Ordered K-theory

Peter Vámos, Exeter

Padova March 2015

"The task of Algebra is to find Invariants"
– Algebraist

"The nicest and most natural
invariants are additive and non-negative"
– So say all of us

Examples of Numerical Invariants

Cardinality	Length	Dimension Rank
Probability	Area	Euler Char. Multiplicity
Measure	Volume	State Weight

Throughout, by a category $\mathcal{C}$ we'll mean a full small subcategory of an Abelian category, closed under isomorphisms and containing the zero.
Actually, the ambient Abelian category will, with a few exceptions, be a category of (right) modules over a ring.

Definitions

Let F be an Abelian group. A function $f\colon \mathcal{C} \to F$ is additive if it is additive over short-exact sequences in $\mathcal{C}$. The pair (F, f) is universal if for every other such pair (G, g), there is a unique homomorphism $h\colon F \to G$ such that g factors through f i.e. $g = hf$.
Let F be a po Abelian group. A function $f\colon \mathcal{C} \to F$ is additive and non-negative if it is additive over short-exact sequences in $\mathcal{C}$ and $f(A) \geq 0$ for all $A \in \mathcal{C}$. The pair (F, f) is universal if for every other such pair (G, g), there is a unique o-homomorphism $h\colon F \to G$ such that g factors through f i.e. $g = hf$.

Examples

1. *Let $\mathcal{C}$ be the category of finitely generated vector spaces over a field/division ring.*

2. *Let $\mathcal{C}$ be the category of finite Abelian groups.*

Examples

1 *Let $\mathcal{C}$ be the category of finitely generated vector spaces over a field/division ring.*

Then $(\mathbb{Z}, \mathrm{dim})$ is universal. Here $\mathbb{Z}$ is ordered naturally with $\leq$.

2 *Let $\mathcal{C}$ be the category of finite Abelian groups.*

Examples

1 *Let* $\mathcal{C}$ *be the category of finitely generated vector spaces over a field/division ring.*

Then $(\mathbb{Z}, \dim)$ *is universal. Here* $\mathbb{Z}$ *is ordered naturally with* $\leq$*.*

2 *Let* $\mathcal{C}$ *be the category of finite Abelian groups.*

Let $\mathbb{Q}_>$ *denote the multiplicative group of positive rational numbers equipped with the divisibility order. Then* $(\mathbb{Q}_>, |\cdot|)$ *is universal.*

The Universal Invariant

Theorem 1

Let $\mathcal{C}$ be a category. Then there is a universal additive invariant $(\mathbf{K_0}(\mathcal{C}), [\cdot])$ such that every additive $g\colon \mathcal{C} \to G$ uniquely factors through $[\cdot]\colon \mathcal{C} \to \mathbf{K_0}(\mathcal{C})$ i.e.

$$\exists!\ h\colon \mathbf{K_0}(\mathcal{C}) \to G \text{ group homomorphism such that } g = h[\cdot].$$

There is also a universal additive and non-negative invariant $(\mathbf{KO}(\mathcal{C}), \langle\cdot\rangle)$ such that every additive and non-negative $g\colon \mathcal{C} \to G$ uniquely factors through $\langle\cdot\rangle\colon \mathcal{C} \to \mathbf{K_0}(\mathcal{C})$ i.e.

$$\exists!\ h\colon \mathbf{KO}(\mathcal{C}) \to G \text{ group o-homomorphism such that } g = h\langle\cdot\rangle.$$

KO theory is OK!

Proposition 2

Suppose that the category $\mathcal{C}$ satisfies

(1) *every exact sequence in $\mathcal{C}$ splits;*

(2) *there is a set of pair-wise non-isomorphic objects $\{S_i\}, i \in I$ in $\mathcal{C}$ such that every $A \in \mathcal{C}$ has a unique representation as a finite direct sum of the S_i:*

$$A \approx \bigoplus_{i \in I} S_i^{n_i},\ n_i \in \mathbb{N}, n_i = 0 \text{ for almost all } i \in I.$$

Then
$(\mathbf{K_0}(\mathcal{C}), [\cdot]) = (\mathbf{KO}(\mathcal{C}), \langle\cdot\rangle) =$ *free po-group on I,*
$[A] = \langle A \rangle = (n_i)$.

Categories satisfying the two conditions above include the category of f.g. semi-simple modules over any ring; the category of f.g. free modules over an IBN ring and the category of finite direct sums of indecomposable injectives over any ring.

Remark

Let F be the free po-group of rank $|I|$ as above in Proposition 2. Then there is an o-homomorphism

$$\nabla F \to \mathbb{Z},\ (n_i) \to \sum_{i \in I} n_i$$

This then yields an integer valued additive and non-negative function on $\mathcal{C}$.

Example 3 (Goldie's Uniform Dimension)

Let $\mathcal{C}$ be the category of finite direct sums of indecomposable injectives over a ring, as in Proposition 2 *above. Then for any module A, let $E(A)$ denote its injective envelope and define*

$$\dim A = \begin{cases} \nabla\langle E(A)\rangle & \textit{if } E(A) \in \mathcal{C}, \\ \infty & \textit{otherwise.} \end{cases}$$

Definitions

Let A be a module over some ring. A chain σ of submodules of A is of the form

$$\sigma\colon 0 = A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n = A. \tag{1}$$

Here n is the length of the chain σ and $A_i/A_{i-1},\ 1 \le i \le n$ are the chain factors of σ. Let τ be another chain for A:

$$\tau\colon 0 = B_0 \subseteq B_1 \subseteq B_2 \subseteq \cdots \subseteq B_m = A.$$

If τ was obtained from σ by inserting extra submodules in σ then we say that τ is a refinement of σ and write $\sigma \le \tau$. The chains σ and τ are equivalent if $n = m$ and there is a permutation π on $\{1, \ldots, n\}$ such that $A_i/A_{i-1} \approx B_{\pi(i)}/B_{\pi(i-1)},\ 1 \le i \le n$.

Theorem 3 (Jordan, Hölder, Schreier)

For a module A any two chains have equivalent refinements.

Definitions

A category $\mathcal{C}$ is said to be semi-closed if it is closed under submodules and factor modules, it is closed (or a Serre category) if, in addition, it is closed under extensions. The category chain $\mathcal{C}$ consists of modules having a chain all whose chain factors are in $\mathcal{C}$.

Observation (Categories)

If $\mathcal{C}$ is semi-closed and and σ is a chain of $A \in \mathcal{C}$ then all the chain factors of σ are in $\mathcal{C}$ and so are those of any refinement of σ. Hence chain $\mathcal{C}$ is closed and indeed it is the smallest closed category containing $\mathcal{C}$, the closed category generated by $\mathcal{C}$.

If A is a module then the segments of A (the sub factors of A) form a semi-closed category so the chain of this category is the closed category generated by A, we'll denote this category by $\mathcal{M}(A)$ and its universal po-group by $\mathbf{KO}(A)$.

Extension by Devissage

Theorem 4 (Devissage)

Let $\mathcal{C}$ be a semi-closed category. Then the homomorphism (resp. o-homomorphism) induced from the inclusion $\mathcal{C} \subseteq \operatorname{chain} \mathcal{C}$ are isomorphisms. In other words every additive (resp. additive and non-negative) function on $\mathcal{C}$ can be uniquely extended to an additive (resp. additive and non-negative) function on $\operatorname{chain} \mathcal{C}$.

Example 4

Let $\mathcal{C}$ be the category of f.g. semi-simple modules over an arbitrary ring R. Then $\operatorname{chain} \mathcal{C}$ consists of modules with finite composition length (equivalent to being both Noetherian and Artinian) and by Proposition 2, $\mathbf{KO}(\operatorname{chain} \mathcal{C})$ *is a free po-group. Then for a module $A \in \operatorname{Mod-} R$ the classical composition length is defined by*

$$\ell(A) = \begin{cases} \nabla \langle A \rangle & \textit{if } A \in \operatorname{chain} \mathcal{C}, \\ \infty & \textit{otherwise.} \end{cases}$$

Observation (Functoriality)

Let $F\colon \mathcal{C} \to \mathcal{D}$ be an exact functor i.e. a functor preserving exact sequences. Then, by the universal property of the groups $\mathbf{K_0}$ and $\mathbf{KO}$, we obtain the unique homomorphism $\mathbf{K_0}(F)\colon \mathbf{K_0}(\mathcal{C}) \to \mathbf{K_0}(\mathcal{D})$ and o-homomorphism $\mathbf{KO}(F)\colon \mathbf{KO}(\mathcal{C}) \to \mathbf{KO}(\mathcal{D})$ respectively.

Now suppose that $\phi\colon R \to S$ is a homomorphism of rings and ${}_RS$, regarded as a left R module is flat.

Let $\mathcal{M}(R)$ and $\mathcal{M}(S)$ be the closed categories generated by R and S respectively and let F be the functor $- \otimes {}_RS$. Since $F(R) = S$, F is an exact functor $\mathcal{M}(R) \to \mathcal{M}(S)$.

Example 5 (Torsion-free and Goldie ranks)

Let S be a (right) Goldie ring e.g. a semiprime right Noetherian ring. Then by Goldie's Theorem S has a classical quotient ring T which is semi-simple Artinian and flat as a left R-module. Let F, as above, be the exact functor $- \otimes_S T$. Since $\mathcal{M}(T)$ is the category of f.g. semi-simple T-modules the function

$$\mathrm{rk}\colon \mathcal{M}(S) \to \mathbb{Z}, \mathrm{rk}(A) = \ell(\mathbf{KO}(F)(A))$$

is a well-defined additive and non-negative function. In the special case when S is on Ore domain (all commutative domains are Ore domains), this is the torsion-free rank of A.
Now let R be a right Noetherian ring and N it's nil radical.Then $S = R/N$ is a Goldie ring and $N^k = 0$ for some $k > 0$. For an R-module A the chain factors of the chain

$$A \supseteq AN \supseteq AN^2 \supseteq \cdots \supseteq AN^k = 0$$

can be viewed as S modules, so by devissage rk extends to $\mathcal{M}(S)$. This is Goldie's rank function on a (right) Noetherian ring.

Extension by Resolution

Let $\mathcal{C}$ be a category closed under finite direct sums and assume that whenever $A \to A''$ is an epimorphism and $A, A'' \in \mathcal{C}$ then so is $\mathrm{Ker}(A \to A'')$. An exact sequence of the form

$$0 \to A_n \to \cdots \to A_1 \to A_0 \to 0, \ A_i \in \mathcal{C}, \ 1 \le i \le n \qquad (2)$$

is called a $\mathcal{C}$-resolution of (of length n) of A_0. If $A_0 \in \mathcal{C}$ as well, then $[A_0] - [A_1] + \cdots + (-1)^n [A_n] = 0$ in $\mathbf{K_0}(\mathcal{C})$. Let res $\mathcal{C}$ denote the category of those (modules) which admit a $\mathcal{C}$-resolution.

Theorem 5 (Grothendieck's Resolution Theorem)

Let $\mathcal{C}$ be as above. Then the natural homomorphism $\mathbf{K_0}(\mathcal{C}) \to \mathbf{K_0}(\mathrm{res}\ \mathcal{C})$ induced by the inclusion $\mathcal{C} \subseteq \mathrm{res}\ \mathcal{C}$ is an isomorphism with an inverse given by

$$[A_0]_{\mathrm{res}} \to [A_1]_{\mathcal{C}} - [A_2]_{\mathcal{C}} + \cdots + (-1)^{n-1}[A_n]_{\mathcal{C}}.$$

where $A_0 \in \mathrm{res}\ \mathcal{C}$ with a $\mathcal{C}$-resolution as in (2) above. Hence any additive map from $\mathcal{C}$ can be uniquely extended to $\mathrm{res}\,\mathcal{C}$.

Euler Characteristic

Let R be a ring with IBN and $\mathcal{C}$ the category of f.g. free R-modules so $\mathbf{K_0}(\mathcal{C}) = \mathbf{KO}(\mathcal{C}) = (\mathbb{Z}, \leq)$. By the Resolution Theorem above $\mathbf{K_0}(\mathrm{res}\,\mathcal{C}) = \mathbb{Z}$. This is called the Euler characteristic $\chi(A)$ of a module A of finite free resolution. Further, if R is commutative, then $\chi(A) \geq 0$, so $\mathbf{KO}(\mathrm{res}\,\mathcal{C}) = \mathbb{Z}$ as well.

Lemma 6

Let R be a commutative ring.

(a) *Let A be an R-module. If P is maximal among the annihilators of non-zero elements of A then P is a prime ideal.*

(b) *Let P be a prime ideal in R and let $\mathcal{C}$ be a closed category of R-modules containing R/P. Then $\langle R/I \rangle = 0$ for an ideal $I \supset P$ in $\mathbf{KO}(\mathcal{C})$.*

Modules with Chain Conditions or Krull Dimension

Lemma 7

Let A be a Noetherian module over a commutative ring R. Then A has a chain of submodules so that all the chain factors are of the form $R/P, P$ a prime ideal of R.

Let R be a commutative ring and let $\mathcal{C}$ be a category of R-modules. Recall that $\operatorname{Spec} R$ denotes the set of prime ideals of the ring R. We write $\operatorname{Spec} \mathcal{C} = \{P \in \operatorname{Spec} R \mid R/P \in \mathcal{C}\}$.

Theorem 8

Let R be a commutative ring and let $\mathcal{C}$ be a closed category of Noetherian R-modules. Then $\mathbf{KO}(\mathcal{C})$ is a free po-group with basis $\langle R/P \rangle, P \in \min \operatorname{Spec} \mathcal{C}$.

The proof uses the following additive, non-negative functions as a 'dual basis'. For $P \in \operatorname{Spec} R$ let the additive function ℓ_P be given by $\ell_P(A) = \ell_{R_P}(A \otimes R_P)$. Then for $P, Q \in \operatorname{Spec} R$:

$$\ell_P(R/Q) = \begin{cases} 0 & \text{if } Q \nsubseteq P, \\ 1 & \text{if } Q = P, \\ \infty & \text{otherwise.} \end{cases}$$

Example 6

Let R be a discrete valuation domain with maximal ideal P, K its quotient field, L a separable extension field of K of finite dimension $[L : K] = n$ and let S be the integral closure of R in L. Then the valuation on R has t extensions to S i.e. there are exactly t maximal ideals of S lying over P; let their respective ramification indices and residue degrees be $e_i, f_i,\ 1 \leq i \leq t$ respectively. Then

$$e_1 f_1 + \cdots + e_t f_t = n.$$

Non-Discrete Valuation Domain

Let R be a rank-one non-discrete valuation domain with valuation $v\colon R \to \mathbb{R}_{\geq} \cup \infty$. For a non-zero ideal $I \subseteq R$ set $v(I) = \inf\{v(r) \mid r \in I\}$.

Lemma 9

Let the situation be as above and $0 \neq I \subseteq J,\ 0 \neq I' \subseteq J'$ be non-zero ideals of R. If $J/I \approx J'/I'$ then $v(I) - v(J) = v(I') - v(J')$.

Example 7 (Northcott-Reufel)

Let the situation be as above, let $\mathcal{T}$ be the torsion modules in $\mathcal{M}(R)$ and let $\mathcal{S}$ be the semi-closed category of torsion segments of R i.e. modules isomorphic to $J/I, 0 \neq I \subseteq J \subseteq R$. Then $L\colon \mathcal{S} \to \mathbb{R},\ L(J/I) = v(I) - v(J)$ defines an additive and non-negative function on $\mathcal{S}$ which uniquely extends to $\operatorname{chain} \mathcal{S} = \mathcal{T}$.

Length Functions

Let L be a non-negative function on a category $\mathcal{C}$ with values in the real numbers and ∞ so $L\colon \mathcal{C} \to \mathbb{R} \cup \{\infty\}$. It is a length function if it is additive and it is sub-additive if $L(A) \leq L(A') + L(A'')$ whenever $0 \to A' \to A \to A'' \to 0$ is an exact sequence in $\mathcal{C}$. The advantage of allowing ∞ to be a value is that we can always consider length functions on all modules of a ring by extending these functions trivially: defining the values outside a closed category to be ∞. However, there is a better way.

Theorem 10

Let L be a sub-additive function on $\mathrm{Mod}\text{-}R$. Then there is a unique additive function $\hat{\mathrm{L}}$ on $\mathrm{Mod}\text{-}R$, called the continuous extension of L, which is minimal among the length functions dominating L.

Definitions

Let $L,\ L_i, i \in I$ be length functions on a category $\mathcal{C}$ and $0 \neq c \in \mathbb{R}$ Then for all $A \in \mathcal{C}$ we define the functions $(cL)(A) = cL(A)$ and $(\sum_i L_i)(A) = \sup\{\sum_{i \in F} L_i(A)\}$ where F runs through the finite subsets of I. It is immediate that cL and $\sum_i L_i$ are again length functions on $\mathcal{C}$. A length function whose only values are 0 or ∞ is called trivial.

If $L \geq L'$ then we can define their difference by setting

$$(L - L')(A) = \begin{cases} L(A) - L'(A) & \text{if } \ L(A) < \infty, \\ \infty & \text{otherwise.} \end{cases}$$

Then $L - L'$ is again a length function.

The length function L is irreducible if it isn't trivial and if $L = L_1 + L_2$ for any two length functions L_1, L_2 implies that either $L_1 = cL$ or $L_2 = cL$ for some $0 \neq c \in \mathbb{R}$.

It is routine to check that the continuous extension (the $\widehat{\ }$ operation) commutes with linear combinations and sums and preserves irreducibility.

Let $\mathcal{C}$ be a semi-closed category and let L be a length function whose domain of definition include $\mathcal{C}$. Let the function $L_\mathcal{C}$ be defined on Mod-R by restricting it to $\mathcal{C}$ and then taking the 0-extension:

$$L_\mathcal{C}(A) = \begin{cases} L(A) & \text{if } \ A \in \mathcal{C}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $L_\mathcal{C}$ is subadditve on Mod-R.

Corollary 11

Let the situation be as described above and let $\hat{L}_\mathcal{C}$ (or just $\hat{L}$ if there is no ambiguity about $\mathcal{C}$) denote the continuous extension of $L_\mathcal{C}$. Then $\hat{L} \leq L$, $\hat{L}$ and L agree on $\mathcal{C}$ and L can be decomposed $L = \hat{L} + (L - \hat{L})$.

Dimension

We now define a dimension in an Abelian category, as an ordinal number similar to the Gabriel-Rentschler dimension (hereinafter just Kdim). On Noetherian modules it will agree with latter but it also provides a useful ordinal for Artinian modules since the definition is self-dual. Nevertheless, the category of objects (modules) having either dimension will be the same.

Let $\mathcal{C}$ be a closed (Serre) category. For a subcategory $\mathcal{B}$ of $\mathcal{C}$ let $\mathcal{B}'$ be the closed subcategory generated by $\mathcal{B}$ and the objects which become simple in $\mathcal{C}/\mathcal{B}$.

The dimension-series, $(\mathcal{C}_\alpha, \mathbf{KO}_\alpha(\mathcal{C}))$ of $\mathcal{C}$, is defined transfinitely:

$$\mathcal{C}_{\text{-}1} = \{0\} \qquad \langle\ \rangle_{\text{-}1} = 0$$

$$\mathcal{C}_\beta = (\mathcal{C}_\alpha)' \qquad \langle\ \rangle_\beta = \langle\ \rangle_{\mathcal{C}_\beta} \colon \mathcal{C}_\beta \to \mathbf{KO}(\mathcal{C}_\beta) \quad \beta = \alpha + 1$$

$$\mathcal{C}_\beta = \Big(\bigcup_{\alpha<\beta} \mathcal{C}_\alpha\Big)' \qquad \langle\ \rangle_\beta = \langle\ \rangle_{\mathcal{C}_\beta} \colon \mathcal{C}_\beta \to \mathbf{KO}(\mathcal{C}_\beta) \quad \beta \text{ limit ordinal.}$$

The dimension of $\mathcal{C}$ and that of an object $A \in \mathcal{C}$ is given by

$$\dim \mathcal{C} = \inf\{\alpha \mid \mathcal{C}_\alpha = \mathcal{C}\} \text{ if there is such an ordinal and } \infty \text{ otherwise}$$

$$\dim A = \inf\{\alpha \mid A \in \mathcal{C}_\alpha\} \text{ if there is such an ordinal and } \infty \text{ otherwise.}$$

Proposition 12

Let A be a Noetherian or Artinian object in a closed category $\mathcal{C}$. Then $\dim A < \infty$. It follows that $\dim A < \infty$ if and only if $\mathrm{Kdim}\, A < \infty$ but in general, these two ordinals are not the same.

Let L be a length function on a closed category $\mathcal{C}$. Then

$$\operatorname{Ker} L = \{A \in \mathcal{C} \mid L(A) = 0\} \quad \text{and} \quad \operatorname{Fin} L = \{A \in \mathcal{C} \mid L(A) < \infty\}$$

are again closed categories. Also, L is called locally discrete if for all $A \in \mathcal{C}$ there are only finitely many values for segments of A i.e. the set $\{L(B) \mid B \text{ is a segment of } A\}$ is finite. For example, if L is integer valued then it is locally discrete.

Proposition 13

Let L be a length function on a closed category. Then L is locally discrete if and only if $\dim(\operatorname{Fin} L / \operatorname{Ker} L) \leq 0$.

The Main Decomposition Theorem

Let $\mathcal{C}$ be a closed category and L be a length function on $\mathcal{C}$. Suppose that $\dim(\operatorname{Fin} L/\operatorname{Ker} L) = \gamma < \infty$. We want to show that L decomposes, uniquely, as a sum of irreducible length functions. By passing to the quotient category $\mathcal{C}/\operatorname{Ker} L$, we may assume that $\operatorname{Ker} L = \{0\}$. Let $\mathcal{C}_\alpha,\ \alpha \leq \gamma$ be the dimension series of $\mathcal{C}$ as defined above, $\mathcal{C}_\gamma = \mathcal{C}$.

We proceed by transfinite induction. For $\alpha = 0, \mathcal{C}_0$ consists of objects of finite composition length. Let Ω_0 be the set of representatives of the isomorphism classes of simple $\mathcal{C}_0$ objects, one from each isomorphism class. Then the restriction of L to $\mathcal{C}_0$ is the unique linear combination of the irreducible length function associated to the elements of Ω_0, let L^0 be the continuous extension of this to $\operatorname{Fin} L$ (or indeed to $\mathcal{C}$). Then by Corollary 11 $L = L^0 + (L - L^0)$, note that $L - L^0$ vanishes on $\mathcal{C}_0$. Now repeat this for $L - L^0$ on $\operatorname{Fin} L/\mathcal{C}_0$ and continue this way by transfinite induction to obtain

Theorem 14 (Main Decomposition Theorem)

Let the situation be as described above. Then

$$L = \sum_{\alpha \leq \gamma} L^{\alpha},\ L^{\alpha} = \sum_{i \in \Omega_{\alpha}} c_i^{\alpha} L_i^{\alpha},\ \alpha \leq \gamma,\ L^{\alpha} \text{ vanishes on } \mathcal{C}_{\beta}, \beta < \alpha$$

and each L_i^{α} is the irreducible length function corresponding to an object in $A_i^{\alpha} \in \Omega_{\alpha}, L_i^{\alpha}(A_i^{\alpha}) = 1, c_i^{\alpha} = L^{\alpha}(A_i^{\alpha})$. Moreover, this representation of L as a sum of irreducible length functions is unique and L is irreducible if and only if it is a positive constant multiple of one of the L_i^{α}.

Corollary 15

Suppose that $\dim(A) < \infty, A \in \mathcal{C}$ (in particular if A is Noetherian or Artinian). Then there is a length function L on $\mathcal{C}$ such that $0 < L(A) < \infty$.

Theorem 16

With the same notation as in the Main Decomposition Theorem above, assume additionally, that $\mathrm{Fin}\, L$ *(or* $\mathcal{C}$*) consists of Noetherian objects. Then each* A_i^α *can be chosen so that every proper factor of it belongs to some* $\mathcal{C}_\beta, \beta < \alpha$ *so its injective envelope,* $E(A_i^\alpha)$ *is indecomposable and* $L_i^\alpha(-) = \ell_S(\mathrm{Hom}_{\mathcal{C}}(-, E(A_i^\alpha)))$ *where* S *is the endomorphism ring of* $E(A_i^\alpha)$*.*

If our category is Noetherian modules over a commutative ring then we can recover Theorem 8.

Proposition 17

Let R *be a commutative ring. Then there is a one-to-one correspondence between those prime ideals* P *of* R *for which* R/P *is Noetherian and indecomposable injective* R*-modules containing a non-zero Noetherian module given by* $P \leftrightarrow E(R/P)$*. Moreover, if* P *is such a prime ideal then the length functions* $\ell_{R_P}(- \otimes R_P)$ *and* $\ell_S(\mathrm{Hom}_R(-, E(R/P))),\ S = \mathrm{End}_R(E(R/P))$ *are equal.*

Example 8

Let V *be a countable dimensional vector space over some field* F *with basis* $\{b_i\}_{i=1}^{\infty}$. *Define linear transformations* $\{\phi_j\}_{j=1}^{\infty}$ *of* V *by*

$$\phi_j(b_i) = \begin{cases} b_i & i < j, \\ b_{i+1} & i = j, \\ 0 & i > j, \end{cases} \qquad 1 \leq i, j \leq \infty.$$

Let R *be the subring of* $\mathrm{End}_F(V)$ *generated by* F *and the* ϕ_j^{s}. *Then* V *is an* R*-module and its only submodules are:* $Rb_1 \supset Rb_2 \supset \cdots \supset Rb_n \supset \cdots$. *Hence* ${}_RV$ *is Noetherian and* $S_i = Rb_i/Rb_{i+1} \not\approx Rb_j/Rb_{j+1} = S_j, i \neq j$ *are non-isomorphic simple segments of* ${}_RV$. *Let* $\mathcal{C} = \mathcal{M}({}_RV)$, ℓ_i *the classical length function associated to* S_i *and* $L = \sum_i 2^{-i}\ell_i$. *Then* $L({}_RV) = 1$, $\mathrm{Ker}\, L = \{0\}, \mathrm{Fin}\, L = \mathcal{C}$ *and* $\dim \mathrm{Fin}\, L/\mathrm{Ker}\, L = 1 > 0$.

Proposition and Definition 18

For a length function L on $\mathrm{Mod}\text{-}R$ the following are equivalent:

(i) $L(A) = \sup\{L(F) \mid F \subseteq A$ *finitely generated* $\}$;

(ii) *if* $A = \cup_{i \in I} A_i$ *for a direct system* $\{A_i\}_{i \in I}$ *of submodules of* A *then* $L(A) = \sup_i L(A_i)$;

(iii) *if* $A = \cup_{i \in I} A_i$ *for submodules* $\{A_i\}_{i \in I}$ *totally ordered by inclusion, then* $L(A) = \sup_i L(A_i)$.

If L satisfies the equivalent conditions above then it is said to be upper continuous. In this case L is completely determined by its values on finitely generated modules.

Theorem 19

If $\dim R < \infty$ then an upper continuous length function on $\mathrm{Mod}\text{-}R$ can be uniquely written as a linear combination of length functions associated to indecomposable injective R-modules as in Theorem 16.

Rank Rings

Recall that for an object/module A, $\mathcal{M}(A)$ is the closed category generated by A consisting of those object which have a chain where every chain factor is a segment of A, see Observation (Categories). This means that for all $x \in \mathbf{KO}(A)$ there is a natural number n such that $-n\langle A\rangle \leq x \leq n\langle A\rangle$. In this situation we say that $\langle A\rangle$ is an order-unit in $\mathbf{KO}(A)$.
We say that the ring R is a rank-ring if there is a non-trivial length function L on $\mathcal{M}(R)$ (equivalently, if $L(R) = 1$). From our previous results we see that R is a rank ring if it has Krull-dimension, in particular if it is (right) Noetherian, or a (Ore) domain or a factor ring of a non-discrete rank one valuation domain.

Proposition 20

The ring R is a rank ring if, and only if, $\mathbf{KO}(R) \neq 0$.

States and von Neumann Regular Rings

Let R be a ring, we now focus on $\mathcal{M}(R)$ and $\mathcal{P}(R)$, the category of f.g. projective R-modules. A length function L (on either categories) is a state if $L(R) = 1$. Let $\mathbf{S}(\mathcal{C})$ be the set of states on $\mathcal{C}$ where $\mathcal{C}$ stands for either of these categories. Then $\mathbf{S}(\mathcal{C})$ is a convex compact subset of a real topological vector space. Moreover, a length function with $\mathcal{C} \subseteq \operatorname{Fin} L$ is irreducible precisely when its normalised state is an extreme point in $\mathbf{S}(\mathcal{C})$.

States and von Neumann Regular Rings

Let R be a ring, we now focus on $\mathcal{M}(R)$ and $\mathcal{P}(R)$, the category of f.g. projective R-modules. A length function L (on either categories) is a state if $L(R) = 1$. Let $\mathbf{S}(\mathcal{C})$ be the set of states on $\mathcal{C}$ where $\mathcal{C}$ stands for either of these categories. Then $\mathbf{S}(\mathcal{C})$ is a convex compact subset of a real topological vector space. Moreover, a length function with $\mathcal{C} \subseteq \operatorname{Fin} L$ is irreducible precisely when its normalised state is an extreme point in $\mathbf{S}(\mathcal{C})$.

Definition

Let R be a ring. A function $\rho\colon R \to [0,1]$ is a pseudo-rank function if it satisfies:

(a) $\rho(1) = 1$;

(b) $\rho(ab) \leq \min(\rho(a), \rho(b))$;

(c) $\rho(e + f) = \rho(e) + \rho(f)$ for orthogonal idempotents $e, f \in R$.

If, in addition, ρ satisfies

(d) $\rho(a) > 0$ for all $0 \neq a \in R$

then ρ is a rank function on R.

Theorem 21 (Goodearl - Handelman, Bergman)

Let R *be a von Neumann regular ring. Then every pseudo-rank function on* R *is induced by a unique state on* $\mathcal{P}(R)$ *(i.e. every pseudo-rank function can be uniquely extended to a state on* $\mathcal{P}(R)$*). In fact the set of pseudo-rank functions on* R *is affinely homeomorhic to the states on* $\mathcal{P}(R)$*.*

Theorem 21 (Goodearl - Handelman, Bergman)

Let R be a von Neumann regular ring. Then every pseudo-rank function on R is induced by a unique state on $\mathcal{P}(R)$ (i.e. every pseudo-rank function can be uniquely extended to a state on $\mathcal{P}(R)$). In fact the set of pseudo-rank functions on R is affinely homeomorhic to the states on $\mathcal{P}(R)$.

Theorem 22

Let R be a von Neumann regular ring. Then every state on $\mathcal{P}(R)$ can be uniquely extended to a state on $\mathcal{M}(R)$. In fact the set of states on $\mathcal{P}(R)$ is affinely homeomorhic to the states on $\mathcal{M}(R)$. Moreover, these states on $\mathcal{M}(R)$ are upper continuous.
Also, in view of Theorem 21 *above, every pseudo-rank function comes from a sate on $\mathcal{M}(R)$.*

Bibliography

Hyman Bass, *Algebraic K-theory*, W. A. Benjamin, Inc., New York-Amsterdam, 1968. MR MR0249491 (40 #2736)

Pierre Gabriel, *Des catégories abéliennes*, Bull. Soc. Math. France **90** (1962), 323–448. MR 0232821 (38 #1144)

K. R. Goodearl, *von Neumann regular rings*, Pitman (Advanced Publishing Program), Boston, Mass., 1979. MR 80e:16011

Irving Kaplansky, *Commutative rings*, revised ed., The University of Chicago Press, Chicago, Ill.-London, 1974. MR 49 #10674

D. G. Northcott and M. Reufel, *A generalization of the concept of length*, Quart. J. Math. Oxford Ser. (2) **16** (1965), 297–321. MR 33 #4101

Peter Vámos, *Additive functions and duality over Noetherian rings*, Quart. J. Math. Oxford Ser. (2) **19** (1968), 43–55. MR 0223434 (36 #6482)

———, *Length functions on modules*, Ph.D. thesis, Department of Pure mathematics, The University of Sheffield, July 1968.