

Geometry of KAM tori for nearly integrable Hamiltonian systems

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Abstract

We obtain a global version of the Hamiltonian KAM theorem for invariant Lagrangean tori by glueing together local KAM conjugacies with help of a partition of unity. In this way we find a global Whitney smooth conjugacy between a nearly integrable system and an integrable one. This leads to preservation of geometry, which allows us to define all nontrivial geometric invariants of an integrable Hamiltonian system (like monodromy) for a nearly integrable one.

Introduction

Classical Kolmogorov-Arnold-Moser (KAM) theory deals with Hamiltonian perturbations of an integrable Hamiltonian system and proves the persistence of quasi-periodic (Diophantine) invariant Lagrangean tori. In [29] Pöschel proved the existence of Whitney smooth action angle variables on a nowhere dense union of tori having positive Lebesgue measure, see also [23, 10]. This version of the KAM theorem can be formulated as a kind of structural stability restricted to a union of quasi-periodic tori, which is referred to as *quasi-periodic stability* [5, 6]. In this context, the conjugacy between the integrable system and its perturbation is smooth in the sense of Whitney.

Our goal is to establish a global quasi-periodic stability result for fibrations of Lagrangean tori, by glueing together local conjugacies obtained from the classical ‘local’ KAM theorem. This glueing uses a partition of unity [20, 31] and the fact that invariant tori of the unperturbed integrable system have a natural affine structure [1, 12, 17]. The global conjugacy is obtained as an appropriate convex linear combination of the local conjugacies. This construction is reminiscent of the one used to build connections or Riemannian metrics in differential geometry.

In AN appendix we show that the Whitney extension theorem [34, 24, 27, 31] can be globalized to manifolds.

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1.1 Motivation

A motivation for globalizing the KAM theorem is the nontriviality of certain torus fibrations in Liouville integrable systems, for example the spherical pendulum. Here an obstruction to the triviality of the foliation by its Liouville tori is given by monodromy, See [15, 28] for a geometrical discussion of all the obstructions for a toral fibration of an integrable Hamiltonian system to be trivial. A natural question is whether (nontrivial) monodromy also can be defined for nonintegrable perturbations of the spherical pendulum. Answering this question is of interest in the study of semiclassical versions of such classical systems, see [25, 14, 13]. The results of the present paper imply that for an open set of Liouville integrable Hamiltonian systems, under a sufficiently small perturbation, the geometry of the fibration is largely preserved by a (Whitney) smooth diffeomorphism. This implies that monodromy can be defined in the nearly integrable case. In particular, our approach applies to the spherical pendulum. For a similar result in two degrees of freedom near a focus-focus singularity see [30]. We expect that a suitable reformulation of our results will be valid in the general Lie algebra setting of [26, 6].

1.2 Formulation of the global KAM theorem

We now give a precise formulation of our results, where ‘smooth’ means ‘of class C^∞ ’. Consider a $2n$ -dimensional, connected, smooth symplectic manifold (M, σ) with a surjective smooth map $\pi : M \rightarrow B$, where B is an n -dimensional smooth manifold. We assume that the map π defines a smooth locally trivial fibre bundle, whose fibres are Lagrangean n -tori. See note 1. In the following $\mathbb{T}^n = \mathbb{R}^n / (2\pi\mathbb{Z}^n)$ is the standard n -torus.

By the Liouville-Arnold integrability theorem [1, 12] it follows that for every $b \in B$ there is a neighbourhood $U^b \subseteq B$ and a symplectic diffeomorphism

$$\varphi^b : V^b = \pi^{-1}(U^b) \rightarrow \mathbb{T}^n \times A^b, \quad m \mapsto (\alpha^b(m), a^b(m)),$$

with $A^b \subseteq \mathbb{R}^n$ an open set and with symplectic form $\sum_{j=1}^n d\alpha_j^b \wedge da_j^b$ such that $a^b = (a_1^b, a_2^b, \dots, a_n^b)$ is constant on fibers of π . We call (α^b, a^b) angle action variables and (V^b, φ^b) an angle action chart.

Now consider a smooth Hamiltonian function $H : M \rightarrow \mathbb{R}$, which is constant on the fibers of π , that is, H is an integral of π . Then the Hamiltonian vector field X_H defined by $\sigma \lrcorner X_H = dH$ is tangent to these fibers. This leads to a vector field

$$(\varphi_*^b X_H)(\alpha, a) = \sum_{j=1}^n \omega_j^b(a) \frac{\partial}{\partial \alpha_j^b} = \sum_{j=1}^n \frac{\partial(\varphi_*^b H)}{\partial a_j^b} \frac{\partial}{\partial \alpha_j^b}$$

on $\mathbb{T}^n \times A^b$ with frequency vector $\omega^b(a) = (\omega_1^b(a), \dots, \omega_n^b(a))$. We call $\omega^b : A^b \rightarrow \mathbb{R}^n$ the local frequency map. We say that H is a *globally nondegenerate* integral of π , if for a collection $\{(V^b, \varphi^b)\}_{b \in B}$ of angle action charts whose domains V^b cover M , each local frequency map $\omega^b : A^b \rightarrow \mathbb{R}^n$ is a diffeomorphism onto its image. See note 2.

Suppose that H is a globally nondegenerate integral on B . If $B' \subseteq B$ is a relatively compact subset of B , that is, the closure of B' is compact, then there is a finite subcover $\{U^b\}_{b \in \mathcal{F}}$ of $\{U^b\}_{b \in B}$ such that for every $b \in \mathcal{F}$ the local frequency map ω^b is a diffeomorphism onto its image. Accordingly, let $M' = \pi^{-1}(B')$ and consider the corresponding bundle $\pi' : M' \rightarrow B'$.

We shall consider perturbations of H in the C^∞ -topology on M . For $M = \mathbb{R}^{2n}$ this topology is defined as follows. For any compact domain $K \subseteq \mathbb{R}^{2n}$, any $m \in \mathbb{N}$ and $\varepsilon > 0$ consider

$$\mathcal{V}_{K,m,\varepsilon} = \{F : M \rightarrow \mathbb{R} \mid \|F\|_{K,m} < \varepsilon\},$$

where $\|F\|_{K,m}$ is the C^m -norm of F on K . The C^∞ -topology is generated by all such sets $\mathcal{V}_{K,m,\varepsilon}$. On a manifold this definition has to be adapted by a partition of unity, compare with [20, 27].

We now state our main result.

Theorem 1. (KAM-GLOBAL) Let (M, σ) be a smooth $2n$ -dimensional symplectic manifold with $\pi : M \rightarrow B$ a smooth locally trivial Lagrangean n -torus bundle. Let $B' \subseteq B$ be an open and relatively compact subset and let $M' = \pi^{-1}(B')$. Suppose that $H : M \rightarrow \mathbb{R}$ is a smooth integral of π , which is globally nondegenerate. Finally let $F : M \rightarrow \mathbb{R}$ be a smooth function. If $F|_{M'}$ is sufficiently small in the C^∞ -topology, then there is a subset $C \subseteq B'$ and a map $\Phi : M' \rightarrow M'$ with the following properties.

1. The subset $C \subseteq B'$ is nowhere dense and the measure of $B' \setminus C$ tends to 0 as the size of the perturbation F tends to zero;
2. The subset $\pi^{-1}(C) \subseteq M'$ is a union of Diophantine X_H -invariant Lagrangean n -tori;
3. The map Φ is a C^∞ -diffeomorphism onto its image and is close to the identity map in the C^∞ -topology;
4. The restriction $\widehat{\Phi} = \Phi|_{\pi^{-1}(C)}$ conjugates X_H to X_{H+F} , that is,

$$\widehat{\Phi}_* X_H = X_{H+F}.$$

Note that the Hamiltonian $H + F$ need not be an integral of π . See notes 3–7.

Corollary 1. If H is globally nondegenerate and $H + F$ is integrable, then the dynamics of H and $H + F$ are conjugate by a global smooth torus bundle isomorphism (which in general is not symplectic). See note 6.

2 Proof of the global KAM theorem

This section is devoted to proving theorem 1. The idea is to glue together the local conjugacies, as obtained by the standard ‘local’ KAM theorem [29, 6, 5], with the help of a partition of unity. The adjective ‘local’ refers to the fact that this theorem deals with local trivializations of the global n -torus bundle.

2.1 The ‘local’ KAM theorem

In the ‘local’ KAM theorem the phase space is $\mathbb{T}^n \times A$, where $A \subseteq \mathbb{R}^n$ is an open, connected set. Also

$$(\alpha, a) = (\alpha_1, \dots, \alpha_n, a_1, \dots, a_n)$$

are angle action coordinates on $\mathbb{T}^n \times A$ with symplectic form $\sum_{j=1}^n d\alpha_j \wedge da_j$. Suppose that we are given a smooth Hamiltonian function $h : \mathbb{T}^n \times A \rightarrow \mathbb{R}$, which is a nondegenerate integral, that is, h does not depend on the angle variables α and the $n \times n$ matrix $\left(\frac{\partial^2 h}{\partial a_i \partial a_j}\right)_{i,j=1}^n$ has rank n . In the present local setting we denote Hamiltonian functions $h, h + f$, etc. by lower-case letters. We now define some concepts needed to formulate the ‘local’ KAM theorem.

The Hamiltonian vector field X_h corresponding to the Hamiltonian h has the form

$$X_h(\alpha, a) = \sum_{i=1}^n \omega_i(a) \frac{\partial}{\partial \alpha_i}, \quad (1)$$

with $\omega(a) = \partial h / \partial a(a)$. The nondegeneracy assumption means that the local frequency map $\omega : a \in A \mapsto \omega(a) \in \mathbb{R}^n$ is a diffeomorphism onto its image.

Next we need the concept of Diophantine frequencies. Let $\tau > n - 1$ and $\gamma > 0$ be constants. Set

$$D_{\tau, \gamma}(\mathbb{R}^n) = \{\omega \in \mathbb{R}^n \mid |\langle \omega, k \rangle| \geq \gamma |k|^{-\tau}, \text{ for all } k \in \mathbb{Z}^n \setminus \{0\}\}. \quad (2)$$

Elements of $D_{\tau, \gamma}(\mathbb{R}^n)$ are called (τ, γ) -*Diophantine* frequency vectors. Let $\Gamma = \omega(A)$. We also consider the shrunken version of Γ defined by

$$\Gamma_\gamma = \{\omega \in \Gamma \mid \text{dist}(\omega, \partial\Gamma) > \gamma\}.$$

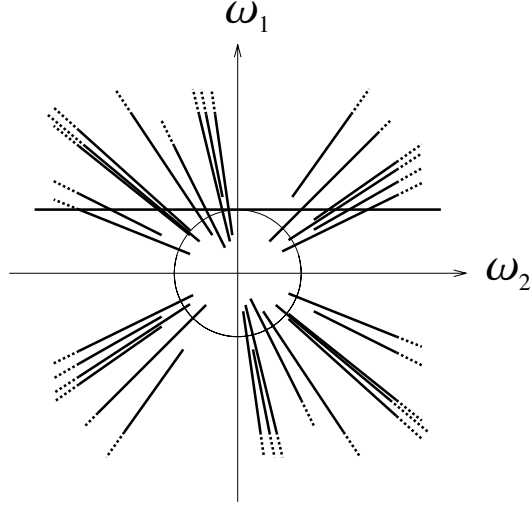


Figure 1: Sketch of the set $D_{\tau, \gamma}(\mathbb{R}^2)$, where the raylike structure is indicated. Compare with note 10.

Let $D_{\tau, \gamma}(\Gamma_\gamma) = \Gamma_\gamma \cap D_{\tau, \gamma}(\mathbb{R}^n)$. From now we take γ to be sufficiently small so that $D_{\tau, \gamma}(\Gamma_\gamma)$ is a nowhere dense set of positive measure. Recall that the measure of $\Gamma \setminus D_{\tau, \gamma}(\Gamma_\gamma)$ tends to zero in Γ as $\gamma \downarrow 0$, [29, 6, 5]. Finally define the shrunken domain $A_\gamma = \omega^{-1}(\Gamma_\gamma) \subseteq A$ as well as its nowhere dense counterpart $D_{\tau, \gamma}(A_\gamma) = \omega^{-1}(D_{\tau, \gamma}(\Gamma_\gamma)) \subseteq A_\gamma$, where the measure of $A \setminus D_{\tau, \gamma}(A_\gamma)$ tends to zero as $\gamma \downarrow 0$.

As said before, we measure smooth perturbations $h + f$ in the C^∞ -topology on $\mathbb{T}^n \times A$, assuming that all functions have smooth extensions to a neighbourhood of the closure $\mathbb{T}^n \times \bar{A}$.

We now are ready to formulate the standard ‘local’ KAM theorem, [29, 6].

Theorem 2. (KAM-‘LOCAL’) Suppose that the smooth Hamiltonian function $h : \mathbb{T}^n \times A \rightarrow \mathbb{R}$ is a nondegenerate integral of $\pi : \mathbb{T}^n \times A \rightarrow A$. Then there exists a C^∞ -neighbourhood \mathcal{V} of h , such that for all $h + f \in \mathcal{V}$ THERE IS A MAP $\Phi^A : \mathbb{T}^n \times A \rightarrow \mathbb{T}^n \times A$ with the following properties.

1. Φ^A is a C^∞ -diffeomorphism onto its image, which is close to the identity map in the C^∞ -topology.
2. The restricted map $\widehat{\Phi}^A = \Phi^A|_{\mathbb{T}^n \times D_{\tau, \gamma}(A_\gamma)}$ conjugates X_h to X_{h+f} , that is, $\widehat{\Phi}^A_* X_h = X_{h+f}$. See notes 8–10.

We can rephrase the ‘local’ KAM theorem as follows. If \mathcal{U} is a C^∞ -neighbourhood of $\text{Id}_{\mathbb{T}^n \times A}$, then there is a C^∞ -neighbourhood $\tilde{\mathcal{V}}$ of X_h such that for all $X_{h+f} \in \tilde{\mathcal{V}}$ there is a diffeomorphism $\Phi \in \mathcal{U}$ which conjugates X_h to X_{h+f} when restricted to $\mathbb{T}^n \times D_{\tau, \gamma}(A_\gamma)$.

2.2 Application of the ‘local’ KAM theorem

Let us return to the global problem of perturbing H to $H + F$ on the relatively compact set $M' = \pi^{-1}(B') \subseteq M$. Note that, in this global setting we denote the Hamiltonians by upper-case letters. Recall that we have an atlas $\{(V^b, \varphi^b)\}_{b \in B}$ of angle action charts on M , where $V^b = \pi^{-1}(U^b) = (\varphi^b)^{-1}(\mathbb{T}^n \times A^b)$. Here we can ensure that all $A^b \subseteq \mathbb{R}^n$ are convex. This convexity is independent of the local coordinates, as long as these are symplectic angle-action variables. See note 1. For each $b \in B$ and a sufficiently small constant $\gamma^b > 0$, the shrunken domain $V_{\gamma^b}^b \subseteq V^b$ defined by $V_{\gamma^b}^b = (\varphi^b)^{-1}(\mathbb{T}^n \times A_{\gamma^b}^b)$ is open and nonempty. This implies that $\{V_{\gamma^b}^b\}_{b \in B}$ still is a covering of M . Given any choice of an open relatively compact set $B' \subseteq B$, there exists a *finite* subcover $\{V_{\gamma^j}^j\}_{j \in \mathcal{J}}$ of $M' = \pi^{-1}(B')$. Note that $V_{\gamma^j}^j \subseteq V^j$, $j \in \mathcal{J}$. Projection by π gives corresponding subsets $U_{\gamma^j}^j \subseteq U^j$, where $\{U_{\gamma^j}^j\}_{j \in \mathcal{J}}$ covers B' . See the following commutative diagram

$$\begin{array}{ccc} V^j & \xrightarrow{\varphi^j} & \mathbb{T}^n \times A^j \\ \downarrow \pi & & \downarrow \\ U^j & \xrightarrow{a^j} & A^j, \end{array}$$

where $\varphi^j = (\alpha^j, a^j)$.

In each angle action chart (V^j, φ^j) where $\varphi^j : V^j = \pi^{-1}(U^j) \rightarrow \mathbb{T}^n \times A^j$, we have a local perturbation problem. Indeed, defining $h^j = H \circ (\varphi^j)^{-1}$ and $f^j = F \circ (\varphi^j)^{-1}$, we are in the setting of the ‘local’ KAM theorem 2 on the phase space $\mathbb{T}^n \times A^j$. We fix $\gamma = \gamma^j$ sufficiently small for theorem 2 to be applicable.

We now construct the nowhere dense set $C \subseteq B'$. Consider $D_{\tau, \gamma^j}(A_{\gamma^j}^j) \subseteq A_{\gamma^j}^j \subseteq A^j$. Inside $D_{\tau, \gamma^j}(A_{\gamma^j}^j)$ we define a subset $D_{\tau, \gamma^j}^*(A_{\gamma^j}^j)$, by eliminating a measure zero set. Towards this goal, we say that $p \in \mathbb{R}^n$ is a *density point* of $D_{\tau, \gamma}(\mathbb{R}^n)$, if for every smooth function $F : \mathbb{R}^n \rightarrow \mathbb{R}$ whose restriction to $D_{\tau, \gamma}(\mathbb{R}^n)$ vanishes, its infinite jet at p also vanishes. The set of all density points, denoted $D_{\tau, \gamma}^*(\mathbb{R}^n) \subseteq D_{\tau, \gamma}(\mathbb{R}^n)$, is a closed subset of full measure, that inherits a raylike structure. See note 11 and [8]. The inverse image under the frequency map of the intersection $D_{\tau, \gamma}^*(\mathbb{R}^n) \cap \Gamma_\gamma$ is a subset $D_{\tau, \gamma^j}^*(A_{\gamma^j}^j)$ of $D_{\tau, \gamma^j}(A_{\gamma^j}^j)$ of full measure. Next let

$$F^j = \{u \in U_{\gamma^j}^j \mid a^j(u) \notin D_{\tau, \gamma^j}^*(A_{\gamma^j}^j)\},$$

then we define

$$C = B' \setminus \bigcup_{j \in \mathcal{J}} F^j,$$

which clearly is a nowhere dense set. Since all F^j have small measure, also the finite union $\bigcup_{j \in \mathcal{J}} F^j$ has small measure. Accordingly, the set $\pi^{-1}(C) \subseteq M'$ is nowhere dense and $M' \setminus \pi^{-1}(C)$ has small measure.

We now continue the global perturbation argument. Given the appropriate (equicontinuous) smallness conditions on the collection of perturbing functions $\{f^j\}_{j \in \mathcal{J}}$, by the ‘local’ KAM theorem 2 we obtain diffeomorphisms $\Phi^j = \Phi^{A^j} : \mathbb{T}^n \times A^j \rightarrow \mathbb{T}^n \times A^j$, close to the identity in the C^∞ -topology, whose restriction $\widehat{\Phi}^j$ to $D_{\gamma^j}^d(A_{\gamma^j}^j)$ satisfies

$$(\widehat{\Phi}^j)_* X_{h^j} = X_{h^j + f^j}.$$

Using the above construction we see that each X_H -invariant Diophantine torus $T \subseteq \pi^{-1}(C)$ is diffeomorphic by $(\varphi^j)^{-1} \circ \Phi^j \circ \varphi^j$ to an invariant torus $(T')^j$ of X_{H+F} , $j \in \mathcal{J}$. This map (locally) conjugates the quasi-periodic dynamics of X_H with that of X_{H+F} . To proceed further we need the following uniqueness theorem.

Theorem 3 (UNICITY) Using the set up of the ‘local’ KAM theorem, Theorem 2, there exist C^∞ -neighbourhoods \mathcal{U}_2 and \mathcal{V}_2 such that if $\Phi \in \mathcal{V}_2$ is a conjugacy between the vector fields X_H and X_{H+F} , which lie in \mathcal{U}_2 , then the conjugacy Φ , restricted to $\mathbb{T}^n \times D_{\tau, \gamma}^*(A_\gamma)$, is unique up to a torus translation.

Proof: See [8]. □

We now combine the global nondegeneracy of H and the unicity theorem 3 to obtain the following result. For sufficiently small F , the correspondence which associates the torus T to the torus T' is unique, and hence is independent of the index $j \in \mathcal{J}$ used to define the conjugacy. Here we have used the fact that the frequency map ω^j is a diffeomorphism on A^j for all $j \in \mathcal{J}$. Loosely speaking one might say that on the union $\pi^{-1}(C)$ of Diophantine tori the action coordinates match under the transition maps $\varphi^j \circ (\varphi^i)^{-1}$. See note 12.

For the angle coordinates matching is quite different. First, in cases where the n -torus bundle $\pi : M \rightarrow B$ is nontrivial, the angle coordinates on $\pi^{-1}(C)$ do not have to match under the transition maps $\varphi^j \circ (\varphi^i)^{-1}$. Second, the angle components of ‘local’ KAM conjugacies Φ^i and Φ^j do not have to match either.

2.3 Affine Structure and Partition of Unity

To overcome the problem of matching the angles we glue the local KAM conjugacies together by taking a convex linear combination on the integrable n -tori. Here we use the natural affine structure of these tori and a partition of unity.

We begin by reconsidering the notion of quasi-periodicity and the way this induces a natural affine structure, compare with [8]. Consider the constant vector field

$$\mathbb{X}_\omega = \sum_{j=1}^n \omega_j \frac{\partial}{\partial \alpha_j},$$

on the standard torus $\mathbb{T}^n = \mathbb{R}^n / (2\pi\mathbb{Z}^n)$ with non-resonant frequency vector $\omega = (\omega_1, \omega_2, \dots, \omega_n)$. The self-conjugacies of \mathbb{X}_ω are exactly the translations of \mathbb{T}^n . This directly follows from the fact that each trajectory of \mathbb{X}_ω is dense. Note that these translations determine the affine structure on \mathbb{T}^n . For an arbitrary vector field X with an invariant n -torus T , we may define quasi-periodicity of $X|_T$ by requiring the existence of a smooth conjugacy $\phi : T \rightarrow \mathbb{T}^n$ with a vector field \mathbb{X}_ω on \mathbb{T}^n , i.e., such that $\phi_*(X|_T) = \mathbb{X}_\omega$. In that situation, the self-conjugacies of $X|_T$ determine a natural affine structure on T . Note that the translations on T and on \mathbb{T}^n are self conjugacies and that therefore the conjugacy ϕ itself is unique modulo torus translations.

Finally observe that in the present integrable Hamiltonian case, the (local) angle-action variables (α, a) give rise to X -invariant tori T_a . Also the involutive integrals give rise to an affine structure [1, 12], that is defined for each of the conditionally periodic (or parallel) invariant tori. We note that on the quasi-periodic tori T_a the latter structure coincides with the one we introduced above. Indeed, in the angle coordinates α the vector field X becomes constant. For details see [8].

Next we introduce a partition of unity, to be used for the explicit construction of a convex linear combination. Recall that we are restricting to a relatively compact subset M' of the regular n -torus bundle M .

Lemma 1. (PARTITION OF UNITY) Subordinate to the covering $\{V_{\gamma^b}^b\}_{b \in B'}$ of M' there is a finite partition of unity $\{(V_{\gamma^j}^j, \xi^j)\}_{j \in \mathcal{J}}$ by C^∞ functions $\xi^j : M' \rightarrow \mathbb{R}$ such that, for every $j \in \mathcal{J}$ we have

1. The support of ξ^j is a compact subset of $V_{\gamma^j}^j$;
2. The function ξ^j takes values in the interval $[0, 1]$;
3. The function ξ^j is constant on the fibers of the bundle projection $\pi' : M' \rightarrow B'$;
4. $\sum_{j \in \mathcal{J}} \xi^j \equiv 1$ on M' .

The lemma follows by carrying out the standard partition of unity construction [20, 31] on B' and pulling everything back by π .

2.4 Glueing the ‘local’ KAM conjugacies

We now conclude the proof of theorem 1. For each $j \in \mathcal{J}$ consider the near-identity map $\Psi^j = (\varphi^j)^{-1} \circ (\Phi^j)^{-1} \circ \varphi^j$, which takes a perturbed torus T' to its unperturbed (integrable) counterpart T and thereby conjugates the quasi-periodic dynamics. Note that the ‘local’ KAM conjugacy Φ^j maps $\mathbb{T}^n \times A_{\gamma^j}^j$ into $\mathbb{T}^n \times A^j$. Therefore $(\Psi^j)^{-1}$ maps $V_{\gamma^j}^j$ into V^j . We make the following assertions about the chart overlap maps on M' .

Lemma 2. (OVERLAP) Consider the perturbed Hamiltonian $H + F$ of theorem 1. Assume that the function F is sufficiently small in the C^∞ -topology. Then on the overlaps $V^i \cap V^j$:

1. The transition map $\Psi^i \circ (\Psi^j)^{-1}$ is close to the identity in the C^∞ topology;
2. For each integrable quasi-periodic n -torus $T \subseteq V_{\gamma^i}^i \cap V_{\gamma^j}^j$, which by $(\Psi^i)^{-1}$ and $(\Psi^j)^{-1}$ is diffeomorphic to T' , the maps $\Psi^i|_{T'}$ and $\Psi^j|_{T'}$ only differ by a small translation.

Proof. The first item follows directly from the equicontinuity condition regarding the size of the $\{f^j\}_{j \in \mathcal{J}}$ and by the chain rule. The second item follows from the fact that a near-identity affine torus transformation is necessarily a translation. See notes 13, 14 and [8]. \square

Proof of theorem 1. The global conjugacy of theorem 1 is obtained by taking the convex linear combination

$$\Phi^{-1} = \sum_{j \in \mathcal{J}} \xi^j \cdot \Psi^j. \quad (3)$$

This expression is well-defined on the union of Diophantine tori T' under consideration, and here is a conjugacy between X_{H+F} and X_H . Indeed, notice that by taking inverses we work on the integrable Diophantine tori $T \subseteq \pi^{-1}(C)$ with the canonical affine structure. In each fiber T of the integrable torus bundle, by lemma 3, the maps Ψ^i and Ψ^j with $i, j \in \mathcal{J}$, only differ by a small translation. The finite convex linear combination Φ^{-1} then is well-defined, implying that Φ is a global conjugacy on the nowhere dense set $\pi^{-1}(C)$. We now have to show that Φ is Whitney smooth and admits a global Whitney extension as a smooth map.

We establish Whitney smoothness of the map Φ , which is characterized by the local extendability of Φ as a smooth map [34]. We check extendability per

action angle chart, considering $\varphi^{j_0} : V^{j_0} \rightarrow \mathbb{T}^n \times A^{j_0}$, for a fixed $j_0 \in \mathcal{J}$. A brief computation yields that, restricted to the Diophantine tori,

$$\varphi^{j_0} \circ \Phi^{-1} \circ (\varphi^{j_0})^{-1} = \sum_{j \in \mathcal{J}} \xi^j \circ (\varphi^{j_0})^{-1} \cdot (\varphi^{j_0} \circ (\varphi^j)^{-1}) \circ (\Phi^j)^{-1} \circ (\varphi^j \circ (\varphi^{j_0})^{-1}). \quad (4)$$

Note that the ‘local’ KAM conjugacies Φ^j (as well as their inverses $(\Phi^j)^{-1}$) can be extended as smooth maps; in fact, by theorem 2 they were already given as extensions. Also note that whenever a ‘local’ KAM map $(\Phi^j)^{-1}$ is not defined, the corresponding bump function $\xi^j \circ (\varphi^{j_0})^{-1}$ vanishes. Now we can extend (4) as a smooth map from $\mathbb{T}^n \times A^{j_0}$ to itself, as a convex linear combination, taken on the product $\mathbb{T}^n \times A^{j_0} \subseteq \mathbb{T}^n \times \mathbb{R}^n$, which has a natural affine structure. For exactly this reason, in section 2.2 we chose A^{j_0} to be convex. In the \mathbb{T}^n -component we are dealing with the near-identity maps $(\Phi^j)^{-1}, j \in \mathcal{J}$. Since all the ingredients we have used are smooth, the final result (4) is also.

We still have to prove that Φ^{-1} , defined by (3), admits a global Whitney extension. In Theorem 4 in the appendix we prove that the local extensions can be glued together to obtain a global Whitney extension of Φ^{-1} , again using a partition of unity.

Finally, the fact that for small F the map Φ is C^∞ -close to the identity map, follows by Leibniz’s rule. This proves the global KAM theorem 1. See note 14. \square

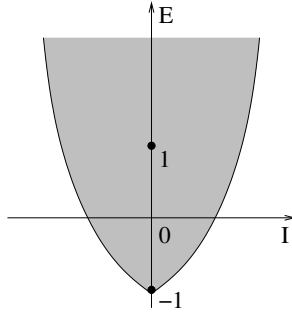


Figure 2: Range of the energy-momentum map of the spherical pendulum.

3 Example: the spherical pendulum

In this section we consider the spherical pendulum [1, 15, 12, 30]. Dynamically, the spherical pendulum is the motion of a unit mass particle restricted to the unit

sphere in \mathbb{R}^3 in a constant vertically downward gravitational field. The configuration space of the spherical pendulum is the 2-sphere $S^2 = \{q \in \mathbb{R}^3 \mid \langle q, q \rangle = 1\}$ and the phase space its cotangent bundle $\widetilde{M} = T^*S^2 = \{(q, p) \in \mathbb{R}^6 \mid \langle q, q \rangle = 1 \text{ and } \langle q, p \rangle = 0\}$. Here $q = (q_1, q_2, q_3)$ and $p = (p_1, p_2, p_3)$, while $\langle \cdot, \cdot \rangle$ denotes the standard inner product in \mathbb{R}^3 .

The spherical pendulum is a Liouville integrable system. By Noether's theorem [1] the rotational symmetry about the vertical axis gives rise to angular momentum $I(q, p) = q_1 p_2 - q_2 p_1$, which is a second integral of motion, in addition to the energy $H(q, p) = \frac{1}{2} \langle p, p \rangle + q_3$. The energy-momentum map of the spherical pendulum is

$$\mathcal{EM} : T^*S^2 \rightarrow \mathbb{R}^2 : (q, p) \mapsto (I, E) = (q_1 p_2 - q_2 p_1, \frac{1}{2} \langle p, p \rangle + q_3).$$

Its fibers corresponding to regular values give rise to a fibration of phase space by Lagrangian 2-tori. The image \widetilde{B} of \mathcal{EM} is the closed part of the plane lying in between the two curves meeting at a corner, see figure 1. The set of singular values of \mathcal{EM} consists of the two boundary curves and the points $(I, E) = (0, \pm 1)$. The points correspond to the equilibria $(q, p) = (0, 0, \pm 1, 0, 0, 0)$; whereas the boundary curves correspond to the horizontal periodic motions of the pendulum discovered by Huygens [22]. Therefore the set B of regular \mathcal{EM} -values consists of the interior of \widetilde{B} minus the point $(I, E) = (0, 1)$, corresponding to the unstable equilibrium point $(0, 0, 1, 0, 0, 0)$. This point is the centre of the nontrivial monodromy. The corresponding fiber $\mathcal{EM}^{-1}(1, 0)$ is a once pinched 2-torus. Note that $\mathcal{EM} : \widetilde{M} \rightarrow \widetilde{B}$ is a singular foliation in the sense of Stefan-Sussmann [32, 33].

On B one of the two components of the frequency map is single valued while the other is multi-valued [15, 12]. In [21] Horozov established global nondegeneracy of H on B . Thus the global KAM theorem 1 can be applied to any relatively compact open subset $B' \subseteq B$. Consequently, the integrable dynamics on the 2-torus bundle $\mathcal{EM}' : M' \rightarrow B'$ of the spherical pendulum is quasi-periodically stable.

This means that any sufficiently small perturbation of the spherical pendulum has a nowhere dense union of Diophantine invariant tori of large measure, which has a smooth interpolation by a push forward of the integrable bundle $\mathcal{EM}' : M' \rightarrow B'$. In section 4 we shall argue that this allows for a definition of (nontrivial) monodromy for the perturbed torus bundle.

Let us briefly return to the case where the perturbation remains within the world of integrable systems. See note 4. For example, in the case of the spherical pendulum, let the perturbation preserve the axial symmetry, whence by Noether's theorem [1, 12] the system remains Liouville integrable. As said before, this yields global structural stability of the X_H -dynamics on the relatively compact set M' , where the conjugacy to the perturbed integrable X_{H+F} -dynamics is a C^∞

isomorphism of 2-torus bundles. This statement just uses global nondegeneracy of H .

4 Monodromy in the nearly integrable case

In this section we show how to define the concept of monodromy for a nearly integrable nowhere dense torus bundle as obtained in this paper. Our construction, however, is independent of any integrable approximation.

4.1 A regular union of tori

Let M be a manifold endowed with a (smooth) metric ϱ . Then for any two subsets $A, B \subseteq M$ we define $\varrho(A, B) = \inf_{x \in A, y \in B} \varrho(x, y)$. (Note that in general this does not define a metric on the set of all subsets.) Let $M' \subseteq M$ be compact and $\{T_\lambda\}_{\lambda \in \Lambda}$ a collection of pairwise disjoint n -tori in M . We require the following *regularity* properties. There exist positive constants ε and δ such that

1. For all $\lambda \in \Lambda$, each continuous map $h_\lambda : T_\lambda \rightarrow T_\lambda$ with $\varrho(x, h_\lambda(x)) < 2\varepsilon$ for all $x \in T_\lambda$, is homotopic with the identity map Id_{T_λ} ;
2. For each $\lambda, \lambda' \in \Lambda$, such that $\varrho(T_\lambda, T_{\lambda'}) < \delta$, there exists a homeomorphism $h_{\lambda', \lambda} : T_\lambda \rightarrow T_{\lambda'}$, such that $\varrho(x, h_{\lambda', \lambda}(x)) < \frac{1}{2}\varepsilon$ for all $x \in T_\lambda$;
3. For each $x \in M'$ there exists $\lambda \in \Lambda$ such that $\varrho(\{x\}, T_\lambda) < \frac{1}{2}\delta$.

Note that the homeomorphism $h_{\lambda', \lambda}$, as required to exist by item 2, by item 1 is unique modulo homotopy. We also observe the following. In the situation of the present paper, we started with a globally nondegenerate Liouville integrable Hamiltonian system, such that an open and dense part of the phase space M is foliated by invariant Lagrangean n -tori. Let M' be a compact union of such tori. Then, under sufficiently small, non-integrable perturbation, the remaining Lagrangean KAM tori in M' as discussed in this paper, form a regular collection in the above sense.

4.2 Construction of a \mathbb{Z}^n -bundle

We now construct a \mathbb{Z}^n -bundle, first only over the union $M'' = \cup_\lambda T_\lambda$, which is assumed to be regular in the sense of section 4.1. For each point $x \in T_\lambda$ the fibre is defined by $F_x = H_1(T_\lambda)$, the first homology group of T_λ . Using the regularity property 2 and the fact that $h_{\lambda', \lambda}$ is unique modulo homotopy, it follows that this bundle is locally trivial. Let E'' denote the total space of this bundle and $\pi'' : E'' \rightarrow M''$ the bundle projection.

The above bundle is extended over $M' \cup M''$ as follows. For each $x \in M'$ we consider $\Lambda(x) = \{\lambda \in \Lambda \mid \varrho(\{x\}, T_\lambda) < \frac{1}{2}\delta\}$. Then consider the set of pairs (λ, α) , where $\lambda \in \Lambda(x)$ and $\alpha \in H_1(T_\lambda)$. On this set we have the following equivalence relation:

$$(\lambda, \alpha) \sim (\lambda', \alpha') \iff (h_{\lambda', \lambda})_* \alpha = \alpha',$$

where h_* denotes the action of h on the homology. The set of equivalence classes is defined as the fibre F_x at x . The fibre F_x is isomorphic to $F_{x'}$ in a natural (and unique) way, for any $x' \in T_\lambda$ with $\lambda \in \Gamma(x)$. This extended bundle again is locally trivial.

We conclude this section by observing that the monodromy of the torus bundle exactly is the obstruction against global triviality of this \mathbb{Z}^n -bundle. Moreover, by the global KAM theorem 1, in the integrable case one obtains the same \mathbb{Z}^n -bundle as in the nearly integrable case.

5 Conclusions

In this paper, we obtained a global quasi-periodic stability result for bundles of Lagrangean invariant tori, under the assumption of global Kolmogorov nondegeneracy on the integrable approximation. We emphasize that our approach works in arbitrarily many degrees of freedom and that it is independent of the integrable geometry one starts with. The global Whitney smooth conjugacy $\widehat{\Phi}$ between Diophantine tori of the integrable and the nearly integrable system can be suitably extended to a smooth map Φ , which serves to smoothly interpolate the nearly integrable, only Whitney smooth Diophantine torus bundle $\Phi(\pi^{-1}(C))$, defined over the nowhere dense set C . The interpolation thereby becomes bundle isomorphic with its integrable counterpart $\pi' : M' \rightarrow B'$. This means that the global geometry is as in the integrable case. The reason for this is that the extended diffeomorphisms are sufficiently close to the identity, and hence are isotopic to the identity, see the appendix. In this sense, the lack of unicity of Whitney extensions plays no role and we can generalize concepts like monodromy directly to the nearly integrable case. For a topological discussion of the corresponding n -torus bundles, see [28, 15].

The present global quasi-periodic stability result directly generalizes to the setting of [6, 5], where a general unfolding theory of quasi-periodic tori, based on [26, 29], is developed. Within the world of Hamiltonian systems this leads to applications at the level of lower dimensional isotropic tori. However, our approach also applies to dissipative, volume preserving, or reversible systems, see

[4]. In these cases, the torus bundles live in the product of phase space and a parameter space.

As we showed above, classical monodromy also exists in the nearly integrable case. As far as we know quantum monodromy only is well-defined for integrable systems, explaining certain defects in the quantum spectrum [13, 14]. It is an open question whether quantum monodromy can be defined in nearly integrable cases, and whether the present approach would be useful there. Here it should be mentioned that in a number of applications [13], the classical limit is not integrable, but only nearly so.

6 Notes

1. Often the regular n -torus bundle is part of a larger structure containing singularities, see section 3 for an example and compare with Stefan-Sussman [32, 33].
2. According to a remark of Duistermaat [15] there is a natural affine structure on B . Using the affine structure on the space of actions B , for each $b \in B$ the second derivative of $h^b = (H|_{V^b}) \circ (\varphi^b)^{-1}$ on $\mathbb{T}^n \times A^b$ is well-defined. By global nondegeneracy we mean that D^2h^b has maximal rank everywhere on V^b for each $b \in B$. Because of the affine structure on B , it follows that on overlapping angle action charts the rank is independent of the choice of chart.

This affine structure (straight lines) also covers the symplectic coordinate independence of local convexity.

Comments on theorem 1

3. The map Φ generally is not symplectic.
4. The conclusion of theorem 1 that restricted to $\pi^{-1}(C)$ the diffeomorphism Φ is a conjugacy between X_H and X_{H+F} , is expressed by saying that X_H is quasi-periodically stable on M' . Note that, by the smoothness of Φ , the measure of the nowhere dense set $\Phi(\pi^{-1}(C))$, which is a union of perturbed tori, is large. Also the set $\pi^{-1}(C)$ has a raylike structure, which is carried over by Φ , see [8].
5. The restriction of Φ to $\pi^{-1}(C) \subseteq M'$ preserves the affine structure of the quasi-periodic tori, see section 2.3. In the complement $M' \setminus \pi^{-1}(C)$ the diffeomorphism Φ has no dynamical meaning. Still, the push forward of the integrable bundle $\pi' : M' \rightarrow B'$ by Φ is a smooth n -torus bundle which interpolates the tori in $\Phi(\pi^{-1}(C))$.

6. It is of interest to consider theorem 1 in the case where the perturbation $H + F$ is integrable, which leads to corollary 1. Then the proof can be simplified by omitting the KAM theory. Indeed, the local conjugacies are given directly by the nondegeneracy assumption, compare with [6], §3. Hence there is no need of any Diophantine non-resonance conditions. In this case Φ is a C^∞ diffeomorphism defined on the whole torus bundle, which is a global conjugacy. It is even an isomorphism of torus bundles. Consequently the restriction of H to the relatively compact set $M' = \pi^{-1}(B')$ is structurally stable under small integrable perturbation. As far as we know this result is new.
7. There are direct generalizations of theorem 1 to the world of C^k -systems endowed with the (weak) Whitney C^k -topology for k sufficiently large, see [20]. For C^k -versions of the classical KAM theorem, see [29, 6, 5]. Here also formulations can be found for real-analytic systems, endowed with the compact-open topology on holomorphic extensions [9].

Comments on theorem 2

8. Observe that Φ^A maps $\mathbb{T}^n \times A_\gamma$ into $\mathbb{T}^n \times A$.
9. In general, the map Φ^A is not symplectic. Notice that theorem 2 asserts that the integrable system is quasi-periodically stable.
10. The nowhere dense set $D_{\tau,\gamma}(\mathbb{R}^n)$ has a raylike structure, also called ‘closed half line structure’ [8]. Indeed, whenever $\omega \in D_{\tau,\gamma}(\mathbb{R}^n)$, we also have $s\omega \in D_{\tau,\gamma}(\mathbb{R}^n)$ for all $s \geq 1$, see figure 1. In the ray direction of A_γ the map Φ^A is smooth. Moreover, smooth dependence on extra parameters is preserved in the following sense: if this smooth dependence holds for $h + f$, then it also holds for Φ^A .

The set $D_{\tau,\gamma}(A_\gamma)$, by the (inverse of) the frequency map, inherits the raylike structure of $D_{\tau,\gamma}(\mathbb{R}^n)$ up to a diffeomorphism. Similarly, for the perturbed tori, this raylike structure is carried over by the KAM diffeomorphism Φ^A .

Other comments

11. The raylike structure of $D_{\tau,\gamma}(\mathbb{R}^n)$, see note 10, largely carries over to the set $C \subseteq B'$ of theorem 1 and thus, by the global KAM diffeomorphism, automatically also to the perturbed torus bundle. To see this consider $j, \ell \in \mathcal{J}$ such that $U_{\gamma^j}^j \cap U_{\gamma^\ell}^\ell \neq \emptyset$ and consider the corresponding frequency maps $\omega^j : U^j \rightarrow A^j$ and $\omega^\ell : U^\ell \rightarrow A^\ell$. For $u \in U_{\gamma^j}^j \cap U_{\gamma^\ell}^\ell$, it is well-known that

$\omega^j(u) = S\omega^\ell(u)$, for a matrix $S \in Gl(n, \mathbb{Z})$, where S is locally constant. See notes 13, 14 and [8].

When considering (2), it follows that, if $\omega \in \mathbb{R}^n$ is (τ, γ) -Diophantine, then $S\omega$ is (τ, γ') -Diophantine, where $\gamma' = c(n)\|S^T\|^{-\tau}\gamma$, for a positive constant $c(n)$, only depending on n and where S^T is the transpose of S . Indeed, for any $\omega \in D_{\tau, \gamma}(\mathbb{R}^n)$ one has

$$|\langle S\omega, k \rangle| = |\langle \omega, S^T k \rangle| \geq |S^T k|^{-\tau} \geq c(n)\|S^T\|^{-\tau}\gamma|k|^{-\tau},$$

for all $k \in \mathbb{Z} \setminus \{0\}$.

Our claim now follows from the fact that we can take the γ^j , $j \in \mathcal{J}$, sufficiently small and that \mathcal{J} is finite.

12. In ‘matching the actions’, we have used the fact that the number of actions equals the number of frequencies. In the more general setting of [6, 5], for example, in the case of lower dimensional isotropic tori, the number of ‘parameters’ exceeds that of the frequencies and the corresponding frequency map is just required to be a submersion. In this situation the ‘matching of the actions’ requires more attention.
13. The affine transformations of the standard n -torus \mathbb{T}^n are all of the form $x \mapsto Sx + c$, where S is an integer $n \times n$ -matrix with $\det S = \pm 1$ and $c \in \mathbb{T}^n$, see [1, 12, 17]. It follows that near-identity affine torus transformations are small translations. This plays a role in transitions between angle-action charts.
14. On an overlap $T \subseteq V_{\gamma^i}^i \cap V_{\gamma^j}^j$, $i \neq j$, consider the angle components α^i and α^j of the chart maps φ^i and φ^j , both taking values in the standard torus \mathbb{T}^n . Here $\alpha^j = S_{i,j}\alpha^i + c_{i,j}$ for $S_{i,j}$ integer $n \times n$ -matrices with $\det S_{i,j} = \pm 1$ and $c_{i,j} \in \mathbb{T}^n$, see above. In the case of nontrivial monodromy we do not always have $S_{i,j} = \text{Id}$, since the transition map $\varphi^i \circ (\varphi^j)^{-1}$ between the local angle action charts is not close to the identity map. We have avoided this problem by considering the near-identity maps Ψ^i instead.
15. The asymptotic considerations of Rink [30] near focus-focus singularities (that is, complex saddles) in Liouville integrable Hamiltonian systems of 2 degrees of freedom, imply that our results on the spherical pendulum hold when B' is a small annular region around the point $(I, E) = (0, 1)$.

The present approach differs from that of [30] in the following way. The latter method interpolates the ‘local’ KAM conjugacies by the identity map,

which may damage the raylike structure of the ‘Cantor set’. Our approach using convex combinations does not. See note 11.

In future work we aim to apply the same methods to deal with (quasi-periodic) bifurcation problems, see [2, 3, 5, 6, 11, 16, 18], where more general (and more complicated) Whitney smooth ‘Cantor stratifications’ occur. Here, apart from half lines, also higher dimensional half spaces do occur. For more references see [7].

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References

- [1] V.I. Arnold, “Mathematical Methods of Classical Mechanics”, GTM **60** Springer-Verlag, New York, 1978.
- [2] H.W. Broer, H. Hanßmann and J. Hoo, The quasi-periodic Hamiltonian Hopf bifurcation, with applications to the Lagrange top with quasi-periodic forcing. Preprint University of Groningen 2004.
- [3] H.W. Broer, J. Hoo and V. Naudot, Normal linear stability of quasi-periodic tori, *J. Differential Equations* (to appear).
- [4] H.W. Broer and G.B. Huitema, Unfoldings of quasi-periodic tori in reversible systems, *Journal of Dynamics and Differential Equations* **7** (1995) 191-212.
- [5] H.W. Broer, G.B. Huitema and M.B. Sevryuk, Quasi-Periodic Motions in Families of Dynamical Systems, Order amidst Chaos, *Lecture Notes in Mathematics*, vol. **1645**, Springer Verlag, New York, 1996.
- [6] H.W. Broer, G.B. Huitema, F. Takens and B.L.J. Braaksma, “Unfoldings and bifurcations of quasi-periodic tori”, *Memoir AMS*, no. **421**, Amer. Math. Soc., Providence, R.I. 1990.

- [7] H.W. Broer and M.B. Sevryuk, KAM Theory: quasi-periodic dynamics in dynamical systems. In: H.W. Broer, B. Hasselblatt and F. Takens (eds.), “Handbook of Dynamical Systems”, Vol. 3, North-Holland (to appear).
- [8] H.W. Broer and F. Takens, Unicity of KAM tori, *Ergod. Th. & Dynam. Sys.*, 2007 (to appear).
- [9] H.W. Broer and F.M. Tangerman, From a differentiable to a real analytic perturbation theory, applications to the Kupka Smale theorems, *Ergod. Th. & Dynam. Sys.*, **6** (1986) 345-362.
- [10] L. Chierchia and G. Gallavotti, Smooth prime integrals for quasi-integrable Hamiltonian systems. *Nuovo Cimento B*(11) **67**(2) (1982) 277-295.
- [11] M.C. Ciocci, A. Litvak-Hinenzon and H.W. Broer, Survey on dissipative KAM theory including quasi-periodic bifurcation theory based on lectures by Henk Broer. In: J. Montaldi and T. Ratiu (eds.), *Geometric Mechanics and Symmetry: the Peyresq Lectures*, LMS Lecture Notes Series **306**. Cambridge University Press, (2005), 303-355.
- [12] R.H. Cushman and L. Bates, “Global aspects of classical integrable systems”, Birkhäuser, Basel, 1997.
- [13] R.H. Cushman, H.R. Dullin, A. Giacobbe, D.D. Holm, M. Joyeux, P. Lynch, D.A. Sadovskii and B.I. Zhilinskiĭ, CO₂ molecule as a quantum realization of the 1 : 1 : 2 resonant swing-spring with monodromy. *Physical Review Letters*, **93**(2) (2004) 024302(4).
- [14] R.H. Cushman and D. A. Sadovskii, Monodromy in the hydrogen atom in crossed fields, *Physica D* **142** (2000) 166-196.
- [15] J.J. Duistermaat, On global action-angle coordinates, *Comm. Pure Appl. Math.* **33** (1980) 687-706.
- [16] K. Efstathiou, Metamorphoses of Hamiltonian Systems with Symmetries, *Lecture Notes in Mathematics*, Vol. **1864**, Springer-Verlag, Berlin, 2005.
- [17] F. Fassò, Hamiltonian perturbation theory on a manifold. *Celest. Mech. and Dyn. Astr.* **62** (1995) 43-69.
- [18] H. Hanßmann, Local and Semi-Local Bifurcations in Hamiltonian Dynamical Systems – Results and Examples, “Habilitationsschrift, RWTH Aachen”, 2004; Lecture Notes in Mathematics (to appear).
- [19] F. Hausdorff, “Set Theory” (2nd edition), Chelsea, 1962. [German original: Veit (Leipzig), 1914.]
- [20] M.W. Hirsch, “Differential Topology”, Springer Verlag, New York, 1976.

- [21] E. Horozov, Perturbations of the spherical pendulum and abelian integrals, *Journal für die Reine und Angewandte Mathematik* **408** (1990) 114-135.
- [22] Chr. Huygens, Appendix I to “De vi centrifuga” (1659) 302-311. In: *Œuvres Complètes de Christiaan Huygens, publiées par la Société Hollandaise des Sciences* **16**, (1929) 306-309, Martinus Nijhoff, The Hague.⁴
- [23] V.F. Lazutkin, Existence of a continuum of closed invariant curves for a convex billiard. (Russian) *Uspehi Mat. Nauk* **27** no.3 (1972) 201-202.
- [24] B. Malgrange, “Ideals of Differentiable Functions”, Oxford University Press. 1966.
- [25] R. Montgomery, The connection whose holonomy is the classical adiabatic angles of Hannay and Berry and its generalization to the non-integrable case, *Commun. Math. Phys.* **120** (1989) 269-294.
- [26] J.K. Moser, Convergent series expansions for quasi-periodic motions, *Math. Ann.* **169** (1967) 136-176.
- [27] J.R. Munkres, “Elementary Differential Topology”, Princeton University Press, 1963.
- [28] N.N. Nekhorosev, Action angle coordinates and their generalizations, *Translations of the Moscow Mathematical Society*, **26** (1972) 180-198.
- [29] J. Pöschel, Integrability of Hamiltonian systems on Cantor sets, *Commun. Pure Appl. Math.* **35** (1982) 653-696.
- [30] B.W. Rink, A Cantor set of tori with monodromy near a focus-focus singularity, *Nonlinearity* **17** (2004) 347-356.
- [31] M. Spivak, “A Comprehensive Introduction to Differential Geometry”, Vol I. Publish or Perish, Berkeley, 1964.
- [32] P. Stefan, Accessible sets, orbits and foliations with singularities, *Proc. London Math. Soc.* **29** (1974) 699-713
- [33] H. Sussmann, Orbits of families of vector fields and integrability of distributions, *Trans. Amer. Math. Soc.* **180** (1973) 171-188.
- [34] H. Whitney, Analytic extensions of differential functions defined in closed sets, *Trans. Amer. Math. Soc.* **36** (1943) 63-89.

⁴The text “De vi centrifuga” was published posthumously in 1704, but the appendix remained unpublished until the edition of the *Œuvres*. So, during Huygens’s lifetime the *Horologium Oscillatorium* of 1673 was the publication which came nearest to his knowledge of circular motion and centrifugal force.

Appendix: The global Whitney extension theorem

The Whitney extension theorem [34, 24, 27, 20] deals with smooth real functions defined on closed subsets $K \subseteq \mathbb{R}^n$. Smoothness, in the sense of Whitney, is characterized completely in terms of the set K . The Whitney extension theorem says that this characterization of smoothness is equivalent to having an extension to a neighbourhood of K that is smooth in the usual sense.

Our present aim is to show that this result carries over to the case where M is a smooth manifold and $K \subseteq M$ is a closed set. In the case of real functions, the above construction with $K \subseteq \mathbb{R}^n$, provides us with local Whitney extensions. Then a partition of unity argument [27, 20, 31] gives a global Whitney extension by gluing together local extensions. Compare with similar arguments in section 2.

This construction directly generalizes to mappings $M \rightarrow \mathbb{R}^n$ and thereby to sections in a vector bundle over M . Finally by using the exponential map $\text{Exp} : T(M) \rightarrow M$ of a smooth Riemannian metric, we generalize the Whitney extension theory to near-identity maps.

Remark. In this paper, the term ‘Whitney smooth’ is only used when no ordinary smoothness is in order; in our case this applies especially for maps defined on nowhere dense sets C . Whitney smoothness on C then means local extendability as a smooth map. Also we speak of Whitney smoothness with regard to torus bundles over (nowhere dense) sets C , which again means that we can locally extend (or interpolate) such bundles as smooth torus bundles.

6.1 Whitney smooth functions and extensions

Let M be a smooth manifold and $K \subseteq M$ a closed subset. A Whitney C^m -function F on K assigns to each point $x \in K$ the m -jet $F(x)$ of a function on M . A main question is: under what condition on F can one find a C^m -function $f : M \rightarrow \mathbb{R}$ (in the usual sense) such that for each $x \in K$ the m -jet of f in x equals $F(x)$, that is, such that $j_x^m(f) = F(x)$. Such an f is called a smooth extension of F . This problem was treated by Whitney in [34]; our reference is [24]. (In the original treatment it was assumed that K is compact and that $M = \mathbb{R}^n$.) We now show that, whenever smooth extensions exist locally, there also exists a global extension.

Theorem 4. (GLOBAL WHITNEY EXTENSION) Let M be a smooth manifold and $K \subseteq M$ a closed set. Let F be a C^m -function on K in the sense of Whitney. Assume that F has locally smooth extensions, in the sense that for each $x \in K$ there exist

1. Open neighbourhoods $V_x \subseteq \bar{V}_x \subseteq U_x$ of x in M ;

2. A compact subset $K_x \subseteq K$ such that $K \cap V_x = K_x \cap V_x$ and $K_x \subseteq U_x$;
3. A C^m -function f_x on U_x such that for each $y \in K_x$ the m -jet of f_x in y equals $F(y)$, that is, $j_y^m(f_x) = F(y)$.

Then, there exists a C^m -function $f : M \rightarrow \mathbb{R}$ such that $j_x^m(f) = F(x)$ for all $x \in K$, that is, f is a globally smooth extension of F .

Proof. First we claim that there is a countable and locally finite open cover $\{V_i\}_{i=0}^\infty$ of a neighbourhood of K in M such that there are compact subsets $K_i \subseteq K$ and open subsets $U_i \subseteq M$ such that:

1. $\bar{V}_i \subset U_i$ and \bar{U}_i is compact;
2. $K \cap V_i = K_i \cap V_i$ and $K_i \subseteq U_i$;
3. There exists a C^m -function $f_i : U_i \rightarrow \mathbb{R}$, such that $j_x^m(f_i) = F(x)$, for each $x \in K_i$.

This cover is obtained from the cover $\{V_x\}_{x \in K}$ by the fact that this cover has a locally finite refinement (because $V = \bigcup_{x \in K} V_x$ is paracompact). Since the topology of V has a countable basis, such a locally finite cover is automatically countable. We can extend this cover of a neighbourhood of K to a cover of all of M by adding $V_0 = M \setminus K$. Let $\{\xi_i\}_{i=0}^\infty$ be a partition of unity for this cover of M , that is:

1. The support of ξ_i is contained in V_i ;
2. Each function ξ_i takes values in the interval $[0, 1]$;
3. For each $x \in M$ we have $\sum_{i=0}^\infty \xi_i(x) = 1$.

For details about the existence of locally finite covers and partitions of unity, compare with [27, 20, 31]. For similar arguments in a simpler setting, see section 2.3.

Finally, we define the global extension by

$$f(x) = \sum_{i=0}^{\infty} \xi_i(x) \cdot f_i(x) \text{ for } x \in M, \quad (5)$$

where we take $f_0(x) \equiv 0$ and $f_i(x) = 0$ for $x \notin U_i$. Since the cover $\{V_i\}_{i=0}^\infty$ is locally finite, the infinite sum, defining f , is well-defined.

To prove that f is a smooth extension of F , we have to show that the m -jet of f , defined by (5), at each point $x \in K$, equals $F(x)$. So let $x \in K$ and let

$$J_x = \{i \mid x \in K_i \cap V_i\} = \{i \mid x \in V_i\}.$$

Then, in a small neighbourhood of x we have that $f = \sum_{i \in J_x} \xi_i \cdot f_i$. The m -jet of this function is determined by the functions ξ_i and the m -jets of f_i in x for $i \in J_x$. By construction, the m -jets $j_x^m(f_i)$, for $i \in J_x$, are all equal. This means that, for the calculation of the m -jet $j_x^m(f)$, we may replace all the f_i by one and the same f_{i_0} for some $i_0 \in J_x$. Then we make use of the fact that near x the sum $\sum_{i \in J_x} \xi_i$ is identically equal to 1. So $j_x^m(f) = j_x^m(f_{i_0}) = F(x)$ by construction. This means that f is indeed a smooth extension of F as desired. \square

6.2 Near identity maps

Again let M be a smooth (C^∞) manifold. For the following construction we need to fix a smooth Riemannian metric on M (not necessarily complete). With such a metric the exponential map $\text{Exp} : T(M) \rightarrow M$ is defined, at least in a neighbourhood of the 0-section in $T(M)$. The definition is as follows: For $v \in T_x(M)$ one considers the (maximal) geodesic $\gamma_v : (-a, b) \rightarrow M$ such that $\gamma_x(0) = x$ and $\gamma'_x(0) = v$. For v small enough, 1 is in the domain $(-a, b)$. Then we define $\text{Exp}(v) = \gamma_v(1)$.

It is well-known [31] that Exp is smooth and that Exp induces a diffeomorphism from a neighbourhood of the 0-section in $T(M)$ to a neighbourhood of the diagonal in $M \times M$ by mapping $v \in T_x(M)$ to the pair $(x, \text{Exp}(v))$. We denote such a neighbourhood of the 0-section in $T(M)$ by W_1 and the corresponding neighbourhood of the diagonal in $M \times M$ by W_2 . Without loss of generality we may assume that for each x the intersection $T_x(M) \cap W_1$ is convex. We say that a map Φ on M is *near the identity* if for each $x \in M$ the pair $(x, \Phi(x))$ belongs to W_2 . In fact this definition fixes a C^0 -neighbourhood of the identity in the strong Whitney topology [20, 27]. We say that a section X in $T(M)$ is small if, for each $x \in M$, $X(x)$ belongs to W_1 . From this construction it follows that there is a one to one correspondence between small sections X in $T(M)$ and near identity maps on N : to X corresponds the map $x \mapsto \text{Exp}(X(x))$.

In section 6.1 we dealt with Whitney extensions of functions. From this we obtain a direct generalization to mappings with values in \mathbb{R}^n , and subsequently to sections in a vector bundle over M . By the above construction the Whitney extension theorem then generalizes to near identity maps. The only observation to justify the latter generalization is that due to the convexity of each $T_x(M) \cap W_1$ as above, a section $X(x) = \sum_i \xi_i(x) \cdot X_i(x)$, where the ξ_i form a partition of unity as described before, is small whenever all the sections X_i are small.