

Proof. The proof of these estimates is tedious but straightforward. Lemma 3.1 and the way we write $\tilde{\varphi}$ with a dependence in x, $\hat{\xi}$ and $\Gamma_\delta(\xi)$ is a simple way to check these computations.

By direct computations, we have

$$D_X\tilde{\psi}_\delta(X,Y) = -2D_X\tilde{\varphi}_\delta|Y|Y_n - K\delta e_n,$$
$$D_Y\tilde{\psi}_\delta(X,Y) = 2Y - D_Y\tilde{\varphi}_\delta|Y|Y_n - 2\tilde{\varphi}_\delta\frac{Y}{|Y|}Y_n - 2\tilde{\varphi}_\delta|Y|e_n + 2A_1Y_ne_n,$$
$$D^2_{XX}\tilde{\psi}_\delta(X,Y) = -2D_{XX}\tilde{\varphi}_\delta|Y|Y_n,$$
$$D^2_{YY}\tilde{\psi}_\delta(X,Y) = 2\mathrm{Id} - 2D_{YY}\tilde{\varphi}_\delta|Y|Y_n - 4D_Y\tilde{\varphi}_\delta\frac{Y}{|Y|}Y_n - 4D_Y\tilde{\varphi}_\delta|Y|e_n$$
$$- 4\tilde{\varphi}_\delta\frac{Y}{|Y|}e_n - 4\tilde{\varphi}_\delta D_Y\frac{Y}{|Y|}Y_n + 2A_1e_n\otimes e_n,$$

where e_n is the nth vector the canonical basis of $\mathbb{R}^n$. In this case, we also have $n(x) = -e_n$, for all $x \in \partial O$.

We estimate the first and second derivatives of $\tilde{\varphi}_\delta$. By using Lemma 3.1 and the estimates on the first and second derivatives of Γ_δ, we have

$$\left|D_X\tilde{\varphi}_\delta(X,Y)\right| = \left|D_X\tilde{\varphi}\left(X, \frac{Y}{|Y|}, \Gamma_\delta(Y)\right)\right| \leqslant K,$$
$$\left|D^2_{XX}\tilde{\varphi}_\delta(X,Y)\right| = \left|D^2_{XX}\tilde{\varphi}\left(X, \frac{Y}{|Y|}, \Gamma_\delta(Y)\right)\right| \leqslant \frac{K}{\Gamma_\delta},$$
$$\left|D_Y\tilde{\varphi}_\delta(X,Y)\right| = \left|D_\xi\tilde{\varphi}D_Y\frac{Y}{|Y|} + D_\zeta\tilde{\varphi}D_Y\Gamma_\delta\right| \leqslant \frac{K}{|Y|},$$
$$\left|D^2_{YY}\tilde{\varphi}_\delta(X,Y) = \right|D^2_{\xi\xi}\tilde{\varphi}\left(D_Y\frac{Y}{|Y|}\right)^2 + D_\xi\tilde{\varphi}D^2_{YY}\frac{Y}{|Y|} + 2D^2_{\xi\zeta}\tilde{\varphi}D_Y\frac{Y}{|Y|}\otimes D_Y\Gamma_\delta$$

$$+ D^2_{\zeta\zeta}\tilde{\varphi}(D_Y\Gamma_\delta)^2 + D_\zeta\tilde{\varphi}D^2_{YY}\Gamma_\delta \Big|$$
$$\leqslant \frac{K}{|Y||Y_n|}\left(\frac{1}{\eta}+1\right).$$

By combining the above estimates, we obtain

$$\big|D_X\tilde{\psi}_\delta(X,Y)\big| \leqslant K|Y|^2 + O(\delta), \qquad \big|D_Y\tilde{\psi}_\delta(X,Y)\big| \leqslant K|Y|,$$
$$\big|D^2_{XX}\tilde{\psi}_\delta(X,Y)\big| \leqslant \eta^{-1}K|Y|\big(|Y|^2+\delta^2\big)^{1/2}, \qquad \big|D^2_{YY}\tilde{\psi}_\delta(X,Y)\big| \leqslant K.$$

Next we estimate $\langle D_Y\tilde{\psi}_\delta(X,Y), Y\rangle$ and $\langle D^2_{YY}\tilde{\psi}_\delta Y, Y\rangle$. Tedious but straightforward computations show that

$$\big|\langle D_Y\Gamma_\delta, Y\rangle\big| \leqslant K\eta, \qquad \big|\langle D_{YY}\Gamma_\delta Y, Y\rangle\big| \leqslant K\eta. \tag{67}$$

Moreover,

$$\begin{aligned}
\big\langle D_Y\tilde{\psi}_\delta(X,Y), Y\big\rangle &= 2|Y|^2 - 4\tilde{\varphi}_\delta|Y|Y_n + 2A_1Y_n^2 - D_\zeta\tilde{\varphi}_\delta\big(\langle D_Y\Gamma_\delta, Y\rangle\big)|Y|Y_n \\
&= 2\tilde{\psi}_\delta + O(\delta) + O(\eta)|Y|^2; \\
\big\langle D^2_{YY}\tilde{\psi}_\delta Y, Y\big\rangle &= 2|Y|^2 - 4\tilde{\varphi}_\delta|Y|Y_n + 2A_1Y_n^2 \\
&\quad - 2D_\zeta\tilde{\varphi}_\delta\big\langle D^2\Gamma_\delta Y, Y\big\rangle|Y|Y_n - 4D_\xi\tilde{\varphi}_\delta\big(\langle D\Gamma_\delta, Y\rangle\big)|Y|Y_n \\
&\quad - D^2_{\xi\xi}\tilde{\varphi}_\delta\big(\langle D\Gamma_\delta, Y\rangle\big)^2|Y|Y_n \\
&= 2\tilde{\psi}_\delta + O(\delta) + O(\eta)|Y|^2.
\end{aligned}$$

These properties complete the proof of Proposition 3.8. □

We turn to the properties of ψ_δ with respect to the boundary condition.

Proposition 3.9. *If $|x-y|$ is small enough and K_1 is large enough, then we have, for some $K>0$,*

$$G\big(x, D_x\psi_\delta(x,y)\big) > -K|x-y|^2 \quad \text{if } x_n = 0, \tag{68}$$
$$G\big(y, -D_y\psi_\delta(x,y)\big) < K|x-y|^2 \quad \text{if } y_n = 0. \tag{69}$$

Proof. We only check (68) the other case being similar.

If $x \in \partial O$ then $d(x) = x_n = 0$; moreover since $y \in \overline{O}$, $d(y) = y_n \geqslant 0$. Thus we have

$$
\begin{aligned}
D_x\psi_\delta(x,y) = {} & 2(x-y) - 2\tilde{\varphi}_\delta\left(\frac{x+y}{2}, x-y\right)|x-y|e_n \\
& + 2\left(\frac{1}{2}D_X\tilde{\varphi}_\delta\left(\frac{x+y}{2}, x-y\right) + D_Y\tilde{\varphi}_\delta\left(\frac{x+y}{2}, x-y\right)\right)|x-y|y_n \\
& + 2\tilde{\varphi}_\delta\left(\frac{x+y}{2}, x-y\right)\frac{x-y}{|x-y|}y_n - 4A_1y_ne_n - K_1\delta e_n.
\end{aligned}
$$

By taking in account that $n(x) = -e_n$, we have

$$
D_x\psi_\delta(x,y) = p + q + r,
$$

where

$$
\begin{aligned}
p &= 2(x-y) + 2\varphi(x, x-y)n(x), \\
q &= 2\left[\tilde{\varphi}_\delta\left(\frac{x+y}{2}, x-y\right)|x-y| - \varphi(x, x-y)\right]n(x) \\
&\quad + \left(D_X\tilde{\varphi}_\delta\left(\frac{x+y}{2}, x-y\right) + 2D_Y\tilde{\varphi}_\delta\left(\frac{x+y}{2}, x-y\right)\right)|x-y|y_n \\
&\quad + 2\tilde{\varphi}_\delta\left(\frac{x+y}{2}, x-y\right)\frac{x-y}{|x-y|}y_n, \\
r &= (4A_1y_n + K_1\delta)n(x).
\end{aligned}
$$

We first notice that, taking in account the definition of $\tilde{\varphi}_\delta$, since φ is homogeneous of degree 1 with respect to p and satisfies (G2) we have

$$
\begin{aligned}
& \left|\tilde{\varphi}_\delta\left(\frac{x+y}{2}, x-y\right)|x-y| - \varphi\left(x, \frac{x-y}{|x-y|}\right)|x-y|\right| \\
& \quad \leqslant K|x-y|\big(|x-y| + \Gamma_\delta\big) \leqslant K\big[|x-y|^2 + \delta + (y_n - x_n)\big] \leqslant K|x-y|^2 + K\delta + Ky_n.
\end{aligned}
$$

Moreover, we have

$$
\left|\tilde{\varphi}_\delta\left(\frac{x+y}{2}, x-y\right)\frac{x-y}{|x-y|}y_n\right| \leqslant K|x-y|.
$$

By using the fact that $G(x,p) = 0$ and combining the above estimates with the properties of G, we get

$$
G(x, D_x\tilde{\psi}_\delta) \geqslant G(x, p+r) - K|q| \geqslant \lambda(2A_1y_n + K_1\delta) - K|x-y|^2 - K\delta - Ky_n.
$$

Thus if A_1 and K_1 are large enough we get

$$
G(x, D_x\tilde{\psi}_\delta) \geqslant -K|x-y|^2 + K\delta. \qquad \square
$$

Step 3. Construction of the function ϕ_0 and its main properties.

For $\overline{C}, A_2 > 0$ large enough, we introduce the following function:

$$\phi_0(x,y) = |x-x_0|^4 - 2\overline{C}|x-x_0|^3\big(d(x)-d(x_0)\big) + A_2\big(d(x)-d(x_0)\big)^4. \tag{70}$$

In order to check that the function Θ_δ satisfy the right boundary conditions, we premise the following lemma whose proof can be found in [5]. To formulate it, we use the following notation: for $p \in \mathbb{R}^n$ and $x \in \partial O$, $\mathcal{T}(p) := p - \langle p, n(x)\rangle n(x)$. $\mathcal{T}(p)$ represents the projection of p on the tangent hyperplane to ∂O at x.

Lemma 3.3. *Assume that* (G1) *and* (G2) *hold and that, for some $x \in \partial O$, and $\tilde{p} \in \mathbb{R}^n$, we have $G(x,\tilde{p}) \leqslant 0$ (respectively $G(x,\tilde{p}) \geqslant 0$), then there exists a constant $\overline{K}$ (depending on ν and K in* (G1)–(G2)) *such that, if $\langle p, n(x)\rangle \leqslant -\overline{K}|\mathcal{T}(p)|$, then*

$$G(x, \tilde{p} + p) \leqslant 0$$

(respectively if $\langle p, n(x)\rangle \geqslant \overline{K}|\mathcal{T}(p)|$, then $G(x,\tilde{p}+p) \geqslant 0$).

The connection with ϕ_0 is given by the following result.

Lemma 3.4. *For $\overline{C}$ large enough and for all $x \in \partial O$, ϕ_0 satisfies*

$$\frac{\partial \phi_0}{\partial n}(x) \geqslant \overline{K}\big|\mathcal{T}(D\phi_0)\big|,$$

where $\overline{K} > 0$ is the constant appearing in Lemma 3.3.

Proof. We first compute the normal derivative of ϕ_0 and we use the usual property linking the distance function and n; this yields

$$\begin{aligned}
\big\langle D_x\phi_0(x), n(x)\big\rangle &= \big\langle 4|x-x_0|^2(x-x_0), n(x)\big\rangle + 2\overline{C}|x-x_0|^3 \\
&= 4|x-x_0|^2\big(d(x_0) + O\big(|x-x_0|^2\big)\big) + 2\overline{C}|x-x_0|^3 \\
&\geqslant 2\overline{C}|x-x_0|^3 + O\big(|x-x_0|^4\big).
\end{aligned}$$

On the other hand, we clearly have $|\mathcal{T}(D\phi_0)| \leqslant 4|x-x_0|^3$. Thus, if $\overline{K}$ is the constant given in Lemma 3.3, by choosing $\overline{C}$ large enough and x close to x_0, we have

$$\big|\mathcal{T}(D\phi_0)\big|^{-1}\big\langle D_x\phi_0(x), n(x)\big\rangle \geqslant \frac{2\overline{C}|x-x_0|^3 + O(|x-x_0|^4)}{4|x-x_0|^3} > \overline{K}. \qquad \square$$

Now we can prove the following result.

Proposition 3.10. *If* $|x-y| \ll 1$, *then we have*

$$G\big(x, D_x\Theta_\delta(x,y)\big) > 0 \quad \text{if } x_n = 0, \tag{71}$$

$$G\big(y, -D_y\Theta_\delta(x,y)\big) < 0 \quad \text{if } y_n = 0. \tag{72}$$

Proof of Proposition 3.10. Again we only check the first property (71). First of all we observe that, because of the assumption (G1) and the property (68) of ψ_δ we have

$$\begin{aligned} G\big(x, D_x\big(e^{-\tilde{K}(y_n)}\big(\psi_\delta(x,y)\big)^{\alpha/2}\big)\big) &= G\big(x, \tilde{K}e^{-\tilde{K}y_n}\big(\tilde{\psi}_\delta(x,y)\big)^{\alpha/2}n(x) \\ &\quad + \frac{\alpha}{2}e^{-\tilde{K}y_n}\big(\tilde{\psi}_\delta(x,y)\big)^{\alpha/2-1}D_x\psi_\delta(x,y)\big) \\ &\geqslant \mu\tilde{K}e^{-\tilde{K}(y_n)}|x-y|^\alpha - Ke^{-\tilde{K}(y_n)}|x-y|^\alpha. \end{aligned}$$

Thus, if $\tilde{K}$ is sufficiently large, we obtain

$$G\big(x, D_x\big(e^{-\tilde{K}(x_n+y_n)}\big(\psi_\delta(x,y)\big)^{\alpha/2}\big)\big) > K|x-y|^\alpha,$$

for some constant K.

Now by combining Lemmas 3.3 and 3.4, we get

$$G\big(x, D_x\Theta_\delta(x,y)\big) = G\big(x, CD_x\big(e^{-\tilde{K}(x_n+y_n)}\big(\psi_\delta(x,y)\big)^{\alpha/2}\big) + LD_x\phi_0\big) > K|x-y|^\alpha.$$

And the result is proved. □

References

[1] M. Bardi, I. Capuzzo Dolcetta, Optimal Control and Viscosity Solutions of Hamilton–Jacobi–Bellman Equations, Birkhäuser, Boston, 1997.
[2] G. Barles, Solutions de viscosite des equations de Hamilton–Jacobi, Math. Appl., vol. 17, Springer-Verlag, 1994.
[3] G. Barles, Interior gradient bounds for the mean curvature equation by viscosity solutions methods, Differential Integral Equations 4 (2) (1991) 263–275.
[4] G. Barles, Nonlinear Neumann boundary conditions for quasilinear degenerate elliptic equations and applications, J. Differential Equations 154 (1999) 191–224.
[5] G. Barles, F. Da Lio, A geometrical approach to front propagation problems in bounded domains with Neumann-type boundary conditions, Interfaces and Free Boundaries 5 (2003) 239–274.
[6] G. Barles, P.E. Souganidis, Space–time periodic solutions and long-time behavior of solutions to quasi-linear parabolic equations, SIAM J. Math. Anal. 32 (6) (2001) 1311–1323.
[7] M.G. Crandall, H. Ishii, P.L. Lions, User's guide to viscosity solutions of second order partial differential equations, Bull. Amer. Math. Soc. 27 (1992) 1–67.
[8] W.H. Fleming, H.M. Soner, Controlled Markov Processes and Viscosity Solutions, Appl. Math., Springer-Verlag, New York, 1993.
[9] H. Ishii, Fully nonlinear oblique derivative problems for nonlinear second-order elliptic PDE's, Duke Math. J. 62 (1991) 663–691.
[10] H. Ishii, P.L. Lions, Viscosity solutions of fully nonlinear second-order elliptic partial differential equations, J. Differential Equations 83 (1) (1990) 26–78.

[11] C.E. Kenig, N.S. Nadirashvili, On optimal estimates for some oblique derivative problems, J. Funct. Anal. 187 (1) (2001) 70–93.
[12] G.M. Lieberman, Local estimates for subsolutions and supersolutions of oblique derivative problems for general second order elliptic equations, Trans. Amer. Math. Soc. 304 (1) (1987) 343–353.
[13] G.M. Lieberman, Pointwise estimates for oblique derivative problems in nonsmooth domains, J. Differential Equations 173 (1) (2001) 178–211.
[14] G.M. Lieberman, N. Trudinger, Nonlinear oblique boundary value problems for nonlinear elliptic equations, Trans. Amer. Math. Soc. 295 (2) (1986) 509–546.
[15] P.L. Lions, Neumann type boundary conditions for Hamilton–Jacobi equations, Duke Math. J. 52 (1985) 793–820.
[16] P.L. Lions, A.S. Sznitman, Stochastic differential equations with reflecting boundary conditions, Comm. Pure Appl. Math. XXXVII (1984) 511–537.
[17] P.L. Lions, N.S. Trudinger, Linear oblique derivative problems for the uniformly elliptic Hamilton–Jacobi–Bellman equation, Math. Z. 191 (1) (1986) 1–15.
[18] N.S. Nadirashvili, Some estimates in a problem with an oblique derivative, Izv. Akad. Nauk SSSR Ser. Mat. 52 (5) (1988) 1082–1090, 1119 (in Russian); translation in Math. USSR-Izv. 33 (2) (1989) 403–411.
[19] N.S. Nadirashvili, On a problem with an oblique derivative, Mat. Sb. (N.S.) 127(169) (3) (1985) 398–416 (in Russian).