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The $\bar{\partial}$-Neumann problem

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Abstract. The $\bar{\partial}$-Neumann problem is probably the most important and natural example of a non-elliptic boundary value problem, arising as it does from the Cauchy-Riemann system. The main tool to prove regularity of solution in of study of this problem are L^2-estimates : subelliptic estimates, superlogarithmic estimates, compactness estimates...
In the first part of this note, we give motivation and classical results on this problem. In the second part, we introduce general estimates for "gain of regularity" of solutions of this problem and relate it to the existence of weights with large Levi-form at the boundary.
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1 The $\bar{\partial}$-Neumann problem and classical results

1.1 The $\bar{\partial}$- Neumann problem

Let $z_1, ..., z_n$ be holomorphic coordinates in $\mathbb{C}^n$ with $x_j = \text{Re}(z_j), y_j = \text{Im}(z_j)$. Then the holomorphic and anti-holomorphic vector fields are

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - \sqrt{-1}\frac{\partial}{\partial y_j}\right) \qquad \text{and} \qquad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + \sqrt{-1}\frac{\partial}{\partial y_j}\right).$$

Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with smooth boundary $b\Omega$. Given functions $\alpha_1, ..., \alpha_n$ on Ω, the problem of solving the equations

$$\frac{\partial v}{\partial \bar{z}_j} = \alpha_j, \qquad j = 1, ..., n \tag{1.1}$$

and studying the regularity of the solution is called *the inhomogeneous Cauchy-Riemann equations.*

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We must assume that the $(\alpha_j)_{j=1}^n$ satisfy **the compatibility condition**

$$\frac{\partial \alpha_i}{\partial \bar{z}_j} - \frac{\partial \alpha_j}{\partial \bar{z}_i} = 0 \qquad \text{for all } i, j = 1, ..., n \tag{1.2}$$

Let $\alpha = \sum \alpha_j d\bar{z}_j$ be (0,1)-form, we rewrite (1.1) and by $\bar{\partial} v = \alpha$ and (1.2) by $\bar{\partial}\alpha = 0$. The inhomogeneous Cauchy-Riemann equations are also called *the $\bar{\partial}$-problem.*

Question: Is there a solution $v \in C^\infty(U \cap \bar{\Omega})$ if the datum α_j belongs to $C^\infty(U \cap \bar{\Omega})$ (*local regularity*)?

The regular properties of v in the interior are well known (see next section). Regularity of v on the boundary is more delicate. Notice that not all solutions are smooth : in fact, not all holomorphic functions in Ω are smoothly extended to $\bar{\Omega}$. If h is such a function and v is a smooth solution, then $v + h$ is also *a solution* and not smooth on the closed domain since $\bar{\partial}(v + h) = \bar{\partial} v = \alpha$. So we do not look for *a solution* but for *the solution.* The optimal solution (the one of smallest in L_2-norm) is the solution orthogonal to the holomorphic functions; this is called the *canonical solution.* It is not known whether the canonical is smooth even if there is a smooth solution.

Moreover, the regularity of canonical solution of the $\bar{\partial}$-problem has some applications in SCV such as : Levi problem, Bergman projection, Holomorphic mappings, ...

The aim of the $\bar{\partial}$-Neumann problem is to study the canonical solution of $\bar{\partial}$-problem.

Before stating the $\bar{\partial}$-Neumann problem, we need definition of $\bar{\partial}^*$ the L_2-adjoint of $\bar{\partial}$. The $\bar{\partial}^*$ is defined as follows: Let

$$u = \sum u_j d\bar{z}_j \in \mathrm{Dom}(\bar{\partial}^*) \cap C^\infty(\Omega)^1,$$

and $\bar{\partial}^* u = g$ if

$$(w, g) = (\bar{\partial} w, u) \quad \text{for all} \quad w \in C^\infty(\bar{\Omega}).$$

We see that $\bar{\partial} w = \sum \frac{\partial w}{\partial \bar{z}_j} d\bar{z}_j$ and

$$(\bar{\partial} w, u) = \sum (\frac{\partial w}{\partial \bar{z}_j}, u_j) \overset{\text{Stokes}}{=} -\sum (w, \frac{\partial u_j}{\partial z_j}) + \int_{b\Omega} \sum w \overline{u_j \frac{\partial r}{\partial z_j}} dS.$$

Hence $\sum w \overline{u_j \frac{\partial r}{\partial z_j}} = 0$ on $b\Omega$ for all w. i.e. $u_j \frac{\partial r}{\partial z_j} = 0$ on $b\Omega$. The condition $u \in \mathrm{Dom}(\bar{\partial}^*)$ impies boundary condition on u.

We now state the $\bar{\partial}$-Neumann problem : given $\alpha \in L_2(\Omega)^1$, find $u \in L_2(\Omega)^1$ such that

$$\begin{cases} (\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial})u = \alpha \\ u \in \mathrm{Dom}(\bar{\partial}) \cap \mathrm{Dom}(\bar{\partial}^*) \\ \bar{\partial} u \in \mathrm{Dom}(\bar{\partial}^*), \bar{\partial}^* u \in \mathrm{Dom}(\bar{\partial}). \end{cases} \tag{1.3}$$

The $\bar\partial$-Neumann problem is a boundary value problem; the Laplacian $\Box = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ equation is elliptic, but the boundary conditions (i.e. $u \in \mathrm{Dom}(\bar\partial^*); \bar\partial u \in \mathrm{Dom}(\bar\partial^*)$) make the problem not elliptic. If (1.3) has a solution for every α, then one defines the $\bar\partial$-Neumann operator $N := \Box^{-1}$, this commutes both with $\bar\partial$ and $\bar\partial^*$.

If $\bar\partial\alpha = 0$, we define $v = \bar\partial^* N\alpha$ then v is the canonical solution of the $\bar\partial$-problem. In fact,

$$\bar\partial v = \bar\partial\bar\partial^* N\alpha = \bar\partial\bar\partial^* N\alpha + \bar\partial^*\bar\partial N\alpha = \Box N\alpha = \alpha$$

and $(v, h) = (\bar\partial^* N\alpha, h) = (N\alpha, \bar\partial h) = 0$ for any h holomorphic. (i.e. $v \perp \mathrm{Ker}\bar\partial$).

Question: What geometric conditions on $b\Omega$ guarantee the existence and regularity of solution of $\bar\partial$-Neumann problem?

1.2 Existence and regularity

Let us introduce the weighted L_2-norms. If $\phi \in C^\infty(\Omega)$ and $u \in L_2(\Omega)^1$ define

$$\|u\|_\phi^2 = (u,u)_\phi = \|ue^{-\frac{\phi}{2}}\|^2 = \int_\Omega |u|^2 e^{-\phi} dV.$$

Denote $\bar\partial_\phi^*$ the adjoint of $\bar\partial$ in this weighted inner product. Using integration by part, we obtain the basic identity:

Theorem 1.1 [Morry-Kohn-Hormander]

$$\begin{aligned} \|\bar\partial u\|_\phi^2 + \|\bar\partial_\phi^* u\|_\phi^2 = & \sum_{ij} \|\frac{\partial u_i}{\partial \bar z_j}\|_\phi^2 + \int_\Omega \sum \frac{\partial^2\phi}{\partial z_i \partial \bar z_j} u_i \bar u_j e^{-\phi} dV \\ & + \int_{b\Omega} \sum \frac{\partial^2 r}{\partial z_i \partial \bar z_j} u_i \bar u_j e^{-\phi} dS, \end{aligned} \tag{1.4}$$

for any $u = \sum u_j d\bar z_j \in \mathrm{Dom}(\bar\partial^*) \cap C^\infty(\bar\Omega)^1$.

Then if Ω is pseudoconvex, i.e., $\sum \frac{\partial^2 r}{\partial z_i \partial \bar z_j} u_i \bar u_j \geq 0$ on $b\Omega$, and $\phi = C|z|^2$ for some suitable constant C. we obtain

$$(\Box u, u) = \|\bar\partial u\|^2 + \|\bar\partial^* u\|^2 \gtrsim \|u\|^2, \tag{1.5}$$

and hence

$$\|\Box u\| \gtrsim \|u\|.$$

Thus, the $\bar\partial$-Neumann problem is solvable in L_2-norm on any smooth, bounded, pseudo-convex domain.

Denote $Q(u,u) = \|\bar\partial u\|^2 + \|\bar\partial^* u\|^2$.

Regularity in the interior: For $u \in C_0^\infty(\Omega)$, then $\|\frac{\partial u}{\partial \bar{z}_j}\|^2 = \|\frac{\partial u}{\partial z_j}\|^2$. For $\phi = 0$, (1.4) implies that

$$Q(u,u) \gtrsim \|\frac{\partial u}{\partial \bar{z}_j}\|^2 = \frac{1}{2}\Big(\|\frac{\partial u}{\partial \bar{z}_j}\|^2 + \|\frac{\partial u}{\partial z_j}\|^2\Big)$$

Combining with (1.5), we get $Q(u,u) \geq \|u\|_1^2$ (*elliptic estimate*).

Then we get regularity property in the interior. So our interest is confined to boundary $b\Omega$.

Regularity at the boundary? The main methods used in investigating of the regularity at the boundary of the solution of $\bar{\partial}$-Neumann problem consist in non-elliptic estimates : subelliptic, superlogarithmic and compactness estimates.

Let $z_0 \in b\Omega$. Suppose U is a neighborhood of z_0. Consider the local boundary coordinates (defined on U) denote by $(t,r) = (t_1, ..., t_{2n-1}, r) \in \mathbb{R}^{2n-1} \times \mathbb{R}$ where r is defining function of Ω.

For $\varphi \in C^\infty(\bar{\Omega} \cap U)$, the tangential Fourier transform of φ, defined by

$$\mathcal{F}_t\varphi(\xi, r) = \int_{\mathbb{R}^{2n-1}} e^{-i\langle t,x\rangle}\varphi(t,r)dt.$$

The standard tangential pseudo-differential operator is expressed by

$$\Lambda\varphi(t,r) = \mathcal{F}_t^{-1}\Big((1+|\xi|^2)^{1/2}\mathcal{F}_t\varphi(\xi,r)\Big).$$

Classes of non-elliptic estimates for the $\bar{\partial}$-Neum. prob. in a neighborhood U of $z_0 \in b\Omega$ are defined by

Definition 1.2

(i) Subelliptic estimate: there is a positive constant ϵ such that

$$||\Lambda^\epsilon u||^2 \lesssim Q(u,u);$$

(ii) Superlogarithmic estimate: for any $\eta > 0$ there is a positive constant C_η such that

$$||\log \Lambda u||^2 \leq \eta Q(u,u) + C_\eta \|u\|_0^2;$$

(iii) Compactness estimate: for any $\eta > 0$ there is a positive constant C_η such that

$$||u||^2 \lesssim \eta Q(u,u) + C_\eta \|u\|_{-1}^2;$$

for any $u \in C_c^\infty(\bar{\Omega} \cap U) \cap \mathrm{Dom}(\bar{\partial}^*)$.

Remark that subelliptic estimate $\Rightarrow$ Superlogarithmic estimate $\Rightarrow$ Compactness estimate.

Let us recall the following results.

Theorem 1.3

(i) [Folland-Kohn 72] *Subelliptic estimate implies local regularity. Moreover,* $\Box u \in H^s(V) \Rightarrow u \in H^{s+2\epsilon}(V')$ *for* $V' \subset V$*, where* ϵ *is order of subellipticity.*

(ii) [Kohn 02] *Superlogarithmic estimate implies local regularity. Moreover,* $\Box u \in H^s(V) \Rightarrow u \in H^s(V')$ *for* $V' \subset V$*.*

(iii) [Kohn-Nirenberg 65] *Compactness estimate over a covering* $\cup U_j \supset b\Omega$*, implies global regularity. Moreover* $\Box u \in H^s(\bar{\Omega}) \Rightarrow u \in H^s(\bar{\Omega})$*.*

Remark: Compactness estimate $\not\Rightarrow$ local regularity (see [Ch02]).

1.3 Geometric condition

When Ω is pseudoconvex, a great deal of work has been done about subelliptic estimates. The most general results have been obtained by Kohn and Catlin.

Theorem 1.4 [Kohn, Ann. of Math, 63-64] *Strongly pseudoconvex* $\Leftrightarrow$ $\frac{1}{2}$*-subelliptic estimate.*

Strongly pseudoconvex : $\partial\bar{\partial}r > 0$, on $b\Omega$ when $u \in \mathrm{Dom}(\bar{\partial}^*)$.

Definition 1.5 Finite type (D'Angelo finite type):

$$D(z_0) = \sup \frac{\mathrm{ord}_{z_0}(r(\phi))}{\mathrm{ord}_0 \phi}$$

where supremum is taken over all local holomorphic curves $\phi : \Delta \to \mathbb{C}^n$ with $\phi(0) = z_0$.

Examples of finite type :

(a) Strongly pseudoconvex $\Leftrightarrow D(z_0) = 2$.

(b) $r = \mathrm{Re} z_n + \sum |z_j|^{2m_j}$ in $\mathbb{C}^n$, then $D(z_0) = 2\max\{m_j\}$.

(c) $r = \mathrm{Re} z_3 + |z_2^b - z_1|^2$, then $D_(z_0) = \infty$, since $\mathcal{C} : \{z_2^b - z_1 = 0, z_3 = 0\} \subset b\Omega$.

(d) $r = \mathrm{Re} z_3 + |z_1|^{2a} + |z_2^b - z_1|^2$, then $D(z_0) = 2ab$.

(e) $r = \mathrm{Re} z_n + \exp(-\frac{1}{|z_j|^s})$ in $\mathbb{C}^n$, then $D(z_0) = \infty$.

Theorem 1.6 [Kohn 79, Acta Math.] *Let* Ω *be pseudoconvex+real analytic+finite type at* $z_0 \in \Omega$*. Then subelliptic estimate hold at* z_0*.*

Theorem 1.7 [Caltin 84-87, Ann. of Maths] *Let* Ω *be pseudoconve+finite type at* $z_0 \in \Omega$*. Then subelliptic estimate hold at* z_0*. Moreover*

$$\epsilon \leq \frac{1}{D(z_0)}.$$

Denote $S_\delta = \{z \in \mathbb{C}^n : -\delta < r < 0\}$.

One of main steps in Catlin's proof is the following reduction:

Theorem 1.8 *Suppose that $\Omega \subset\subset \mathbb{C}^n$ is a pseudoconvex domain defined by $\Omega = \{r < 0\}$, and that $z_o \in b\Omega$. Let U is a neighborhood of z_o. Suppose that there is a family smooth real-valued function $\{\Phi^\delta\}_{\delta>0}$ satisfying the properties:*

$$\begin{cases} |\Phi^\delta| \leq 1 \text{ on } U, \\ \partial\bar{\partial}\Phi^\delta \gtrsim 0 \text{ on } U, \\ \partial\bar{\partial}\Phi^\delta \gtrsim \delta^{-2\epsilon} \text{ on } U \cap S_\delta \end{cases}$$

Then there is a subelliptic estimate of order ϵ at z_o.

Remark 1.9 Superlogarithmic estimate for the $\bar{\partial}$-Neumann problem was first introduced by [Koh02]. Supperlogarithmic estimate might hold on some class of infinite type domains.

Remark 1.10 A great deal of work has been done on pseudoconvex domains, however, not much is known in the non-pseudoconvex case except from the results related to the celebrated $Z(k)$ condition [Hor65], [FK72] and the case of top degree $n-1$ of the forms due to Ho [Ho85].

2 f-estimates on q-pseudoconvex/concave domain

2.1 The q-pseudoconvex/concave domain

Let $\lambda_1 \leq \cdots \leq \lambda_{n-1}$ be the eigenvalue of $\partial\bar{\partial}r$ on $b\Omega$, for a pair of indices q_0, q ($q_0 \neq q$) suppose that

$$\sum_{j=1}^{q} \lambda_j - \sum_{j=1}^{q_0} r_{jj} \geq 0 \qquad \text{on } b\Omega. \tag{2.1}$$

Definition 2.1

(i) If $q > q_0$, we say that Ω is q-pseudoconvex at z_0.

(ii) If $q < q_0$, we say that Ω is q-pseudoconcave at z_0.

Special case:

(a) $q_0 = 0, q = 1 : \lambda_1 \geq 0$, this means $\partial\bar{\partial}r \geq 0$, that is, 1-pseudoconvex $\equiv$ pseudoconvex.

(b) $q_0 = n-1, q = n-2$, then (n-2)-pseudoconcave $\equiv$ pseudoconcave.

2.2 Main Theorem

Theorem 2.2 *Let* Ω *be q-pseudoconvex* (resp. *q-pseudoconcave*) *domain at* z_0*. Let* U *is a neighborhood of* z_0*. Assume that there exists a family function* $\{\Phi^\delta\}_{\delta>0}$*, satisfying properties*

$$\begin{cases} |\Phi^\delta| \leq 1 \\ \sum_{j=0}^{q} \lambda_j^{\Phi^\delta} - \sum_{j=1}^{q_0} \Phi^\delta_{jj} \gtrsim f(\frac{1}{\delta})^2 \end{cases} \quad \text{on } S_\delta \cap U$$

Then the estimate

$$||f(\Lambda)u||^2 \lesssim Q(u,u) \tag{2.2}$$

holds for any form with degree $k \geq q$ *(resp.* $k \leq q$*) where* $f(\Lambda)$ *is the operator with symbol* $f((1+|\xi|^2)^{\frac{1}{2}})$ *. Moreover,*

(i) *if* $\lim_{\delta\to 0} \frac{f(\frac{1}{\delta})}{(\frac{1}{\delta})^\epsilon} \geq C > 0$ *then (2.2) implies* ϵ*-subelliptic estimate;*

(ii) *if* $\lim_{\delta\to 0} \frac{f(\frac{1}{\delta})}{\log\frac{1}{\delta}} = +\infty$ *then (2.2) implies superlogarithmic estimate;*

(iii) *if* $\lim_{\delta\to 0} f(\frac{1}{\delta}) = +\infty$ *then (2.2) implies compactness estimates.*

Remark 2.3 Superlogarithmic estimates were never handled in this way, but by Kohn's subelliptic multipliers

Remark 2.4 Catlin only used his weight on *finite type* domains.

Remark 2.5 We get a generalization : Pseudoconvex $\rightsquigarrow$ q-Pseudoconvex/concave

2.3 Construction of the Catlin's weight

Strongly pseudoconvex domain: Let Ω be strongly pseudoconvex at z_0 then $\partial\bar\partial r > 0$ in a n.b.h. of z_0. Define

$$\Phi^\delta = -\log(-\frac{r}{\delta} + 1)$$

Then

$$\partial\bar\partial\Phi^\delta \gtrsim \frac{1}{\delta}\partial\bar\partial r \gtrsim \delta^{-2\frac{1}{2}} \qquad z \in S_\delta.$$

We get $\frac{1}{2}$-subelliptic estimates on this class of domain.

Domain satisties $Z(k)$ condition:

Definition 2.6 [Z(k) condition] Ω satisfies $Z(k)$ condition if $\partial\bar{\partial}r$ has either at least $(n-k)$ positive eigenvalues or at least $(k+1)$ negative eigenvalues.

Theorem 2.7 *Let Ω be a domain of $\mathbb{C}^n$ which satisfies $Z(k)$ condition, then*

$$|||u|||_{1/2}^2 \lesssim Q(u,u)$$

holds for any u of degree k.

This is classical result of non-pseudoconvex domain. This theorem can be found in [FK72]. We give a new way to get $\frac{1}{2}$-subelliptic estimates by construction the family of Catlin's weight functions.

If Ω satisfies $Z(k)$ condition, then Ω is strongly k-pseudoconvex or strongly k-pseudoconcave. We define

$$\Phi^\delta = -\log(\frac{-r}{\delta} + 1).$$

Similarly in the case strongly pseudoconvex, we get $\frac{1}{2}$-subelliptic estimates for any form of degree k.

Decoupled domain:

Theorem 2.8 *Let $\Omega \in \mathbb{C}^2$ be defined by*

$$r = Rew + P(z) < 0$$

where P is a subharmornic non-harmonic function. Further, suppose that there is an invertible function F with $\frac{F(|z|)}{|z|^2}$ increasing such that

$$\partial\bar{\partial}P(z) \gtrsim \frac{F(|z|)}{|z|^2}.$$

Then, f-estimate holds with $f(\delta^{-1}) = (F^{-1}(\delta))^{-1}$.

Example 2.1 If $P(z) = |z|^{2m}$,then $F(\delta) = \delta^{2m} \Rightarrow f(\delta^{-1}) = \delta^{-\frac{1}{2m}} \Rightarrow \frac{1}{2m}$-subelliptic estimate.

Example 2.2 If $P(z) = \exp(-\frac{1}{|z|^s})$, then $F(\delta) = \exp(-\frac{1}{\delta^s}) \Rightarrow f(\delta^{-1}) = \left(\log\frac{1}{\delta}\right)^{1/s} \Rightarrow f$-estimate holds for this f. So, if $0 < s < 1$ then $\lim_{\xi\to\infty}\frac{f(|\xi|)}{\log|\xi|} = \infty$, we get superlogarithmic estimate. Furthermore, we obtain compactness estimate for any $s > 0$.

Sketch of the proof of Theorem 2.8. Define

$$\Phi^\delta = -\frac{r}{\delta} + \log(|z|^2 + f(\delta^{-1})^{-2}).$$

Then on S_δ,

$$\partial\bar\partial\Phi^\delta \gtrsim \delta^{-1}\frac{F(|z|)}{|z|^2} + \frac{f(\delta^{-1})^{-2}}{(|z|^2+f(\delta^{-1})^{-2})^2}.$$

If $|z| \geq f(\delta^{-1})^{-1}$,(e.i. $= f(\delta^{-1})^{-1} = F^{-1}(\delta)$) then

$$(I) \geq \delta^{-1}\frac{F(F^{-1}(\delta))}{f(\delta^{-1})^{-2}} = f(\delta^{-1})^2.$$

Otherwise, if $|z| \leq f(\delta^{-1})^{-1}$, then

$$(II) \gtrsim f(\delta^{-1})^2$$

So that $\partial\bar\partial\Phi^\delta \gtrsim f(\delta^{-1})^2$ on S_δ.

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