Analisi Complessa–Final examination (sample)

Exercise 1. Let D be a region of $\mathbb{C}$ and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions holomorphic on D. Assume that the sequence $g_m = \prod_{k=0}^{m} f_k$ converges compactly to some non identically zero $g \in \mathcal{O}(D)$. Then for every compact subset K of D there exists $n_K \in \mathbb{N}$ such that for $n \geq n_K$ the function f_n has no zero in K; moreover $g(z) = 0$ if and only if $f_n(z) = 0$ for some $n \in \mathbb{N}$.

Exercise 2. State and prove the pole shifting theorem. Let D be an open region, and assume that the polynomials are dense in $\mathcal{O}(D)$, with respect to the topology of compact convergence. Is it true that D is simply connected?

Analisi Complessa–Final examination (sample II)

Exercise 1. Let D be a region of $\mathbb{C}$. Denote by $\mathcal{O}'(D)$ the set of holomorphic functions on D which are derivatives of some function holomorphic on D. Is it true that $\mathcal{O}'(D)$ is closed in $\mathcal{O}(D)$ (in the compact–open topology, of course)? Under which condition on D we have $\mathcal{O}'(D) = \mathcal{O}(D)$?

Fix $c \in D$; prove that for every $f \in \mathcal{O}'(D)$ there exists a unique $I_c f \in \mathcal{O}(D)$ such that $(I_c f)' = f$ and $I_c f(c) = 0$, and show that the map $f \mapsto I_c f$ is continuous.Give a simple description of the map $f \mapsto I_c f'$.

Exercise 2. *(this is extracted from the lecture notes)*

(i) Let A and B be open subsets of $\mathbb{C}$, and let $\varphi : A \to B$ be a homeomorphism of A onto B. Assume that z_j is a sequence in A which converges to a point $a \in \partial A$, and that $\varphi(z_j)$ converges to a point $b \in \mathbb{C}$. Prove that $b \in \partial B$.

(ii) Let $A = D \setminus \{0\} = \{z \in \mathbb{C} : 0 < |z| < 1\}$ be the punctured unit disc, and let B be the annulus $B = \{z \in \mathbb{C} : 1 < |z| < 2\}$. Prove that there exists no holomorphic isomorphism of A onto B (Hint: for a holomorphic mapping $f : A \to B$, 0 is a removable singularity ... use then (i) ...).

(iii) Define explicitly a diffeomorphism of A onto B, with A, B as in (ii).

(iv) State the Riemann mapping theorem, and give a sketch of the proof.

Please put your name on the list if you are coming to the examination on march 23.