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Polynomial approximation from diffused data: unisolvence and stability

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Abstract In this work, we address the problem of polynomial interpolation of non-pointwise data. More specifically, we assume that our input information comes from measurements obtained on diffuse compact domains. Although the nodal and the diffused problems are related by the mean value theorem, such an approach does not provide any concrete insights in terms of well-posedness and stability. We hence develop a different framework in which *unisolvence* can be again recovered from nodal results, for which a wide literature is available. To analyze the stability of the so-obtained diffused interpolation procedure, we characterize the norm of the interpolation operator in terms of a Lebesgue constant-like quantity. After analyzing some of its features, such as invariance properties and sensitivity to support overlapping, we numerically verify the theoretical findings.

Keywords interpolation, reconstruction, Lebesgue constant, unisolvence, integral data

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1 Introduction

Interpolation deals with the reconstruction and approximation of a function $f(x)$ given some samplings $f(x_1), \dots, f(x_N)$ on a set of nodes $x_1, \dots, x_N$ in some domain

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$\Omega \subset \mathbb{R}^n$. Such a set of must be *unisolvent* [23] for an appropriate target (finite dimensional) space; that is, there must exist a unique element of such space that interpolates f at these nodes. This procedure assumes that f admits a point-wise value at such nodes and, in a problem-oriented perspective, that we are also able to capture the corresponding data. In some situations, for instance in climate studies and geophysics [31], physical [62] or statistical [55] problems, optimal transport [43], mimetic methods [42] or sensor placement [44], such data is not available or may even be meaningless, e.g., if we are trying to interpolate a differential n-form [52] by means of weights [51]. This latter situation arises in the construction of finite element spaces [2, 22] and appears in virtual-elements-related settings [7].

In the aforementioned cases, non-pointwise or *diffused* data can be in fact given and *integral interpolation* [9] supersedes nodal interpolation. In the following, we consider as target space the polynomial space $\mathbb{P}_d(\Omega)$ of total degree d, whose dimension is

$$N := \dim \mathbb{P}_d(\Omega) = \binom{n+d}{n}.$$

As a result, the usual interpolating conditions that allow to project a function f onto $p \in \mathbb{P}_d(\Omega)$ are replaced by

$$\int_{K_i} f(x)\,\mathrm{d}x = \int_{K_i} p(x)\,\mathrm{d}x, \qquad i = 1, \ldots, N, \tag{1}$$

where $\{K_i\}_{i=1}^N$ is an appropriate collection of compact sets supported in Ω. In some applications, such as [10], the conditions (1) are replaced by the *average-matching* conditions

$$\frac{1}{\mu(K_i)} \int_{K_i} f(x)\,\mathrm{d}x = \frac{1}{\mu(K_i)} \int_{K_i} p(x)\,\mathrm{d}x, \qquad i = 1, \ldots, N, \tag{2}$$

which are obtained by normalizing (1) by the Lebesgue measure $\mu(K_i)$ of K_i. Such settings are of course equivalent as long as the measures are bounded away from zero. This latter formulation makes also evident the relationship between histopolation and *ENO/WENO* schemes [58], and their relative variants [27], provided that the supports K_i appropriately partition the domain Ω. In such a context, which arises in the treatment of conservation laws by the Finite Volume Method (FVM), the averaging-matching conditions (2) are known as *cell averages*, and are required to approximate the numerical flux between neighboring cells, see, e.g., [33]. It is worth pointing out here that this offers also a connection with kernel methods [1].

Remark 1 Any compact set K_i defining the conditions (1) is understood to be the closure of an open and connected set U of finite and non-vanishing Lebesgue measure: $0 < \mu(K_i) = \mu(U) < +\infty$. The same implicit assumption will be made on other compact sets K.

Conditions in (1) are called *area-matching* [55] and the corresponding technique is known as *histopolation* [53], as (at least when the dimension of the ambient space is $n = 1$) they correspond to asking that the areas spanned by some histograms associated with $f(x)$ and $p(x)$ coincide. When supports K_i are tensor products of segments, several spline-based techniques have been developed, see e.g. [30], but for less regular domains, histopolation appears to be rather unexplored.

Now, suppose that the information at the left hand side of (1) indeed represents some data extracted from a sensor. For such an object, we may decide both its shape and its placement in the domain Ω. Of course, we want to place these N sensors in Ω so that for each $f(x)$ there exists a unique polynomial $p(x)$ of total degree at most d satisfying (1); further, we wish for a distortion as small as possible in the reconstructed polynomial, avoiding Runge phenomena [4] and limiting uncontrollable propagation of errors.

The well-posedness of the interpolation operator $\Pi : C^0(\Omega) \to \mathbb{P}_d(\Omega)$ associated with (1) is related to the *unisolvence* of the supports K_i.

Definition 1 A collection of compact sets $\{K_1, \ldots, K_N\}$ is termed unisolvent for $\mathbb{P}_d(\Omega)$ if $\int_{K_i} p(x)\,\mathrm{d}x = 0$ for $i = 1, \ldots, N$ implies that $p(x) = 0$.

The characterization of unisolvent sets in the present framework is far from being trivial. We face this problem in Section 3, where we devise two techniques for the identification of unisolvent sets. It is worth mentioning that such techniques hinge on the interplay with nodal unisolvence, for which a rich literature is available. Without the presumption of being exhaustive enough, we recall the papers [12, 13, 23, 37, 38] and some of their relevant applications [24, 26, 45]. An approach that well matches our construction is the curve-based one, considered for instance in [14, 32, 34]. Once unisolvence is guaranteed, it is possible to associate with the collection $\{K_1, \ldots, K_N\}$ a "dual" basis $\{\ell_{K_1}, \ldots, \ell_{K_N}\}$.

Definition 2 Let $\{K_1, \ldots, K_N\}$ be a unisolvent collection of compact sets. The Lagrange basis for $\mathbb{P}_d(\Omega)$ is the unique basis satisfying the duality relations

$$\int_{K_i} \ell_{K_j}(x)\,\mathrm{d}x = \delta_{i,j} \qquad i, j = 1, \ldots, N. \tag{3}$$

The Lagrange basis allows us to represent the solution of the problem (1) in the convenient form

$$p(x) = \Pi f(x) = \sum_{i=1}^{N} \left(\int_{K_i} f(t)\,\mathrm{d}t \right) \ell_{K_i}(x), \tag{4}$$

which also makes evident that Π acts as the identity operator when restricted to the space of polynomials $\mathbb{P}_d(\Omega)$: indeed, we will show in Proposition 1 that it satisfies $\Pi p = p$ for each $p \in \mathbb{P}_d(\Omega)$.

The representation (4) separates the contribution of the specific problem (carried by $\int_{K_i} f$), which is here used as a coefficient on the polynomial expansion, and the impact of the choice on the supports K_i, determining the basis functions ℓ_{K_i}. To have a measure of the quality of our supports, we shall hence choose compact sets that yield Lagrange functions with small sup-norm [28]; recall that

$$\|f\|_\infty := \sup_{x \in \Omega} |f(x)|.$$

That this norm might be meaningful also in our approach is suggested by the mean value theorem applied to the set of conditions in Eq. (1), as it transforms the diffused interpolation problem into a nodal one $f(\xi_i) = p(\xi_i)$ for $i = 1, \ldots, N$, on some *unknown* points $\xi_i \in K_i$ for $i = 1, \ldots, N$. To relate the uniform norm with the diffused supports K_i, we may again invoke the mean value theorem to see that

$$\|f\|_\infty := \sup_{x \in \Omega} |f(x)| = \sup_{K \subset \Omega} \frac{1}{\mu(K)} \left| \int_K f(x)\,\mathrm{d}x \right|. \tag{5}$$

After recalling some classical inequalities that enlighten the role of the operator norm

$$\|\Pi\|_\infty := \sup_f \frac{\|\Pi f\|_\infty}{\|f\|_\infty} = \sup_{\|f\|_\infty = 1} \|\Pi f\|_\infty \tag{6}$$

with respect to the uniform norm, in Section 2 we exploit (5) to relate $\|\Pi\|_\infty$ with the underlying diffused supports. This provides a Lebesgue constant [5], thus a measurement of the numerical conditioning, for the diffused problem.

Once the features of this Lebesgue constant are established, it is necessary to detect collections of supports that keep such a quantity under control. As in the nodal framework, this Lebesgue constant turns out to be extremely sensitive to the choice of the supports. The identification of effective sets of diffused supports is also complicated by the large number of parameters needed to represent a compact set (in place of a point), which generally leads to an undetermined problem. If, in univariate histopolation [20], this can be handled by identifying some relevant subclasses of segments, in the multivariate one even this simplification fails. As a consequence, in Section 4 we analyze the counterpart of nodal techniques for the identification of approximated Leja sequences [16] and Fekete sets [15]. These algorithmic techniques, partially motivated by stability results [50], yield in fact a collection of supports with a low Lebesgue constant.

Remark 2 To make the description more readable, the dependence of the functions and the polynomials on the variable x (which also denotes multivariate coordinates $x_1, \ldots, x_n$) will be frequently suppressed.

2 Polynomial approximation

Uniform approximation [28, Chapter 6] aims at controlling the sup-norm of the interpolation procedure, that is, if p is the polynomial that satisfies (1), it deals with the quantity $\|f - p\|_\infty$. This is generally performed via the analysis of two inequalities. The first one controls the stability of the problem: if the data collected are not exact, the perturbation $\|f - \widetilde{f}\|_\infty < \varepsilon$ propagates proportionally to the norm of the interpolation operator:

$$\|\Pi f - \Pi \widetilde{f}\|_\infty = \|\Pi(f - \widetilde{f})\|_\infty \leq \|\Pi\|_\infty \|f - \widetilde{f}\|_\infty. \tag{7}$$

If, in addition, Π is also a projection operator, one has the Lebesgue-like inequality

$$\|f - \Pi f\|_\infty \leq \|f - p\|_\infty + \|\Pi f - \Pi p\|_\infty \leq (1 + \|\Pi\|_\infty) \inf_{p \in \mathbb{P}_d(\Omega)} \|f - p\|_\infty. \tag{8}$$

Remark 3 Whenever $\xi_1, \ldots, \xi_N$ are unisolvent nodes for interpolation, the norm of the *nodal* Lagrange interpolation operator may be characterized in terms of the corresponding cardinal functions $\{\ell_{\xi_i}\}_{i=1}^N$ verifying $\ell_{\xi_i}(\xi_j) = \delta_{i,j}$. One indeed has $\|\Pi\|_\infty = \sup_{x \in \Omega} \sum_{i=1}^N |\ell_{\xi_i}(x)|$. This quantity is highly sensitive to the choice of the nodes ξ_i's and it is used to detect well-conditioned sets of nodes.

The interpolation operator defined by the set of conditions (1) is linear, thus satisfies (7). It is also a projection operator, and hence satisfies (8) as well.

Proposition 1 *Suppose $\{K_i\}_{i=1}^N$ is unisolvent for $\mathbb{P}_d(\Omega)$ in the sense of Definition 1. The corresponding interpolation operator $\Pi : C^0(\Omega) \to \mathbb{P}_d(\Omega)$ defined in (4) is a projection operator onto polynomials $\mathbb{P}_d(\Omega)$, that is, $\Pi p = p$ for each $p \in \mathbb{P}_d(\Omega)$.*

Proof By plugging (3) into (4), one immediately retrieves that $\Pi \ell_{K_i}(x) = \ell_{K_i}(x)$. We then compute

$$\begin{aligned}\Pi\left(\Pi f\right) &= \sum_{j=1}^N \int_{K_j} \left(\sum_{i=1}^N \left(\int_{K_i} f \right) \ell_{K_i} \right) \ell_{K_j} = \sum_{j=1}^N \left(\sum_{i=1}^N \left(\int_{K_i} f \right) \int_{K_j} \ell_{K_i} \right) \ell_{K_j} \\ &= \sum_{j=1}^N \left(\int_{K_j} f \right) \ell_{K_j} = \Pi f,\end{aligned}$$

which shows the claim. □

The inequalities (7) and (8), together with the representation (4), motivate the role of the operator norm $\|\Pi\|_\infty$ as the parameter that controls the numerical conditioning of the interpolation procedure.

2.1 The Lebesgue constant

To select good sets of supports for diffused data, we adapt the quantity proposed in [5] for differential k-forms as

$$\Lambda_d := \sup_{K \subset \Omega} \frac{1}{\mu(K)} \sum_{i=1}^N \mu(K_i) \left| \int_K \ell_{K_i} \right|, \tag{9}$$

where the Lagrange functions ℓ_{K_i} are defined via the duality relation (3). The quantity (9) can be characterized in terms of nodal evaluations.

Lemma 1 *Let $\{K_1, \ldots, K_N\}$ be unisolvent for $\mathbb{P}_d(\Omega)$. One has*

$$\Lambda_d = \sup_{\xi \in \Omega} \sum_{i=1}^N \mu(K_i) \left| \ell_{K_i}(\xi) \right|. \tag{10}$$

Proof To see that (10) holds, we observe that for each K one has

$$\frac{1}{\mu(K)} \sum_{i=1}^N \mu(K_i) \left| \int_K \ell_{K_i} \right| \leq \frac{1}{\mu(K)} \int_K \sum_{i=1}^N \mu(K_i) \left| \ell_{K_i} \right| = \sum_{i=1}^N \mu(K_i) \left| \ell_{K_i}(\xi_K) \right|$$

for some $\xi_K \in K$; the last equality follows from the mean value theorem since $\sum_{i=1}^N \mu(K_i) |\ell_{K_i}|$ is a continuous function. Passing to the supremum, we hence get

$$\sup_{K \subset \Omega} \frac{1}{\mu(K)} \sum_{i=1}^N \mu(K_i) \left| \int_K \ell_{K_i} \right| \leq \sup_{\xi \in \Omega} \sum_{i=1}^N \mu(K_i) \left| \ell_{K_i}(\xi) \right|.$$

To obtain the equality, it is sufficient to consider, for any $\xi \in \Omega$, balls of (sufficiently small) radius ε centered at ξ (possibly intersected with Ω, to handle the cases where Ω has boundary and ξ belongs to the boundary of Ω). These are compact sets contained in Ω, and letting $\varepsilon \to 0$ one obtains the claim. □

Apart from few situations, mostly one-dimensional (see e.g. [53, Theorem 5.2] or [20, Proposition 3.6]), explicit expressions for histopolation-based Lagrange functions ℓ_{K_i} are not available. They may nevertheless be recovered by selecting any convenient basis $\{p_j\}_{j=1}^N$ for $\mathbb{P}_d(\mathbb{R}^n)$ and constructing the square *Vandermonde matrix* of size $N \times N$, being $N = \dim \mathbb{P}_d(\mathbb{R}^n)$,

$$\boldsymbol{V}_{i,j} := \int_{K_i} p_j, \tag{11}$$

so that $\ell_{K_i}(x) = \sum_{j=1}^N \boldsymbol{V}_{i,j}^{-T} p_j(x)$. Note that this requires the invertibility of the Vandermonde matrix, which is independent of the basis chosen for $\mathbb{P}_d(\mathbb{R}^n)$ and is in fact an equivalent condition for unisolvence, see, e.g., [8, Lemma 3.2.2].

Remark 4 The term $\mu(K_i)$ in (9) may be incorporated in the Lagrange functions by modifying (3) as

$$\frac{1}{\mu(K_i)} \int_{K_i} \widetilde{\ell}_{K_j} = \delta_{i,j},$$

obtaining the equivalent expression $\Lambda_d = \sup_{\xi \in \Omega} \sum_{i=1}^N |\widetilde{\ell}_{K_i}(\xi)|$. These can be seen as the Lagrange functions associated with the problem (2). Consistently, in this case, the corresponding Vandermonde matrix scales that defined in Eq. (11) by the diagonal matrix $\operatorname{diag}(\mu(K_1), \ldots, \mu(K_N))$.

Both the characterizations in (9) are interesting: the first one will be used to relate Λ_d with $\|\Pi\|_\infty$, whereas the second one helps in reducing the computational cost in simulations.

Remark 5 By construction, $\Lambda_d \geq 1$ for each d. Indeed, since each K_i is compact, considering $K = K_j$ and exploiting the duality relationship of the Lagrange basis, one has

$$\Lambda_d = \sup_{K \subset \Omega} \frac{1}{\mu(K)} \sum_{i=1}^N \mu(K_i) \left| \int_K \ell_{K_i} \right| \geq \frac{1}{\mu(K_j)} \sum_{i=1}^N \mu(K_i) \left| \int_{K_j} \ell_{K_i} \right| = \frac{\mu(K_j)}{\mu(K_j)} = 1.$$

The next result shows that Λ_d indeed overestimates $\|\Pi\|_\infty$.

Proposition 2 *Let $\{K_i\}_{i=1}^N$ (possibly not disjoint) be unisolvent for $\mathbb{P}_d(\mathbb{R}^n)$. Then*

$$\|\Pi\|_\infty \leq \Lambda_d.$$

Proof Expanding the definition of Π and the right hand side equality in (9), we compute

$$\begin{aligned}
\|\Pi\|_\infty &= \sup_{\|f\|=1} \|\Pi f\|_\infty = \sup_{\|f\|=1} \sup_{\xi\in\Omega} \left| \sum_{i=1}^N \left(\int_{K_i} f \right) \ell_{K_i}(\xi) \right| \\
&\leq \sup_{\|f\|=1} \sup_{K\subset\Omega} \sum_{i=1}^N \left| \int_{K_i} f \right| \left| \int_K \ell_{K_i} \right| \\
&= \sup_{\|f\|=1} \sup_{K\subset\Omega} \sum_{i=1}^N \frac{\mu(K_i)}{\mu(K_i)} \left| \int_{K_i} f \right| \left| \int_K \ell_{K_i} \right| \\
&\leq \sup_{K\subset\Omega} \sum_{i=1}^N \mu(K_i) \left| \int_K \ell_{K_i} \right| = \Lambda_d.
\end{aligned}$$

The claim is proved. □

The numerical experiment depicted in [20, Fig. 3] shows that the distance between $\|\Pi\|_\infty$ and Λ_d increases as the overlap of the supports enlarges. Imposing disjointedness of the supports, we also get the converse inequality.

Theorem 1 *Let the set* $\{K_i\}_{i=1}^N$ *be unisolvent for* $\mathbb{P}_d(\mathbb{R}^n)$*. If* $K_i \cap K_j = \emptyset$ *for* $i \neq j$*, then*

$$\|\Pi\|_\infty = \Lambda_d. \tag{12}$$

Proof Thanks to Proposition 2, we are only left to prove that $\|\Pi\|_\infty \geq \Lambda_d$. To do so, for each $0 < \varepsilon < \Lambda_d$ we exhibit a function $f_\varepsilon \in C^0(\Omega)$ such that $\|\Pi f_\varepsilon\|_\infty \geq \Lambda_d - \varepsilon$. In view of Remark 5, for each K_i we may find an open set $U_i \subset K_i$ such that $\mu(K_i \setminus U_i) \leq \frac{\varepsilon}{\Lambda_d}\mu(K_i)$, or equivalently, $\mu(U_i) = (1 - \frac{\varepsilon}{\Lambda_d})\mu(K_i)$. Notice that such a construction is well-posed due to the assumptions of Remark 1. Using mollifiers, we may define collections of continuous functions

$$f_\varepsilon^{(i)}(x) = \begin{cases} \pm 1 & x \in U_i \\ 0 & x \notin K_i \end{cases}$$

such that $\|f_\varepsilon^{(i)}\|_\infty = 1$. Define $f_\varepsilon := \sum_{i=1}^N f_\varepsilon^{(i)}$. Independently of the choice of the sign made on each $f_\varepsilon^{(i)}$, the function f_ε is continuous since the K_i's are disjoint and the $f_\varepsilon^{(i)}$'s are continuous. It is also clear that, by construction, $\|f_\varepsilon\|_\infty = 1$. Further, for any $K \subset \Omega$, there exists a function f_ε that satisfies

$$\operatorname{sgn} \int_{K_i} f_\varepsilon = \operatorname{sgn} \int_K \ell_{K_i}$$

for each $i = 1, \ldots, N$ (the case $\int_K \ell_{K_i} = 0$ can be arbitrarily treated). By construction, one also has

$$\left| \int_{K_i} f_\varepsilon \right| = \left| \int_{K_i} f_\varepsilon^{(i)} \right| \geq \mu(U_i) = \left(1 - \frac{\varepsilon}{\Lambda_d} \right) \mu(K_i).$$

Using the above two facts and exploiting the equality in (9), we compute

$$\begin{aligned}
\|\Pi\|_\infty \geq \|\Pi f_\varepsilon\|_\infty &= \sup_{K\subset\Omega} \frac{1}{\mu(K)} \left| \int_K \sum_{i=1}^N \left(\int_{K_i} f_\varepsilon \right) \ell_{K_i} \right| \\
&= \sup_{K\subset\Omega} \frac{1}{\mu(K)} \left| \sum_{i=1}^N \left(\int_{K_i} f_\varepsilon \right) \int_K \ell_{K_i} \right| \\
&\geq \sup_{K\subset\Omega} \frac{1}{\mu(K)} \left| \sum_{i=1}^N \mu(U_i)| \int_K \ell_{K_i} \right| \\
&= \sup_{K\subset\Omega} \frac{1}{\mu(K)} \left| \sum_{i=1}^N \left(1 - \frac{\varepsilon}{\Lambda_d}\right) \mu(K_i) \int_K \ell_{K_i} \right| \\
&= \left(1 - \frac{\varepsilon}{\Lambda_d}\right) \sup_{K\subset\Omega} \frac{1}{\mu(K)} \left| \sum_{i=1}^N \mu(K_i) \int_K \ell_{K_i} \right| = \Lambda_d - \varepsilon.
\end{aligned}$$

The claim is proved. □

2.2 Invariance under affine transformations

Following the discussion in [3, Section 6], one sees that Lebesgue constants associated with non point-wise objects may not be invariant under the choice of the reference domain. In this section, we show that the quantity (9) depends only on the positioning of $K_1, \ldots, K_N$ inside Ω and not on the placement of Ω in $\mathbb{R}^n$. This follows from the fact that if Ω is rigidly mapped to another reference domain by a mapping φ, the Lebesgue constant associated with $K_1, \ldots, K_N$ in Ω coincides with the Lebesgue constant associated with $\varphi(K_1), \ldots, \varphi(K_N)$ in $\widehat{\Omega} := \varphi(\Omega)$.

Proposition 3 *Let $\varphi : \Omega \to \widehat{\Omega}$ be an affine transformation $\varphi(x) = Ax + b$ such that* $\det A \neq 0$. *Let*

$$\Lambda_d^\varphi := \sup_{\widehat{K}\in\widehat{\Omega}} \frac{1}{\mu(\widehat{K})} \sum_{i=1}^N \mu(\varphi(K_i)) \left| \int_{\widehat{K}} \ell_{\varphi(K_i)} \right|$$

be the quantity (9) *associated with* $\{\varphi(K_i)\}_{i=1}^N$. *Then*

$$\Lambda_d = \Lambda_d^\varphi.$$

Proof By the transformation formula for multiple integrals, we have

$$\delta_{i,j} = \int_{K_j} \ell_{K_i} = \int_{\varphi(K_j)} \frac{1}{|\det A|} \ell_{K_i} \circ \varphi^{-1}.$$

Put $\ell_{\varphi(K_i)} := |\det A|^{-1} \ell_{K_i} \circ \varphi^{-1}$. Since φ^{-1} is a non-degenerate affine map, $\left\{\ell_{\varphi(K_i)}\right\}_{i=1}^N$ is a basis for polynomials of degree d satisfying the duality constraint (3). As a consequence, it is the Lagrange basis for $\mathbb{P}_d(\widehat{\Omega})$ associated with $\{\varphi(K_i)\}_{i=1}^N$. By the same calculation, we also get that, for each $K \subset \Omega$,

$$\int_K \ell_{K_i} = \int_{\varphi(K)} \ell_{\varphi(K_i)}, \qquad i = 1, \ldots, N.$$

To conclude the proof, it is now sufficient to write $\widehat{K} = \varphi(K)$ and observe that $\mu(\varphi(K)) = |\det A|\mu(K)$ for each $K \subset \Omega$. Applying the above equalities and the definition in Eq. (9), we readily compute

$$\begin{aligned}
\Lambda_d^{\varphi} &:= \sup_{\widehat{K}\subset\widehat{\Omega}} \frac{1}{\mu(\widehat{K})} \sum_{i=1}^{N} \mu(\varphi(K_i)) \left| \int_{\widehat{K}} \ell_{\varphi(K_i)} \right| \\
&= \sup_{\varphi(K)\subset\widehat{\Omega}} \frac{1}{\mu(\varphi(K))} \sum_{i=1}^{N} \mu(\varphi(K_i)) \left| \int_{\varphi(K)} \ell_{\varphi(K_i)} \right| \\
&= \sup_{K\subset\Omega} \frac{1}{|\det A|\mu(K)} \sum_{i=1}^{N} |\det A|\mu(K_i) \left| \int_{K} \ell_{K_i} \right| \\
&= \sup_{K\subset\Omega} \frac{1}{\mu(K)} \sum_{i=1}^{N} \mu(K_i) \left| \int_{K} \ell_{K_i} \right| = \Lambda_d,
\end{aligned}$$

and the proof is concluded. □

2.3 Bridging nodal and diffused interpolation

The mean value theorem and the stability estimate given in [50, Proposition 1] allow us to relate the Lebesgue constant (9) to its nodal counterpart discussed in Remark 3. In particular, the Lebesgue constant associated with histopolation on the sets $\{K_1, \dots, K_N\}$ is close to the nodal Lebesgue constant of some set of nodes $\{\xi_1, \dots, \xi_N\}$, with $\xi_i \in K_i$ for $i = 1, \dots, N$, provided that the sets $\{K_1, \dots, K_N\}$ are *convex* and have a sufficiently small diameter. This is formalized and quantified by the following result, whose proof is omitted: it can be directly deduced from [19, Proposition 5], observing that the diameter of a triangle coincides with its longest edge, hence with its diameter as a compact set.

Proposition 4 *Let $\mathcal{X} := \{\xi_1, \dots, \xi_N\}$ be a set of nodes which is unisolvent for interpolation in $\mathbb{P}_d(\Omega)$, and let $\Lambda_d^{\mathcal{X}}$ denote its nodal Lebesgue constant. Let $\{K_1, \dots, K_N\}$ be a collection of compact convex sets such that $\xi_i \in K_i$ and $\mu(K_i \cap K_j) = 0$ if $i \neq j$. Let $r_{\max} := \frac{1}{2}\max_i \operatorname{diam}(K_i)$, and denote by Λ_d^K the Lebesgue constant* (9) *of $\{K_1, \dots, K_N\}$. For each $\alpha \in (0,1)$ so that $r_{\max} < 2\,c\,d^r \Lambda_d^{\mathcal{X}}\,\alpha$, c being the Markov constant and r an appropriate constant depending on Ω, hence*

$$\Lambda_d^K \leq \frac{1}{1-\alpha}\Lambda_d^{\mathcal{X}}. \tag{13}$$

Remark 6 The role of the Markov constant in Proposition 4 is also addressed in [50]. In particular, if Ω is a convex body, one has $r = 2$. We will work under this assumption in the numerical section.

2.4 Numerical estimation of the Lebesgue constant

So far, we have dealt with the definition of Lebesgue constants on the continuous level. In this section we provide an implementation strategy. Due to the lack of general closed formulae for the Lagrange basis, this technique requires the inversion of the Vandermonde matrix, which may be ill-conditioned [56].

2.4.1 An implementation strategy

To estimate Lebesgue constants associated with some unisolvent sets $\{K_i\}_{i=1}^N$ on a reference domain Ω we discretize Eq. (9). For what concerns the right hand side, this consists in selecting a large but finite family of supports $\mathcal{S} := \{S_\ell\}_{\ell=1}^M$ with $M \gg N$. Let us expand the leftmost equality in (9) as

$$\begin{aligned}\Lambda_d &\approx \sup_{S_\ell \in \mathcal{S}} \frac{1}{\mu(S_\ell)} \sum_{i=1}^N \mu(K_i) \left| \int_{S_\ell} \ell_{K_i} \right| = \sup_{S_\ell \in \mathcal{S}} \frac{1}{\mu(S_\ell)} \sum_{i=1}^N \mu(K_i) \left| \int_{S_\ell} \sum_{j=1}^n \boldsymbol{V}_{i,j}^{-T} p_j \right| \\ &= \sup_{S_\ell \in \mathcal{S}} \frac{1}{\mu(S_\ell)} \sum_{i=1}^N \mu(K_i) \left| \sum_{j=1}^n \boldsymbol{V}_{i,j}^{-T} \int_{S_\ell} p_j \right| ,\end{aligned}$$

where $\{p_j\}_{j=1}^N$ is any basis for $\mathbb{P}_d(\Omega)$. We will discuss in the next Section 2.4.2 a convenient choice. Define the $M \times N$ matrix

$$\boldsymbol{W}_{\ell,j}^{\mathcal{S}} := \int_{S_\ell} p_j$$

and the square diagonal matrices

$$\boldsymbol{S} = \begin{pmatrix} \mu(S_1)^{-1} & 0 & \dots & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \mu(S_M)^{-1} \end{pmatrix} \quad \text{and} \quad \boldsymbol{K} = \begin{pmatrix} \mu(K_1) & 0 & \dots & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \mu(K_N) \end{pmatrix} .$$

It follows that, for each ℓ,

$$\frac{1}{\mu(S_\ell)} \sum_{i=1}^N \mu(K_i) \left| \int_{S_\ell} \ell_{K_i} \right| = \sum_{t=1}^N \left| \left(\boldsymbol{S} \boldsymbol{W}^{\mathcal{S}} \boldsymbol{V}^{-1} \boldsymbol{K} \right)_{\ell,t} \right| ,$$

whence

$$\Lambda_d \approx \| \boldsymbol{S} \boldsymbol{W}^{\mathcal{S}} \boldsymbol{V}^{-1} \boldsymbol{K} \|_\infty , \tag{14}$$

being $\| \cdot \|_\infty$ the ∞-norm for matrices (i.e., the maximum by row-wise sum of absolute values, see e.g., [36, Eq. (2.3.10)]). Even if one has at disposal and uses an exact quadrature formula (as, e.g., those used in [6, 7]) for the elements of $\mathcal{S}$, so that no approximation error is carried on $\boldsymbol{W}^{\mathcal{S}}$, it is clear that the assembly of such a matrix is expensive. We hence discretize also (10) by considering a comparably large set of nodes $\mathcal{X} := \{\xi_i\}_{i=1}^M$, with $M \gg N$. By the very same reasoning, we obtain that

$$\begin{aligned}\Lambda_d &\approx \sup_{\xi_\ell \in \mathcal{X}} \sum_{i=1}^N \mu(K_i) \, |\ell_{K_i}(\xi_\ell)| = \sup_{\xi_\ell \in \mathcal{X}} \sum_{i=1}^N \mu(K_i) \left| \sum_{j=1}^n \boldsymbol{V}_{i,j}^{-T} p_j(\xi_\ell) \right| \\ &= \sup_{\xi_\ell \in \mathcal{X}} \sum_{i=1}^N \mu(K_i) \left| \sum_{j=1}^n \boldsymbol{V}_{i,j}^{-T} p_j(\xi_\ell) \right| .\end{aligned}$$

In this case, defining the $M \times N$ matrix

$$\boldsymbol{W}^{\mathcal{X}}_{\ell,j} := p_j(\xi_\ell)$$

one readily obtains

$$\Lambda_d \approx \|\boldsymbol{W}^{\mathcal{X}} \boldsymbol{V}^{-1} \boldsymbol{K}\|_\infty. \tag{15}$$

Comparing (15) with (14), it is evident that the latter formulation is sensibly cheaper, as it avoids the computation of $M \times N$ integrals, replacing them with the same amount of polynomial evaluations, and does not require the first matrix-matrix product. Since the equality between (9) and (10) holds only on the continuous level, i.e., if all compact sets K and nodes ξ in Ω are considered, (15) and (14) may differ. Numerical verifications show that the distance between such quantities is in fact negligible.

2.4.2 A remark on the conditioning of the Vandermonde matrix

Obstructions in the computation of Lagrange bases for histopolation are well-known [20, 35]. Due to the lack of closed formulae, one is forced to fix a basis $\{p_i\}_{i=1}^N$ for $\mathbb{P}_d(\Omega)$, compute the relative Vandermonde matrix (11) and invert it to obtain the coefficients of (3), as described after Eq. (11). Of course, the inversion of an ill-conditioned matrix may cause a severe instability. This topic is widely studied in the literature; in particular, in the univariate setting, it has been investigated how large the degree can be pushed in order to deal with a reliable result [57].

To avoid the propagation of errors and instability, among easy-to-write bases, we select the one that shows the smallest conditioning of the resulting Vandermonde matrix. Two natural candidates are thus the monomial $\mathcal{M}_d := \{x^\alpha y^\beta\}$ and the Chebyshev $\mathcal{T}_d := \{T_\alpha(x) T_\beta(y)\}$ bases, with $\alpha + \beta \leq d$.

In Figure 1 we depict the trend of the conditioning of the Vandermonde matrix $\boldsymbol{V}$ with respect to the monomial basis (blue) and the Chebyshev basis, for discs of fixed radius centered at Halton points [39] (which are quasi-random points) and points defined in [11] (due to their construction, we will refer to such points as Chebyshev orbits). In both cases Chebyshev polynomials lead to a better conditioned Vandermonde matrix; in the numerical section we will thus stick with this choice.

3 Two approaches to unisolvence

The detection of unisolvent sets is the starting step of any interpolation problem. In the multidimensional setting, already in the nodal framework, few explicit unisolvent sets of points for the polynomial space $\mathbb{P}_d(\Omega)$ are known. Nevertheless, it is well known that any collection of nodes is *generically* unisolvent: the measure of the manifold defined by non-unisolvent sets for polynomial interpolation is zero.

Lemma 2 *Let $\{\xi_1, \ldots, \xi_N\}$ be nodes such that*

$$\det \left[p_j(\xi_i)\right]_{i,j} = 0.$$

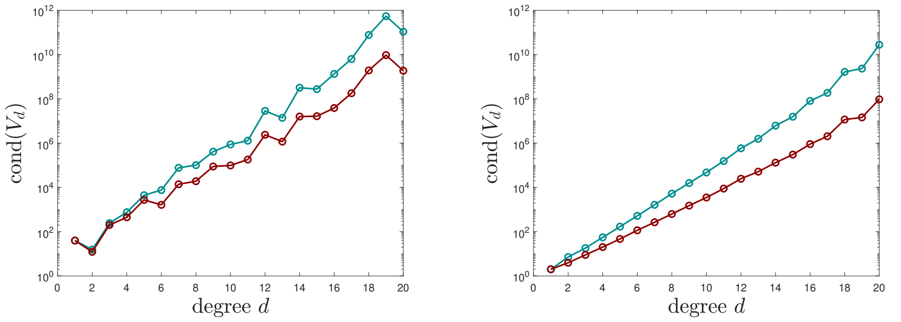

Fig. 1 Conditioning of the Vandermonde matrix: Chebyshev polynomials (red) vs monomials (blue). Supports are discs centered at Halton points (left) and Chebyshev points in the reference disc (right).

For each $\varepsilon > 0$*, there exists a collection of nodes* $\{\widetilde{\xi}_1, \dots, \widetilde{\xi}_N\}$*, with* $|\xi_i - \widetilde{\xi}_i| < \varepsilon$ *for* $i = 1, \dots, N$*, such that*

$$\det \left[p_j(\widetilde{\xi}_i) \right]_{i,j} \neq 0.$$

Lemma 2 is an immediate consequence of the properties of the Zariski topology. Roughly speaking, it can be read as follows: there is a "small perturbation" of any collection of non-unisolvent nodes that gives unisolvent nodes. This fact also carries over to the present framework: any collection of compact sets $\{K_1, \dots, K_N\}$ can be perturbed to a unisolvent collection for diffused interpolation in $\mathbb{P}_d(\mathbb{R}^n)$.

Proposition 5 *Let* $\{K_1, \dots, K_N\}$ *be a collection of compact sets such that*

$$\det \left[\int_{K_i} p_j(x) \, \mathrm{d}x \right]_{i,j=1}^{N} = 0.$$

Then, for any $\varepsilon > 0$*, there exists a collection of compact sets* $\{\widetilde{K}_1, \dots, \widetilde{K}_N\}$*, with Hausdorff distance* $\mathrm{dist}_H(K_i, \widetilde{K}_i) < \varepsilon$ *for* $i = 1, \dots, N$*, such that*

$$\det \left[\int_{\widetilde{K}_i} p_j(x) \, \mathrm{d}x \right]_{i,j=1}^{N} \neq 0.$$

Proof Let $\varepsilon > 0$. For each collection $\xi_1, \dots, \xi_N$, with $\xi_i \in B(0, \varepsilon)$ for each i, consider the matrix

$$[\boldsymbol{V}_\xi]_{i,j} := \int_{K_i + \xi_i} p_j(x) \, \mathrm{d}x.$$

Due to the transformation formula for multiple integrals, the map

$$q : \quad (\xi_1, \dots, \xi_N) \mapsto \det \boldsymbol{V}_\xi$$

is a polynomial in the variables $\xi_1, \dots, \xi_N$. Suppose, by contradiction, that $\det \boldsymbol{V}_\xi = 0$ for each $(\xi_1, \dots, \xi_N) \in (B(0, \varepsilon))^N$. Since q vanishes on an open and non empty set, q is the zero polynomial, which means that $\det \boldsymbol{V}_\xi$ vanishes for all translates of

$\{K_1, \ldots, K_N\}$. Then we can pick $(\xi_1, \ldots, \xi_N) \in (\mathbb{R}^n)^N$ such that $(K_i + \xi_i) \cap (K_j + \xi_j) = \emptyset$ for $i \neq j$; further, one may assume such sets to be well-separated. Considering the collection $\{K_1 + \xi_1, \ldots, K_N + \xi_N\}$, one finds distinct nodes $\{\widetilde{\xi}_1, \ldots, \widetilde{\xi}_N\}$ verifying the mean value theorem. Due to the continuity of polynomials, a small perturbation of the compact sets $\{K_1 + \xi_1, \ldots, K_N + \xi_N\}$ produces a small perturbation of the nodes $\{\widetilde{\xi}_1, \ldots, \widetilde{\xi}_N\}$, which hence remain distinct, and by Lemma 2 we reach a contradiction. As a consequence, there exist $\{\xi_1, \ldots, \xi_N\} \in (B(0, \varepsilon))^N$, i.e. $\{K_1 + \xi_1, \ldots, K_N + \xi_N\}$ with $\mathrm{dist}_H(K_i, \widetilde{K}_i) < \varepsilon$ for $i = 1, \ldots, N$, such that $\det \boldsymbol{V}_\xi \neq 0$. □

3.1 Unisolvence by translation

Despite the density result of Proposition 5, the identification of classes of unisolvent supports is a hard task, if not connected with nodal results. Also, such class of results is of limited direct applicability, as the mean value theorem does not allow for a precise localization of the nodes. If we fix a reference compact set K and consider its translates, we may relate with nodal interpolation results. To begin with, we extend [25, Lemma 3.12] to the case of generic compact sets. We adopt the usual notation $K + \xi := \{x + \xi \mid x \in K\}$.

Lemma 3 *Let $K \subset \mathbb{R}^n$ be a compact set and let $p \in \mathbb{P}_d(\mathbb{R}^n)$. If*

$$\int_{K+\xi} p(x)\, \mathrm{d}x = 0 \tag{16}$$

for each $\xi \in \mathbb{R}^n$, then $p(x) = 0$.

Proof Suppose, by contradiction, that p is not the zero polynomial. Let us denote by $\mathcal{Z}[p]$ the zero locus of p, which is an algebraic variety. Hence, the complement of $\mathcal{Z}[p]$ in $\mathbb{R}^n$ consists of disjoint open sets. At least one such regions, say U, has an infinite radius, that is, for any $r > 0$ there exists $x \in U$ such that the ball centered at x and of radius r is completely contained in U. Hence there exists $\xi \in \mathbb{R}^n$ such that $(K+\xi) \subset U$, which implies that $(K+\xi) \cap \mathcal{Z}[p] = \emptyset$. As a consequence, $\mathrm{sgn}\,(p)$ is constant in $K+\xi$ and its interior, which is open and non empty, being the measure of K finite and greater than zero. Thus Eq. (16) implies that $p(x) = 0$ for each x in such an open set and therefore $p(x) = 0$ for each $x \in \mathbb{R}^n$. □

Proposition 6 *Let $K \subset \mathbb{R}^n$ be a compact set. Adopting the usual convention*

$$AK + \xi := \{Ax + \xi \mid x \in K\},$$

suppose that there exists a non-degenerate family of affinities $\varphi_i(x) = Ax + \xi_i$ such that $K_i := \varphi_i(K) = AK + \xi_i$ for $i = 1, \ldots, N$. If

(i) $\{\xi_i\}_{i=1}^N$ is a unisolvent set for nodal interpolation in $\mathbb{P}_d(\Omega)$,

then

(ii) $\{K_i\}_{i=1}^N$ is a unisolvent set for integral interpolation in $\mathbb{P}_d(\Omega)$.

Proof We shall prove that

$$\int_{K_i} p(x)\,\mathrm{d}x = 0 \quad i = 1, \ldots, N \qquad \Longrightarrow \qquad p(x) = 0 \quad \forall x \in \Omega.$$

Consider the map

$$q: \quad \xi \mapsto \int_{AK+\xi} p(x)\,\mathrm{d}x.$$

Since $\xi \mapsto AK + \xi$ is an affinity, by the change of variable formula one deduces that $q(\xi)$ is a polynomial map. Further, since $p \in \mathbb{P}_d$, also $q \in \mathbb{P}_d$. By construction, q vanishes at $\{\xi_i\}_{i=1}^N$, i.e., $q(\xi_i) = 0$ for $i = 1, \ldots, N$, which is a unisolvent set for $\mathbb{P}_d$, so q is the null polynomial; namely $q(\xi) = 0$ for each $\xi \in \mathbb{R}^n$. Thus

$$0 = q(\xi) = \int_{AK+\xi} p(x)\,\mathrm{d}x \qquad \forall \xi \in \mathbb{R}^n.$$

Since A is fixed, the claim follows from Lemma 3. □

Remark 7 In the univariate framework, the converse implication $(ii) \Longrightarrow (i)$ holds. Indeed, if $\int_{K_i} p(x)\,\mathrm{d}x = 0$ implies that $p(x) = 0$, in particular all the supports K_i must be distinct (otherwise we find a repeated condition and the problem becomes underdetermined). Hence $K_i \neq K_j$ if $i \neq j$. Since $K_i = AK + \xi_i$ and $K_j = AK + \xi_j$, one immediately finds $\xi_i \neq \xi_j$ for $i \neq j$. As a consequence, the set $\{\xi_i\}_{i=1}^N$ contains pairwise distinct points and is therefore unisolvent for nodal interpolation. This partially extends the characterization of unisolvent segments in [20].

The above result makes it possible to invoke the literature on unisolvent sets for multivariate interpolation, for instance [18, 41, 46], at the price of requiring that all the supports are the image through φ of the same reference compact set K. The way the compact set K is selected does not affect unisolvence, provided that it satisfies the assumptions of Remark 1.

3.2 Unisolvence by algebraic varieties

The constraint of fixing the reference compact set K once and for all can be replaced if nodes may be collected into subsets lying on *(affine) algebraic varieties*. To do so, recall that an affine algebraic variety in $\mathbb{R}^n$ is the zero locus of a finite collection of polynomials in n variables [40, p. 3]. In what follows, we will consider algebraic varieties $\mathcal{V}$ defined by just one polynomial $P \in \mathbb{P}_{\dim \mathcal{V}}(\mathbb{R}^n)$, for short we shall write $\mathcal{V} := \{P = 0\}$. Proposition 6 immediately yields the following result.

Corollary 1 *Let $p \in \mathbb{P}_d(\mathbb{R}^n)$ and let $\mathcal{V} := \{P = 0\}$. If $\mathcal{X} = \{\xi_i\}_{i=1}^{\dim \mathbb{P}_d(\mathcal{V})}$ is a nodal unisolvent set for $\mathbb{P}_d(\mathcal{V})$ and*

$$\int_{K_i} p(x)\,\mathrm{d}x = 0 \qquad i = 1, \ldots, \dim \mathcal{V},$$

K_i being defined as in Proposition 6. Then P divides p.

Proof Since $\mathcal{X}$ is unisolvent for $\mathbb{P}_d(\mathcal{V})$, by Proposition 6 we have that

$$\int_{K_i} p(x)\,\mathrm{d}x = 0 \qquad i = 1, \ldots, \dim \mathcal{V},$$

implies that $p(x) = 0$ for each $x \in \mathcal{V}$. Since $\mathcal{V} = \{P = 0\}$, there exists $q \in \mathbb{P}_{d-\dim \mathcal{V}}(\mathbb{R}^n)$ such that $p = Pq$, which shows the claim. □

Corollary 1 allows to split the global interpolation problem into subproblems onto algebraic varieties, provided that they do not have common components. This means that, given $\mathcal{V} = \{P = 0\}$ and $\mathcal{W} = \{Q = 0\}$, there does not exist any polynomial R such that either $P = RQ$ or $Q = RP$.

For each variety $\mathcal{V}_j$, we select a reference compact $\widehat{K}_j$ and we translate it by vectors $\{\xi_j^i\}_{i=1}^{\dim \mathbb{P}(\mathcal{V}_j)}$. This moves the global unisolvence conditions to problems on the single varieties. We shall thus require that collections $\mathcal{X}_j := \{\xi_j^i\}_{i=1}^{\dim \mathbb{P}(\mathcal{V}_j)}$ are unisolvent for the spaces $\mathbb{P}(\mathcal{V}_j)$ (for some polynomial degree).

Clearly, if $\mathcal{V}_j = \mathbb{R}^n$, the next result is equivalent to Proposition 6. In Remark 12 we instead give an example in which the split interpolation problem is sensibly simpler than the global one.

Theorem 2 *Let $p \in \mathbb{P}_d(\mathbb{R}^n)$. Let $\mathcal{V}_1 = \{P_1 = 0\}, \ldots, \mathcal{V}_s = \{P_s = 0\}$ be affine algebraic varieties with pairwise no common components. Suppose that $P_j \in \mathbb{P}_{d_j}(\mathbb{R}^n)$ and $\sum_{j=1}^s d_j > d$. Suppose that, for each $\mathcal{V}_j$, there exists a collection of nodes $\mathcal{X}_j := \{\xi_j^1, \ldots, \xi_j^{N_j}\}$ which is unisolvent for $\mathbb{P}_{d-\sum_{i<j} d_i}(\mathcal{V}_j)$. If, for $j = 1, \ldots, s$,*

$$\int_{K_j^i} p(x)\,\mathrm{d}x \ = 0 \qquad i = 1, \ldots, N_j := \dim \mathbb{P}_{d-\sum_{i<j} d_i}(\mathcal{V}_j),$$

being $K_j^i := \varphi_j^i(K) = A_j x + \xi_j^i$ with $\det A_j \neq 0$, then $p(x) = 0$.

Proof Up to relabelling, we may reorder the varieties $\mathcal{V}_i$ so that $j < k$ if $d_j > d_k$. Consider $j = 1$. Since

$$\int_{K_1^i} p(x)\,\mathrm{d}x = 0 \qquad i = 1, \ldots N_1,$$

by Corollary 1 we have that P_1 divides p; equivalently, we may write $p = P_1 q$ for some $q \in P_{d-d_1}$. We may now iterate this process up to $j = s-1$, to get that

$$p = \left(\prod_{j=1}^{s-1} P_j \right) q_{s-1}.$$

This formula holds because the varieties $\mathcal{V}_j$ have no common components, hence at the j-th step the polynomial P_j, which divides p by Corollary 1, must divide q_{j-1} and not $\prod_{k=1}^{j-1} P_k$. By the above reasoning, when $j = s$,

$$\int_{K_s^i} p(x)\,\mathrm{d}x = 0 \qquad i = 1, \ldots, N_s$$

implies that P_s divides q_s. But

$$\deg q_s = d - \sum_{j=1}^{s-1} d_j < d_s = \deg P_s,$$

which implies that q_s is the zero polynomial, and so is p. This proves the claim. □

Remark 8 Theorem 2 uses Corollary 1 (hence Proposition 6) on each algebraic variety $\mathcal{V}_j$ independently. This means that, for each variety, one may choose a different reference set K.

Remark 9 Compact sets K_j^i (associated with the j-th variety) need not intersect $\mathcal{V}_j$, but may intersect other supports and other varieties.

Remark 10 The condition $\sum_{j=1}^{s} d_j > d$ can be replaced by the equality $\sum_{j=1}^{s} d_j = d$ plus one spare information (e.g. an evaluation or another integral) on a support that preserves unisolvence. In such a case, the proof reads exactly the same except for the last step. We give an example of this in Proposition 7.

3.3 An application to balls

We make the result of Theorem 2 more concrete, replacing algebraic varieties with orbits, and considering balls centered at the origin as reference compact sets. In this case, centers are localized on varieties $\mathcal{V}_j$ (relating to the nodal results in [13]); in turn, we obtain an easier algorithmic description.

The algebraic variety

$$\mathcal{V}(r) := \left\{ x = (x_1, \dots, x_n) \in \mathbb{R}^n \;\mid\; \sum_{i=1}^{n} x_i^2 = r^2 \right\} \tag{17}$$

describes the orbit of a point at distance r from 0 under the action of the special orthogonal group $SO(n)$. Note that $\mathcal{V}(r) = S_r^{n-1} := rS^{n-1}$. Being orbits, $\mathcal{V}_j := \mathcal{V}(r_j)$ does not intersect $\mathcal{V}_k := \mathcal{V}(r_k)$ if $r_j \neq r_k$. Since the polynomial

$$P_j(x_1, \dots, x_n) := -r_j^2 + \sum_{i=1}^{n} x_i^2 \tag{18}$$

describing any $\mathcal{V}_j$ has degree 2, the dimension of the space of polynomials on $\mathcal{V}_j$ is

$$\dim \mathbb{P}_d(\mathcal{V}(r)) = \dim \mathbb{P}_d(\mathbb{R}^n) - \dim \mathbb{P}_{d-2}(\mathbb{R}^n) = \binom{d+n}{n} - \binom{d+n-2}{n}, \tag{19}$$

see [13, Lemma 2.1]. As reference compact K, we consider the unit n-ball $B := B(0,1)$ and we take the linear part A of the affinity φ_j to be a scalar multiple of the identity. Hence, $B_j^i = \varphi_j^i(B) = B(\xi_j^i, r_j)$. Further, since B is centered at 0, every B_j^i is centered at a point in $\mathcal{V}_j$. Unisolvence for the integral interpolation problem on each variety $\mathcal{V}_j$ is thus granted if centers ξ_j^i, for $i = 1, \dots, N_j$ and each N_j being determined by (19), are placed at unisolvent nodes on the sphere $S_{r_j}^{n-1}$; for an account of such nodes see, for instance, [49, 64]. Corollary 1 may thus be simplified as follows.

Lemma 4 *Let $p \in \mathbb{P}_d(\mathbb{R}^n)$ and let $\mathcal{X} = \{\xi_i\}_{i=1}^N$ be unisolvent for nodal interpolation in $\mathbb{P}_d\left(\mathcal{V}(r)\right)$. If, for some $\bar{r} > 0$,*

$$\int_{B(\xi_i,\bar{r})} p(x)\,\mathrm{d}x = 0 \qquad \text{for each } \xi_i \in \mathcal{X},$$

then there exists $q \in \mathbb{P}_{d-2}(\mathbb{R}^n)$ such that $p = Pq$.

Our next construction follows Algorithm 1, whose well posedness will be a consequence of Proposition 7. The idea consists in selecting an orbit, selecting unisolvent nodes for nodal interpolation of the restriction of $\mathbb{P}_d$ on this orbit, constructing balls with radius r_1, passing to the next orbit, selecting unisolvent nodes for nodal interpolation of the restriction of $\mathbb{P}_{d-2}$ on this orbit (the reduction of the degree being a consequence of Lemma 4 and Eq. (19)), constructing balls with radius r_2, and iterating the procedure.

Algorithm 1 Construction of the unisolvent set by orbits

```
Require: The total degree $d$
Ensure: A collection of unisolvent balls
  $\mathbf{r} = []$                                      ▷ Vector of the radii
  while $d \geq 1$ do
      choose $r \notin \mathbf{r}$ and add $r$ to $\mathbf{r}$          ▷ Guarantees disjointedness of orbits
      fix $r_j$                                          ▷ Radius of the balls on this orbit
      compute $d_j := \dim \mathbb{P}_d(\mathcal{V}_j)$ by (19)     ▷ Number of balls to place on the $j$-th orbit
      choose $c_j$ which are $d_j$ unisolvent points on the orbit $\mathcal{V}_j$
      construct $d_j$ balls of radius $r_j$ centered at $c_j$
      $d = d - 2$                                        ▷ Every time reduces by 2 the total degree
  end while
  if $d = 0$ then                                        ▷ Only the degenerate orbit is left
      construct a ball of any radius centered at 0
  end if
```

Note that, if d is the total degree of the polynomial p, unisolvence on each variety implies the convergence of the loop in $d^* := \lceil \frac{d}{2} + 1 \rceil$ iterations, regardless of the dimension of the ambient space. An iterative application of Lemma 4 for $d_j = 0, 2, \dots, d$ if d is even or $d_j = 1, 3, \dots, d$ if d is odd is in fact the core of the proof of the next result.

Proposition 7 *Let $p \in \mathbb{P}_d(\mathbb{R}^n)$. Let $\mathcal{X} = \{\xi^i\}_{i=1}^{\dim \mathbb{P}_d(\mathbb{R}^n)}$ be a collection of points such that $\mathcal{X}_j := \mathcal{V}_j \cap \mathcal{X}$ is unisolvent for $\mathbb{P}_{d_j}(\mathcal{V}_j)$. For each j, fix $r_j > 0$ and put $B_j^i := B(\xi_j^i, r_j)$. If*

$$\int_{B_j^i} p(x)\,\mathrm{d}x = 0 \qquad i = 1, \dots, N_j := \dim \mathbb{P}_{d_j}(\mathcal{V}_j),$$

for each $j = 1, \dots, d^*$, then $p(x) = 0$.

Proof For $j = 1, \dots, d^*$, order orbits $\mathcal{V}_j$ so that $j < k$ if $\#\left(\mathcal{X}_j\right) > \#\left(\mathcal{X}_k\right)$. Consider $j = 1$. Since

$$\int_{B_1^i} p(x)\,\mathrm{d}x = 0 \qquad \text{for each } \xi_1^i \in \mathcal{X}_1,$$

and $\mathcal{X}_1$ is a unisolvent set for nodal interpolation in $\mathbb{P}_d(\mathcal{V}_1)$, by Lemma 4 there exists a degree 2 polynomial P_1 as in (18) such that $p = P_1 q_1$, whence $\deg q_1 = d-2$. Iterating this up to $j = d^* - 1$, we get

$$p = \left(\prod_{j=1}^{d^*-1} P_j \right) q_{d^*-1}. \tag{20}$$

Since varieties $\mathcal{V}_j$ are the orbits of the same group action, they do not intersect and thus have no common components. Hence, for $j = 1, \dots, d^* - 1$, P_j does not divide $\prod_{k=1}^{\ell} P_k$ for any $\ell < j$ and thus q_{d^*-1} is a polynomial of degree $\bar{d} \leq 1$. With respect to the last orbit $\mathcal{V}_{d^*}$, we have

$$p\big|_{\mathcal{V}_{d^*}} = c q_{d^*-1},$$

with $c = \prod_{j=1}^{d^*-1} (r_j - r_{d^*})^2$ being a constant. We show, separating the cases $\bar{d} = 0, 1$, that

$$\int_{B^i_{d^*}} q_{d^*-1}(x)\,\mathrm{d}x = 0 \qquad \text{for all } \xi^i_{d^*} \in \mathcal{X}_{d^*}$$

implies that $q_{d^*-1} = 0$. This will complete the proof.

Let us first consider the case $\bar{d} = 0$. The polynomial q_{d^*-1} is therefore a constant, say c', with vanishing integral on a full measure set. Hence $c' = 0$ and, by Eq. (20), $p(x) = 0$.

If we assume $\bar{d} = 1$ the claim is a consequence of Lemma 4: the algebraic variety $\mathcal{V}_{d^*}$ described by P_{d^*} has degree 2 and, since it divides q_{d^*-1}, it must be $q_{d^*-1} = 0$. Again by Eq. (20), $p(x) = 0$. □

Remark 11 Describing varieties as orbits clarifies that the degenerate orbit in (18) for $r = 0$ preserves unisolvence. We stress that, as a consequence of Proposition 7, this point can in fact be any point not lying on any of the $\mathcal{V}_j$.

Remark 12 The condition on the unisolvence on the algebraic varieties may significantly simplify the problem. For instance, if $n = 2$, it is sufficient to place distinct nodes on each of the circumferences $\mathcal{V}_j$, with $j = 1, \dots, d^*$. The number of nodes to be placed on each $\mathcal{V}_j$ is further prescribed by (19).

4 Numerical experiments

To carry out our numerical experiments, we consider $\Omega \subset \mathbb{R}^2$ as the unit disc. Although this is not essential under the theoretical point of view of Section 3, such a setting matches a well-known topic in approximation theory [46, 63], and also allows to recover some estimates, such as those in [47, 48]. Further, since we showed in Proposition 1 that Π is a projection operator onto polynomials of degree d, the norm $\|\Pi\|_\infty$ is bounded from below by

$$\|\Pi\|_\infty \geq c_n \cdot \begin{cases} \log d, & \text{if } n = 1, \\ d^{\frac{n-1}{2}}, & \text{if } n > 1, \end{cases} \tag{21}$$

as established in [60].

4.1 Lebesgue constants

The numerical scheme for the estimation of the Lebesgue constants has been discussed in Section 2.4.1. From now on, to ease the description of the supports, we consider $K_i = B(\xi_i, r_i)$, so that parameters ξ_i and r_i assume a neat meaning. This also matches some literature, such as [61].

4.1.1 Numerical verification of stability under perturbation

To begin our analysis, we provide a numerical account of Proposition 4, verifying that the quantity Λ_d is little affected by the variation of the size of the supports. To see this, we consider discs centered at the (unisolvent) nodes of the "Chebyshev grid" proposed by Bojanov and Xu in [11]; unisolvence for the histopolation problem associated with such supports can be deduced from both Proposition 6 and Theorem 2. In particular, any collection of discs of fixed radius $r > 0$ centered at $\mathcal{X}$ is unisolvent for the corresponding histopolation problem. Figure 2 shows that the Lebesgue constant (9), represented by the error bars (for different radii), is very close to the nodal Lebesgue constant of the set $\mathcal{X}$.

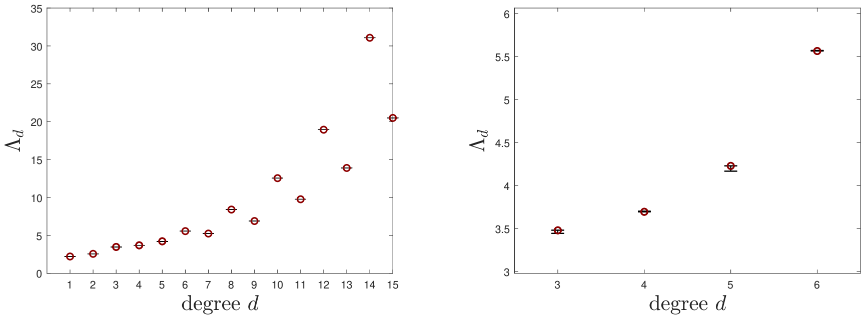

Fig. 2 Lebesgue constants of discs centered at Chebyshev points. The nodal Lebesgue constant is represented by the red circles, while the error bars (which are indeed very small) denote the Lebesgue constant of the discs centered at such points, for different radii. The right panel is a zoom of the left panel for $d = 3, \dots, 6$. Note that the error bars are almost contained in the circles.

Numerically, a further instance of stability emerges. In particular, the location of the centers of the discs appears to impact on the Lebesgue constant more than their radius. To have an account of this, we set up the following experiment. Consider a unisolvent set of nodes $\mathcal{X} = \{\xi_1, \dots, \xi_N\}$ and the discs $K_1, \dots, K_N$ (all sharing the same radius r) centered at $\mathcal{X}$. In Figure 3 we compute Λ_7 as a function of r, expressed as the ratio $r/r_{\max}$, being $r_{\max}$ the largest radius for which the discs are disjoint. Notice that, in this case, it is not guaranteed that the hypotheses of Proposition 4 are fulfilled. Nevertheless, we only observe a small change in the Lebesgue constant as the radius varies.

This justifies, in the next numerical tests, the choice of a fixed radius r for all discs, easing the description of the supports $B(\xi_i, r)$ by only the parameter ξ_i

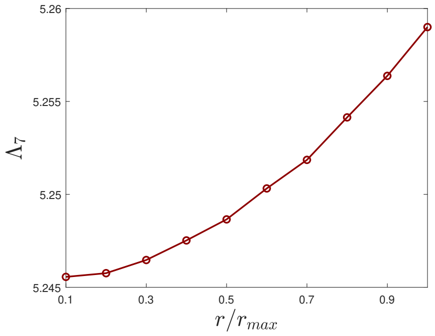

Fig. 3 Lebesgue constants Λ_7 of discs centered at Chebyshev points. Even if the size of the discs varies, the Lebesgue constant remains rather stable (note the scale on the y-axis).

representing the center. Also, this radius will be taken in such a way that the hypotheses of Theorem 1 are fulfilled, so that $\|\Pi\|_\infty = \Lambda_d$.

4.1.2 Approximate Fekete supports and discrete Leja supports

All the preceding techniques hinge on a collection of nodes for the construction of a unisolvent collection of compact sets. There is, however, another possibility, which also yields generally stable supports for interpolation. This procedure mimics Fekete and Leja constructions.

In analogy with the nodal case, we call *Fekete supports* for $\mathbb{P}_d(\Omega)$ the collection of compact sets $\{K_1, \ldots, K_N\}$ that maximize the modulus of the determinant of the (normalized) Vandermonde matrix (11). While the exact problem is independent of the basis chosen for the polynomial space [13], in the approximate case the selection of the supports depends on the basis considered for $\mathbb{P}_d(\Omega)$, see [17, p. 17]. Consistently with the related literature, we consider the basis detected in Section 2.4.2 and, further, we normalize basis functions, as described in Remark 4. This is crucial when considering varying size supports [21].

Supports $\{K_1, \ldots, K_N\}$ that maximize the modulus of the determinant of $\boldsymbol{V}$, indeed provide a controlled growth of the corresponding Lebesgue constant. In fact, since $\|\widetilde{\ell}_{K_i}\|_\infty \leq 1$ for $i = 1, \ldots, N = \dim \mathbb{P}_d(\Omega)$, then

$$\Lambda_d \leq \dim \mathbb{P}_d(\Omega). \tag{22}$$

The identification of Fekete supports is a hard problem [21], and algorithms for the approximate Fekete nodes have been designed by exploiting a greedy procedure on different bi and tri-dimensional domains [15, 18, 59].

Remark 13 From the theoretical point of view [16, Sect. 3], the supports identified by such algorithms may violate the estimate (22), as the search does not involve all possible $K \subset \Omega$. On the numerical side, this effect has been observed in the simplicial framework [19, Fig. 6].

In the nodal case, similar to approximate Fekete points, are the *discrete Leja sequences* [29]. Therefore, we call *Leja (sequences of) supports* the arrays of supports $\{K_1, \ldots, K_N\}$, $N = \dim \mathbb{P}_d(\Omega)$, obtained as follows. Once a basis is fixed,

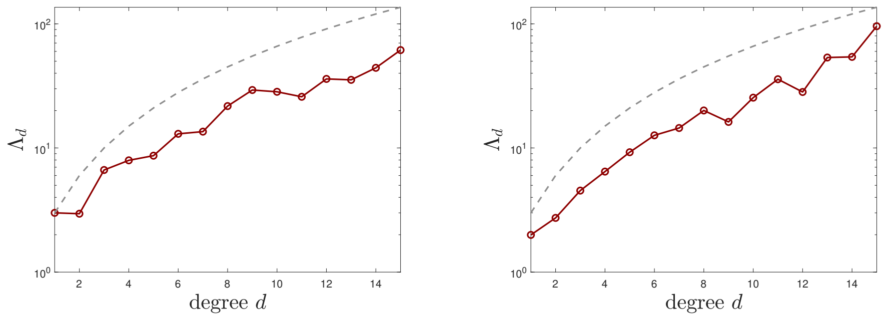

Fig. 4 Lebesgue constants for $d = 1, \dots, 15$: approximate Fekete supports (left, solid red) and discrete Leja supports (right, solid red), compared with the theoretical bound (gray, dashed) given in (22). Supports are extracted from a mesh containing about 8000 discs centered at a uniform grid.

$\{b_1, \dots, b_N\}$ for $\mathbb{P}_d(\Omega)$, the first support K_1 of the sequence is

$$K_1 = \arg\max_{K \subset \Omega} \frac{1}{\mu(K)} \left| \int_K b_1(x)\,\mathrm{d}x \right| .$$

The subsequent supports are chosen iteratively so that, at the ℓ-th step of the procedure, the modulus of the determinant of the partial Vandermonde matrix

$$\left| \det \left[\frac{1}{\mu(K_i)} \int_{K_i} b_j(x)\,\mathrm{d}x \right]_{i,j=1}^{\ell} \right|$$

is maximized (the inductive definition ensures that $\{K_1, \dots, K_{\ell-1}\}$ have been fixed). For greedy algorithms for the construction of nodal (discrete) Leja sequences see [16, 17].

Remark 14 Leja sequences depend on (the ordering of) the basis chosen for the space of polynomials [16, Section 3]. However, once two bases $\{b_i\}_{i=1}^{\dim \mathbb{P}_d(\Omega)}$ and $\{g_i\}_{i=1}^{\dim \mathbb{P}_d(\Omega)}$ satisfy

$$\operatorname{span}\{b_1, \dots, b_k\} = \operatorname{span}\{g_1, \dots, g_k\} \qquad k = 1, \dots, \dim \mathbb{P}_d(\Omega),$$

the corresponding (discrete) Leja sequences coincide [17, p. 17]. In fact, since such a condition holds for $\mathcal{M}_d$ and $\mathcal{T}_d$, the selected Discrete Leja sequences are the same for both bases.

Those algorithms applied to supports provide both the *approximate Fekete supports (AFS)* and the discrete Leja supports (DLS). The greedy maximization of submatrix volumes by the QR factorization with column pivoting of the transposed Vandermonde matrix (11) gives rise to the AFS, while the DLS corresponds to the greedy maximization of nested square submatrix determinants (implemented by a standard LU factorization with row pivoting). The Lebesgue constants of the corresponding sets are shown in Figure 4 for the AFS (left) and for the DLS (right). The gray dashed line represents the bound (22).

4.2 Interpolation tests

In Proposition 6 and Theorem 2 we have provided two methodologies for constructing unisolvent sets for the interpolation problem (1). The first hinges on point-based supports, whereas the second exploits a decomposition into algebraic varieties (or possibly orbits, as in Proposition 7). Further, in Section 4.1.2, we extended algorithms for the search of unisolvent and well-performing points to the case of diffused supports. We now compare these approaches in terms of the resulting interpolation operator.

Concerning point-based unisolvent sets, we shall consider either discs centered at quasi-random Halton points [39] or at the “Chebyshev” points of [11] (for an optimized version, see [46]), whose Lebesgue constant is depicted in Figure 2 for the degree 7. Notice that this latter collection also fits the class of orbit-based supports.

We consider the function

$$f_1(x, y) = e^x \sin(x + y).$$

In Figure 5 we depict the interpolation error $\|f_1 - \Pi f_1\|_\infty$, Π being defined as in (4), with respect to the sup-norm, as a function of the total polynomial degree d of the interpolant Π. All the strategies at play behave similarly, showing comparable convergence towards the exact solution.

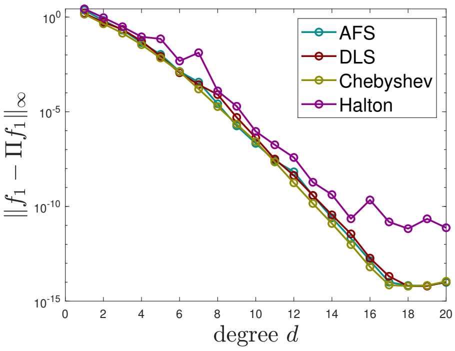

Fig. 5 Error $\|f_1 - \Pi f_1\|_\infty$ for $d = 1, \dots, 20$, for different choices of supports. All the radii are constant and chosen to avoid overlaps.

The situation becomes more interesting when considering a Runge-like function [54]

$$f_2(x, y) = \frac{1}{25(x^2 + y^2) + 1}.$$

In this case, the choice of supports seriously affects the reliability of the interpolation. In accordance with the expectations, if supports are not carefully selected, the interpolant captures the function only away from the boundary, where instability becomes evident. This is depicted in Figure 6, where the target Runge function is pictured in light red and the approximating polynomials, for different choices of supports, are depicted in light blue.

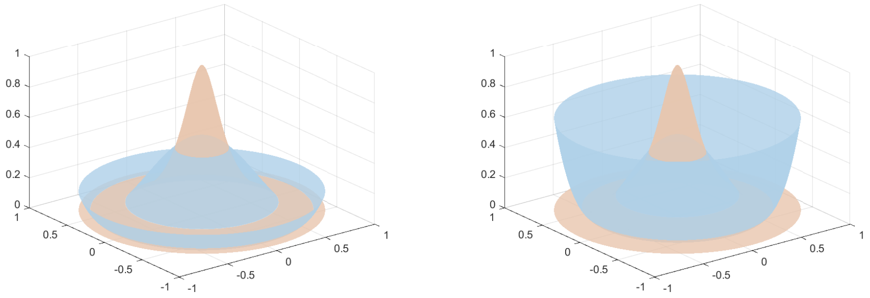

Fig. 6 The exact function f_2 (light red) and two interpolants (light blue). One based on discs with centers driven by Halton points (right), the other with radius based on a Chebyshev distribution (left), for $d = 5$.

Figure 7 reports the errors $\|f_2 - \Pi f_2\|_\infty$ again with respect to the total degree d. This indicates that the convergence towards the exact function is strongly dependent on the compact sets selected as supports of integration. In particular, it shows that the quantity Λ_d defined in Eq. (9) fruitfully predicts the quality of the interpolation supports.

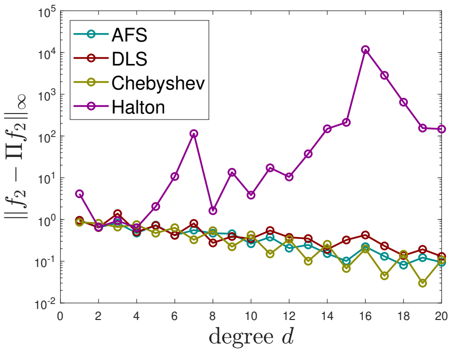

Fig. 7 Error $\|f_2 - \Pi f_2\|_\infty$ for $d = 1, \ldots, 20$, for different choices of supports. All the radii are constant and chosen to avoid overlaps.

5 Conclusions

In this paper, we investigated the existence and features of a Lagrange-like polynomial interpolation scheme based on diffuse data. Such data is measured via integral quantities on compact supports. We showed that the properties of the corresponding projection operator Π closely resemble those of the nodal Lagrange interpolation operator. We introduced the concept of the Lebesgue constant Λ_d, which relates the supports of integration to $\|\Pi\|_\infty$, where the uniform norm is considered. After determining a cost-effective method to compute Λ_d, limiting the use of numerical quadrature only to the Vandermonde matrix, we employed three different strategies to construct unisolvent sets and assessed their efficacy. This also comes with a stability result that allows us to relate supports with some nodes, for which a large literature is available. As an application of this, we matched the estimated Lebesgue constants Λ_d with some interpolation errors coming from Runge-like problems. Numerical results show that the positioning of the compact supports, rather than their size, controls the features of the interpolation operator Π, and that all the strategies proposed for the construction of unisolvent sets of compact supports may be used to extract productive results. In the future, we aim to consolidate the understanding of the trend of the Lebesgue constants by determining explicit bounds for their growth.

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Data availability All the numerical data shown in the present work have been produced by the authors. Open-source codes are available at the repository `https://github.com/gelefant/InterpDISC` (for the interpolation test) and `https://github.com/gelefant/DiscAFS_DLS` (for the extraction of Fekete and Leja supports).

Declaration

Conflict of interest The authors declare that they have no conflict of interest.

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