

On the principal eigenvalue of elliptic operators in \mathbb{R}^N and applications

Henri Berestycki ^{a,*} and Luca Rossi ^{a,b}

^a EHESS, CAMS, 54 Boulevard Raspail, F-75006 Paris, France

^b Università La Sapienza Roma I, Dipartimento di Matematica,
Piazzale Aldo Moro 2, I-00185 Roma, Italy

Abstract

Two generalizations of the notion of principal eigenvalue for elliptic operators in \mathbb{R}^N are examined in this paper. We prove several results comparing these two eigenvalues in various settings: general operators in dimension one; self-adjoint operators; and “limit periodic” operators. These results apply to questions of existence and uniqueness for some semi-linear problems in all of space. We also indicate several outstanding open problems and formulate some conjectures.

1 Introduction

The principal eigenvalue is a basic notion associated with an elliptic operator. For instance, the study of semi-linear elliptic problems in bounded domains often involves the principal eigenvalue of an associated linear operator. To motivate the results of the present paper, let us first recall some classical properties of a class of semi-linear elliptic problems in bounded domains.

Let $-L$ be a linear elliptic operator acting on functions defined on a bounded and smooth domain $\Omega \subset \mathbb{R}^N$:

$$Lu = a_{ij}(x) \partial_{ij} u + b_i(x) \partial_i u + c(x) u$$

(here and throughout the paper, the convention is adopted for summation on repeated indices).

Consider the Dirichlet problem:

$$\begin{cases} -Lu = g(x, u), & x \in \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (1.1)$$

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We are interested in positive solutions of (1.1). Assume that g is a C^1 function such that:

$$g(x, s) < g'_s(x, 0)s, \quad \forall s > 0, \quad \forall x \in \Omega$$

and:

$$\exists M > 0 \quad \text{such that} \quad g(x, s) + c(x)s \leq 0, \quad \forall s \geq M$$

Then, existence of positive solutions of (1.1) is determined by the principal eigenvalue μ_1 of the problem linearized about $u = 0$:

$$\begin{cases} -L \varphi - g_s(x, 0) \varphi = \mu_1 \varphi & \text{in } \Omega \\ \varphi = 0 & \text{on } \partial\Omega \end{cases} \quad (1.2)$$

Recall that μ_1 is characterized by the existence of an associated eigenfunction $\varphi > 0$ of (1.2). It is known indeed that (1.1) has a positive solution if and only if $\mu_1 > 0$ (see e.g. [2]). Under the additional assumption that $s \mapsto g(x, s)/s$ is decreasing, one further obtains a uniqueness result [2].

Problems of the type (1.1) arise in several contexts, in particular in population dynamics. In many of the applications, the problem is set in an unbounded domain, often in \mathbb{R}^N . Clearly, extensions to unbounded domains of the previous result, as well as others of the same type, require one to understand the generalizations and properties of the notion of principal eigenvalue of elliptic operators in unbounded domains. In Section 2, we indicate some new results about such a semi-linear problem, extending the result for (1.1).

Another example of use of principal eigenvalue is the characterization of the existence of the Green function for linear periodic operators (see Agmon [1]). We refer to [18] and to its bibliography for details on the subject. Moreover, the principal eigenvalue of an elliptic operator has been shown to play an important role in some questions in branching processes (see Englander and Pinsky [9], Pinsky [19]). Very recently, the principal eigenvalue of an elliptic operator in \mathbb{R}^N is being introduced in the context of economic models [11].

Some definitions of the notion of principal eigenvalue in unbounded domains have emerged in the works of Agmon [1], Berestycki, Nirenberg and Varadhan [6], Pinsky [18] and others. With a view to applications to semi-linear equations, in particular two definitions have been used in [3], [4], [5]. We will recall these definitions later in this section. In this paper, we examine these definitions and further investigate their properties. In particular, we are interested in understanding when the two definitions coincide or for which classes of operators does one or the other inequality hold. We also further explain the choice of definition. This work grew out of our previous article with François Hamel [5] which already addressed some of these issues. We review the relevant results from [5] in Section 3.

Let us now recall the definitions. In the sequel, we define the class of elliptic operators (in non-divergence form) as the elliptic operators $-L$ with

$$Lu = a_{ij}(x)\partial_{ij}u + b_i(x)\partial_i u + c(x)u, \quad \text{in } \mathbb{R}^N.$$

Self-adjoint elliptic operators $-L$, are defined by:

$$Lu = \partial_i(a_{ij}(x)\partial_j u) + c(x)u, \quad \text{in } \mathbb{R}^N,$$

(we use the usual summation convention on repeated indices). Throughout the paper, $(a_{ij})_{ij}$ will denote a $N \times N$ symmetric matrix field such that

$$\forall x, \xi \in \mathbb{R}^N, \quad \underline{a}|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \bar{a}|\xi|^2, \quad (1.3)$$

where \underline{a} and \bar{a} are two positive constants, $(b_i)_i$ will denote a N dimensional vector field and c a real valued function. We always assume that there exists $0 < \alpha \leq 1$ such that:

$$a_{ij}, b_i, c \in C_b^{0,\alpha}(\mathbb{R}^N), \quad (1.4)$$

in the case of general operators, and

$$a_{ij} \in C_b^{1,\alpha}(\mathbb{R}^N), \quad c \in C_b^{0,\alpha}(\mathbb{R}^N), \quad (1.5)$$

in the self-adjoint case. By $C_b^{k,\alpha}(\mathbb{R}^N)$, we mean the class of functions $\phi \in C^k(\mathbb{R}^N)$ such that ϕ and the derivatives of ϕ up to the order k are bounded and are uniformly Hölder continuous with exponent α . Notice that every self-adjoint operator satisfying (1.5) can be viewed as a particular case of general elliptic operator satisfying (1.4).

It is well known that any elliptic operator $-L$ as defined above admits a unique principal eigenvalue, either in bounded smooth domains associated with Dirichlet boundary conditions, as well as in \mathbb{R}^N provided that its coefficients are periodic in each variable. This principal eigenvalue is the bottom of the spectrum of $-L$ in the appropriate function space, and it admits an associated positive principal eigenfunction. This result follows from the Krein Rutman theory and from compactness arguments (see [14] and [13]).

In this paper, we examine some properties of two different generalizations of the principal eigenvalue in unbounded domains. The first one, originally introduced in [6], reads:

Definition 1.1 *Let $-L$ be a general elliptic operator defined in a domain $\Omega \subseteq \mathbb{R}^N$. We set*

$$\lambda_1(-L, \Omega) := \sup\{\lambda \mid \exists \phi \in C^2(\Omega) \cap C_{loc}^1(\bar{\Omega}), \phi > 0 \text{ and } (L + \lambda)\phi \leq 0 \text{ in } \Omega\}, \quad (1.6)$$

Here, $C_{loc}^1(\bar{\Omega})$ denotes the set of functions $\phi \in C^1(\Omega)$ for which ϕ and $\nabla\phi$ can be extended by continuity on $\partial\Omega$, but which are not necessarily bounded. Berestycki, Nirenberg and Varadhan showed that this is a natural generalization of the principal eigenvalue. Indeed, if Ω is bounded and smooth, then $\lambda_1(-L, \Omega)$ coincides with the principal eigenvalue of $-L$ in Ω , with Dirichlet boundary conditions. As we will see later, the eigenvalue λ_1 does not suffice to completely describe the properties of semi-linear equations in the whole space, in contradistinction with the Dirichlet principal eigenvalue in bounded domains for problem (1.1). For this, we also require another generalization, whose definition is similar to that of λ_1 . This generalization has been introduced in [3], [5] and reads:

Definition 1.2 *Let $-L$ be a general elliptic operator defined in a domain $\Omega \subseteq \mathbb{R}^N$. We set*

$$\begin{aligned} \lambda'_1(-L, \Omega) := \inf\{\lambda \mid \exists \phi \in C^2(\Omega) \cap C_{loc}^1(\bar{\Omega}) \cap W^{2,\infty}(\Omega), \phi > 0 \text{ and } -(L + \lambda)\phi \leq 0 \text{ in } \Omega, \\ \phi = 0 \text{ on } \partial\Omega, \quad \text{if } \partial\Omega \neq \emptyset\}. \end{aligned} \quad (1.7)$$

Several other generalizations are possible, starting from Definition 1.1 and playing on the space of functions or the inf and sup inequalities. We will explain why Definition 1.2 is relevant.

If L is periodic (in the sense that its coefficients are periodic in each variable, with same period) then $\lambda_1(-L, \mathbb{R}^N) \geq \lambda'_1(-L, \mathbb{R}^N)$, as is shown by taking ϕ equal to a positive periodic principal eigenfunction in (1.6) and (1.7). More generally, if there exists a bounded positive eigenfunction φ , then $\lambda_1 \geq \lambda'_1$. But in general, if the operator L is not self-adjoint, equality need not hold between λ_1 and λ'_1 , even if L is periodic (see Section 3). It is then natural to ask what are the relations between λ_1 and λ'_1 in the general case. In Section 3, we review a list of statements, most of them given in [5], which answer this question in some particular cases. In Section 4, we state our new main results as well as some problems which are still open. In Section 5, we motivate our choice of taking (1.6) and (1.7) as generalizations of the principal eigenvalue. The last three sections are dedicated to the proofs of our main results.

2 Positive solutions of semi-linear elliptic problems in \mathbb{R}^N

Let us precisely describe how the eigenvalues λ_1 and λ'_1 are involved in the study of the following class of nonlinear problems:

$$-a_{ij}(x)\partial_{ij}u(x) - b_i(x)\partial_i u(x) = f(x, u(x)), \quad \text{in } \mathbb{R}^N. \quad (2.8)$$

This type of problem arises in particular in biology and in population dynamics. Here and in the sequel, the function $f(x, s) : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$ is assumed to be in $C_b^{0,\alpha}(\mathbb{R}^N)$ with respect to the variable x , locally uniformly in $s \in \mathbb{R}$, and to be locally lipschitz-continuous in the variable s , uniformly in $x \in \mathbb{R}^N$. Furthermore, we always assume that f satisfies:

$$\forall x \in \mathbb{R}^N, \quad f(x, 0) = 0,$$

$$\exists \delta > 0 \text{ such that } s \mapsto f(x, s) \text{ belongs to } C^1([0, \delta]), \text{ uniformly in } x \in \mathbb{R}^N,$$

$$f_s(x, 0) \in C_b^{0,\alpha}(\mathbb{R}^N).$$

We will always denote with L_0 the linearized operator around the solution $u \equiv 0$ associated to the equation (2.8), that is:

$$L_0 u = a_{ij}(x)\partial_{ij}u + b_i(x)\partial_i u + f_s(x, 0)u, \quad \text{in } \mathbb{R}^N.$$

In [3] it is proved that, under suitable assumptions on f , if L_0 is self-adjoint and the functions a_{ij} and $x \mapsto f(s, x)$ are periodic (in each variable) with same period, then (2.8) admits a unique positive bounded solution if and only if the periodic principal eigenvalue of $-L_0$ is less than or equal to zero (see Theorems 2.1 and 2.4 in [3]). This result has been extended in [5] to non-periodic, non-self-adjoint operators, by using $\lambda_1(-L_0, \mathbb{R}^N)$ and $\lambda'_1(-L_0, \mathbb{R}^N)$ instead of the periodic principal eigenvalue of $-L_0$. The assumptions required are:

$$\exists M > 0, \forall x \in \mathbb{R}^N, \forall s \geq M, \quad f(x, s) \leq 0, \quad (2.9)$$

$$\forall x \in \mathbb{R}^N, \forall s \geq 0, \quad f(x, s) \leq f_s(x, 0)s. \quad (2.10)$$

The existence result of [5] is:

Theorem 2.1 *Let L_0 be the linearized operator around zero associated to equation (2.8).*

1) *If (2.9) holds and either $\lambda_1(-L_0, \mathbb{R}^N) < 0$ or $\lambda'_1(-L_0, \mathbb{R}^N) < 0$, then there exists at least one positive bounded solution of (2.8).*

2) *If (2.10) holds and $\lambda'_1(-L_0, \mathbb{R}^N) > 0$, then there is no nonnegative bounded solution of (2.8), other than the trivial one $u \equiv 0$.*

Theorem 2.1 follows essentially from Definitions 1.1, 1.2 and a characterization of λ_1 (see Theorem 5.1 and Propositions 6.1, 6.5 in [5] for details). In [9], Engländer and Pinsky proved a similar existence result for a class of solution of minimal growth (which they define there) for nonlinearities of the type $f(x, u) = b(x)u - a(x)u^2$, with $\inf a > 0$ (see also [8], [19]).

Since the theorem involves both λ_1 and λ'_1 , one does not have a simple necessary and sufficient condition. This is one of the motivations to investigate the properties of these two generalized eigenvalues. In particular, it is useful to determine conditions which yield equality between them or at least that yield an ordering.

From the results we prove in this paper we can deal in particular with the case that the operator is self-adjoint *limit periodic*. The notion of limit periodic operator is defined precisely below in section 4.2. Essentially, it means that the operator is the uniform limit (in the sense of coefficients) of a sequence of periodic operators. In this case, we still have a condition, extending that in theorem 2.1, which is nearly necessary and sufficient.

Theorem 2.2 *If $-L_0$ is a self-adjoint limit periodic operator, then we have:*

1) *If (2.9) holds and $\lambda_1(-L_0, \mathbb{R}^N) < 0$, then there exists at least one positive bounded solution of (2.8). If, in addition, (2.11) below holds, then such a solution is unique.*

2) *If (2.10) holds and $\lambda_1(-L_0, \mathbb{R}^N) > 0$, then there is no nonnegative bounded solution of (2.8), other than the trivial one $u \equiv 0$.*

The same result holds in dimension $N=1$ if L_0 is an arbitrary self adjoint operator.

The case of equality: $\lambda_1(-L_0, \mathbb{R}^N) = 0$ is open.

For uniqueness, in unbounded domains, one needs to replace the classical assumption that $s \mapsto f(x, s)/s$ is decreasing by the following one:

$$\forall 0 < s_1 < s_2, \quad \inf_{x \in \mathbb{R}^N} \left(\frac{f(x, s_1)}{s_1} - \frac{f(x, s_2)}{s_2} \right) > 0. \quad (2.11)$$

The uniqueness result of [5] is more delicate and involves the principal eigenvalue of some limit operators defined there. It becomes simpler to state in the case the coefficients in (2.8) are *almost periodic*, in the sense of the following definition:

Definition 2.3 *A function $g : \mathbb{R}^N \rightarrow \mathbb{R}$ is said to be almost periodic (a.p.) if from any sequence $(x_n)_{n \in \mathbb{N}}$ in \mathbb{R}^N one can extract a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ such that $g(x_{n_k} + x)$ converges uniformly in $x \in \mathbb{R}^N$.*

Theorem 2.4 (Theorem 1.5 in [5]) *Assume that the functions a_{ij} , b_i and $f_s(\cdot, 0)$ are a.p. If (2.11) holds and $\lambda_1(-L_0, \mathbb{R}^N) < 0$, then (2.8) admits at most one nonnegative bounded solution, besides the trivial one: $u \equiv 0$.*

Theorems 2.1 and 2.4 essentially contain the results in the periodic self-adjoint framework (which hold under the same assumptions (2.9), (2.10) and (2.11)). In that case, in fact, $\lambda_1(-L_0, \mathbb{R}^N)$ and $\lambda'_1(-L_0, \mathbb{R}^N)$ coincide with the periodic principal eigenvalue of $-L_0$ (see Proposition 3.3 below) and then the only case which is not covered is when the periodic principal eigenvalue is equal to zero.

3 Some properties of the generalized principal eigenvalues λ_1 and λ'_1 in \mathbb{R}^N

In this section, unless otherwise specified, $-L$ denotes a general elliptic operator. When we say that L is periodic, we mean that there exist N positive constants l_1, \dots, l_N such that

$$\forall x \in \mathbb{R}^N, \forall i \in \{1, \dots, N\}, \quad a_{ij}(x + l_i e_i) = a_{ij}(x), \quad b_i(x + l_i e_i) = b_i(x), \quad c(x + l_i e_i) = c(x),$$

where (e_1, \dots, e_N) is the canonical base of \mathbb{R}^N . The following are some of the known results concerning λ_1 and λ'_1 . Actually, in some statements of [5], the coefficients of L were in $C^{0,\alpha}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and the “test functions” ϕ in the definition of λ'_1 were taken in $C^2(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$ instead of $C^2(\mathbb{R}^N) \cap W^{2,\infty}(\mathbb{R}^N)$. However, one can check that the following results - as well as Theorem 2.1 - can be proved arguing exactly as in the proofs of the corresponding results in [5].

Proposition 3.1 ([6] and Proposition 4.2 in [5]) *Let Ω be a general domain in \mathbb{R}^N and $(\Omega_n)_{n \in \mathbb{N}}$ be a sequence of nonempty open sets such that*

$$\Omega_n \subset \Omega_{n+1}, \quad \bigcup_{n \in \mathbb{N}} \Omega_n = \Omega.$$

Then $\lambda_1(-L, \Omega_n) \searrow \lambda_1(-L, \Omega)$ as $n \rightarrow +\infty$.

Proposition 3.1 yields that $\lambda_1(-L, \mathbb{R}^N) < +\infty$. Furthermore, taking $\phi \equiv 1$ as a test function in (1.6), we see that $\lambda_1(-L, \mathbb{R}^N) \geq -\|c\|_\infty$. Thus, λ_1 is always a well defined real number.

In the case of L periodic, the periodic principal eigenvalue of $-L$ is defined as the unique constant such that there exists a positive periodic $\varphi \in C^2(\mathbb{R}^N)$ satisfying $(L + \lambda)\varphi = 0$ in \mathbb{R}^N . Its existence follows from the Krein Rutman theory.

Proposition 3.2 (Proposition 6.3 in [5]) *If L is periodic, then its periodic principal eigenvalue λ_p coincides with $\lambda'_1(-L, \mathbb{R}^N)$.*

It is known that, in the general non-self-adjoint case, $\lambda_1 \neq \lambda'_1$. Indeed, as an example, consider the one dimensional operator $-Lu = -u'' + u'$, which is periodic with arbitrary positive period. Then it is easily seen that

$$\lambda'_1(-L, \mathbb{R}) = 0 < \frac{1}{4} = \lambda_1(-L, \mathbb{R}^N).$$

In fact, since $\varphi \equiv 1$ satisfies $-L\varphi = 0$, it follows that the periodic principal eigenvalue of $-L$ is 0 and then, by Proposition 3.2, $\lambda'_1(-L, \mathbb{R}) = 0$. On the other hand, for any $R > 0$, the function

$$\varphi_R(x) := \cos\left(\frac{\pi}{2R}x\right) e^{\frac{1}{2}x}$$

satisfies: $-L\varphi_R = \left(\frac{1}{4} + \frac{\pi^2}{4R^2}\right)\varphi_R$, which shows that φ_R is a principal eigenfunction of $-L$ in $(-R, R)$, under Dirichlet boundary conditions. Therefore, by Proposition 3.1,

$$\lambda_1(-L, \mathbb{R}) = \lim_{R \rightarrow \infty} \left(\frac{1}{4} + \frac{\pi^2}{4R^2}\right) = \frac{1}{4} > \lambda'_1(-L, \mathbb{R}).$$

Proposition 3.3 (Proposition 6.6 in [5]) *If the elliptic operator $-L$ is self-adjoint and periodic, then $\lambda_1(-L, \mathbb{R}^N) = \lambda'_1(-L, \mathbb{R}^N) = \lambda_p$, where λ_p is the periodic principal eigenvalue of $-L$.*

For the sequel of this paper it is useful to recall the proof of the last statement.

Proof of Proposition 3.3. First, from Proposition 3.2 one knows that $\lambda_p = \lambda'_1(-L, \mathbb{R}^N)$. Now, let φ_p be a positive periodic principal eigenfunction of $-L$ in \mathbb{R}^N . Taking $\phi = \varphi_p$ in (1.6), it is straightforward to see that $\lambda_1(-L, \mathbb{R}^N) \geq \lambda_p$.

To show the reverse inequality, consider a family of cutoff functions $(\chi_R)_{R \geq 1}$ in $C^2(\mathbb{R}^N)$, uniformly bounded in $W^{2,\infty}(\mathbb{R}^N)$, such that $0 \leq \chi_R \leq 1$, $\text{supp}\chi_R \subset \overline{B}_R$ and $\chi_R = 1$ in B_{R-1} .

Fix $R > 1$ and call λ_R the principal eigenvalue of $-L$ in B_R . It is obtained by the following variational formula:

$$\lambda_R = \min \left\{ \frac{\int_{B_R} (a_{ij}(x)\partial_i v \partial_j v - c(x)v^2)}{\int_{B_R} v^2}; v \in H_0^1(B_R), v \neq 0 \right\}. \quad (3.12)$$

Taking $v = \varphi_p \chi_R$ as a test function in (3.12), and calling $C_R = B_R \setminus B_{R-1}$, we find:

$$\begin{aligned} \lambda_R &\leq -\frac{\int_{B_R} (L(\varphi_p \chi_R)) \varphi_p \chi_R}{\int_{B_R} \varphi_p^2 \chi_R^2} = \frac{\lambda_p \int_{B_{R-1}} \varphi_p^2 - \int_{C_R} (L(\varphi_p \chi_R)) \varphi_p \chi_R}{\int_{B_R} \varphi_p^2 \chi_R^2} \\ &= \lambda_p - \frac{\lambda_p \int_{C_R} \varphi_p^2 \chi_R^2 + \int_{C_R} (L(\varphi_p \chi_R)) \varphi_p \chi_R}{\int_{B_R} \varphi_p^2 \chi_R^2}. \end{aligned}$$

Since $\min \varphi_p > 0$, it follows that there exists $K > 0$, independent of R , such that

$$\int_{B_R} \varphi_p^2 \chi_R^2 \geq \int_{B_{R-1}} \varphi_p^2 \geq K(R-1)^N.$$

Consequently,

$$\lambda_R \leq \lambda_p + K' \frac{R^{N-1}}{(R-1)^N},$$

where K' is a positive constant independent of R . Letting R go to infinity and using Proposition 3.1, we get: $\lambda_1(-L, \mathbb{R}^N) \leq \lambda_p$ and therefore, $\lambda_1(-L, \mathbb{R}^N) = \lambda_p$. \square

The next result is an extension of the previous proposition. It is still about periodic operators, but which are not necessarily self-adjoint. A gradient type assumption on the first order coefficients is required.

Theorem 3.4 (Theorem 6.8 in [5]) *Consider the operator*

$$Lu := \partial_i(a_{ij}(x)\partial_j u) + b_i(x)\partial_i u + c(x)u, \quad x \in \mathbb{R}^N,$$

where a_{ij} , b_i , c are periodic in x , with the same period (l_1, \dots, l_N) , the matrix field $A(x) = (a_{ij}(x))_{1 \leq i, j \leq N}$ is in $C^{1,\alpha}(\mathbb{R}^N)$, elliptic and symmetric, the vector field $b = (b_1, \dots, b_N) \in C^{1,\alpha}(\mathbb{R}^N)$ and $c \in C^{0,\alpha}(\mathbb{R}^N)$. Assume that there is a function $B \in C^{2,\alpha}(\mathbb{R}^N)$ such that $a_{ij}\partial_j B = b_i$ for all $i = 1, \dots, N$ and assume that the vector field $A^{-1}b$ has zero average in the periodicity cell $C = (0, l_1) \times \dots \times (0, l_N)$. Then $\lambda_1(-L, \mathbb{R}^N) = \lambda_p = \lambda'_1(-L, \mathbb{R}^N)$, where λ_p is the periodic principal eigenvalue of $-L$ in \mathbb{R}^N .

Next, the natural question is to ask what happens when we drop the periodicity assumption. Up to now, the only available result had been obtained in [5] in the case of dimension one. It stated:

Proposition 3.5 (Proposition 6.11 in [5]) *Let $-L$ be a self-adjoint operator in dimension one. Then $\lambda_1(-L, \mathbb{R}) \leq \lambda'_1(-L, \mathbb{R})$.*

This type of result will be extended below.

4 Main results and open problems

The goal of this paper is to further explore these properties. We will examine three main classes: self-adjoint operators in low dimension, limit periodic operators and general operators in dimension one. We seek to identify classes of operators for which either equality or an inequality between λ_1 and λ'_1 hold.

4.1 Self-adjoint case

Our first result is an extension of the comparison result of Proposition 3.5 to dimensions $N = 2, 3$ in the self-adjoint framework.

Theorem 4.1 *Let $-L$ be a self-adjoint elliptic operator in \mathbb{R}^N , with $N \leq 3$. Then $\lambda_1(-L, \mathbb{R}^N) \leq \lambda'_1(-L, \mathbb{R}^N)$.*

The assumption $N \leq 3$ in Theorem 4.1 seems to be only technical, as well as it was for the assumption $N = 1$ in Proposition 3.5. This is why we believe that the above result holds in any dimension N . But the problem is open at the moment.

4.2 Limit periodic operators

Next, we examine the class of limit periodic operators which extends that of periodic operators. In a sense, this class is intermediate between periodic and a.p. Here is the definition:

Definition 4.2 1) We say that a general elliptic operator $-L$ is general limit periodic if there exists a sequence of general elliptic periodic operators

$$-L_n u := -a_{ij}^n \partial_{ij} u - b_i^n \partial_i u - c^n u$$

such that $a_{ij}^n \rightarrow a_{ij}$, $b_i^n \rightarrow b_i$ and $c^n \rightarrow c$ in $C_b^{0,\alpha}(\mathbb{R}^N)$ as n goes to infinity.

2) We say that a self-adjoint elliptic operator $-L$ is self-adjoint limit periodic if there exists a sequence of self-adjoint elliptic periodic operators

$$-L_n u := -\partial_i(a_{ij}^n \partial_j u) - c^n u$$

such that $a_{ij}^n \rightarrow a_{ij}$ in $C_b^{1,\alpha}(\mathbb{R}^N)$ and $c^n \rightarrow c$ in $C_b^{0,\alpha}(\mathbb{R}^N)$ as n goes to infinity.

Clearly, if all the coefficients of the operators L_n in Definition 4.2 have the same period (l_1, \dots, l_N) , then L is periodic too. It is immediate to show that the coefficients of a limit periodic operator are in particular a.p. in the sense of Definition 2.3. One of the results we obtain is:

Theorem 4.3 Let $-L$ be a general limit periodic operator. Then $\lambda'_1(-L, \mathbb{R}^N) \leq \lambda_1(-L, \mathbb{R}^N)$.

The other result obtained concerns self-adjoint limit periodic operators. It extends Proposition 3.3.

Theorem 4.4 Let $-L$ be a self-adjoint limit periodic operator. Then $\lambda_1(-L, \mathbb{R}^N) = \lambda'_1(-L, \mathbb{R}^N)$.

In the proofs of Theorems 4.3 and 4.4, we make use of the Schauder interior estimates and the Harnack inequality. One can find a treatment of these results in [10], or one can consult [15], [16] and [21] for the original proofs of the Harnack inequality.

Going back to the nonlinear problem, owing to Theorem 4.4, the existence and uniqueness results in the limit periodic case can be expressed in terms of λ_1 (or, equivalently, λ'_1) only, which is the statement of Theorem 2.2.

4.3 The case of dimension $N = 1$

Our last result establishes a comparison between λ_1 and λ'_1 for general elliptic operators in dimension one:

Theorem 4.5 Let $-L$ be a general elliptic operator in dimension one. Then $\lambda'_1(-L, \mathbb{R}) \leq \lambda_1(-L, \mathbb{R})$.

Notice that, by Theorems 4.3 and 4.5, if $-L_0$ is limit periodic or $N = 1$, then we can state Theorem 2.1 without mentioning λ_1 . Hence, only the sign of λ'_1 is involved in the existence result.

4.4 Open problems

The notions of generalized principal eigenvalue raise several questions which still need an answer. Some of them are:

Open problem 4.6 Does (2.8) admit positive bounded solutions (even in the self-adjoint case) if $\lambda'_1(-L_0, \mathbb{R}^N) = 0$?

Open problem 4.7 Is it true that $\lambda'_1(-L, \mathbb{R}^N) \leq \lambda_1(-L, \mathbb{R}^N)$, for any general elliptic operator $-L$ and any dimension N ?

Conjecture 4.8 If $-L$ is a self-adjoint elliptic operator, then $\lambda_1(-L, \mathbb{R}^N) \leq \lambda'_1(-L, \mathbb{R}^N)$ in any dimension N .

Note that should the answers to both 4.7 and 4.8 be positive, then we would have $\lambda_1 = \lambda'_1$ in the self-adjoint case, in arbitrary dimension.

5 Different definitions of the generalized principal eigenvalue

In this section, we present various definitions which one could consider as generalizations of the principal eigenvalue in the whole space. Then, we explain the choice of (1.6) and (1.7) as the most relevant extensions. Here, $-L$ will always denote a general elliptic operator (satisfying (1.3) and (1.4)).

The quantity λ_1 given by (1.6) is often called the “generalized” principal eigenvalue. It is considered the “natural” generalization of the principal eigenvalue because, as already mentioned, it coincides with the Dirichlet principal eigenvalue in bounded smooth domains. Also, the sign of λ_1 determines the existence or non-existence of a Green function for the operator (see Theorem 3.2 in [17]). The constant λ'_1 has been introduced, more recently, in [3]. If Ω is bounded and smooth, then $\lambda'_1(-L, \Omega) = \lambda_1(-L, \Omega)$. Moreover, as we have seen in Proposition 3.2, in the periodic case λ'_1 coincides with the periodic principal eigenvalue.

The quantity λ_1 is the largest constant λ for which $-(L + \lambda)$ admits a positive subsolution. The definition of λ'_1 is based on that of λ_1 , with two changes: first, we take subsolutions instead of supersolutions (and we replace the sup with the inf); second, we take test functions in $W^{2,\infty}$. If we introduce only one of these changes, we obtain the following definitions:

$$\begin{aligned} \mu_1(-L, \Omega) := \sup\{\lambda \mid \exists \phi \in C^2(\Omega) \cap C^1(\overline{\Omega}) \cap W^{2,\infty}(\Omega), \phi > 0 \text{ and } (L + \lambda)\phi \leq 0 \text{ in } \Omega, \\ \phi = 0 \text{ on } \partial\Omega, \text{ if } \partial\Omega \neq \emptyset\}, \end{aligned} \quad (5.13)$$

or

$$\mu'_1(-L, \Omega) := \inf\{\lambda \mid \exists \phi \in C^2(\Omega) \cap C^1_{loc}(\overline{\Omega}), \phi > 0 \text{ and } -(L + \lambda)\phi \leq 0 \text{ in } \Omega\}, \quad (5.14)$$

The quantity μ_1 is not interesting for us because, as is shown by Remark 6.2 in [5], if we replace λ'_1 with μ_1 , then the necessary condition given by Theorem 2.1 part 2) fails to hold. For completeness, we include this observation here.

Remark 5.1 Consider the equation $-u'' - b(x)u' = 0$ in \mathbb{R} . We show that, for b opportunely chosen, there exists a positive function $\phi \in C^2(\mathbb{R}) \cap W^{2,\infty}(\mathbb{R})$ such that $(L_0 + 1)\phi \leq 0$ in \mathbb{R} ($L_0 u = u'' + b(x)u'$ in this case). Therefore, $\mu_1(-L_0, \mathbb{R}) \geq 1$, but all the functions u identically equal to a positive constant solve $-u'' - b(x)u' = 0$. The function ϕ is defined in $[-1, 1]$ by: $\phi(x) = 2 - x^2$. For $x \in (-1, 1)$, we have that $(L_0 + 1)\phi = -2 - 2b(x)x + \phi \leq -2b(x)x$. Hence, it is sufficient to take $b(x) \leq 0$ for $x \leq 0$ and $b(x) \geq 0$ for $x \geq 0$ to obtain: $(L_0 + 1)\phi \leq 0$ in $(-1, 1)$. Then set

$$\phi(x) := \begin{cases} e^x & \text{if } x \leq -2 \\ e^{-x} & \text{if } x \geq 2. \end{cases}$$

For $x < -2$, $(L_0 + 1)\phi = e^x(2 + b(x))$ and, for $x > 2$, $(L_0 + 1)\phi = e^{-x}(2 - b(x))$. Hence, if $b \leq -2$ for $x < -2$ and $b(x) > 2$ for $x > 2$, we find: $(L_0 + 1)\phi \leq 0$ in $(-\infty, -2) \cup (2, +\infty)$. Clearly, it is possible to define ϕ in $(-2, -1) \cup (1, 2)$ in such a way that $\inf_{[-2, -1]} \phi' > 0$, $\sup_{[1, 2]} \phi' < 0$ and $\phi \in C^2(\mathbb{R}) \cap W^{2,\infty}(\mathbb{R})$. Consequently, taking $b < -M$ in $(-2, -1)$ and $b > M$ in $(1, 2)$, with $M > 0$ large enough, we get: $(L_0 + 1)\phi \leq 0$ in \mathbb{R} .

Neither is the definition (5.14) much meaningful as, in general, $\mu'_1 = -\infty$. This is seen next.

Remark 5.2 Consider the following family of functions:

$$\forall k > 0, \quad \phi_k(x) := e^{kx \cdot v}, \quad x \in \mathbb{R}^N,$$

where v is an arbitrary unit vector in \mathbb{R}^N . Straightforward computation yields:

$$-L\phi_k = -a_{ij}(x)v_i v_j k^2 \phi_k - (b(x) \cdot v)k\phi_k - c(x)\phi_k \leq (-\underline{a} k^2 + \|b\|_\infty k + \|c\|_\infty)\phi_k,$$

where \underline{a} is given by (1.3) and $b(x) = (b_1(x), \dots, b_N(x))$. Therefore, for every $\lambda \in \mathbb{R}$ there exists $k > 0$ large enough such that $-(L + \lambda)\phi_k \leq 0$ and then the quantity $\mu'_1(-L, \mathbb{R})$ defined by (5.14) is equal to $-\infty$.

On the contrary, we have:

Remark 5.3 The quantity $\lambda'_1(-L, \mathbb{R}^N)$ given by (1.7) satisfies: $-\|c\|_\infty \leq \lambda'_1(-L, \mathbb{R}^N) \leq \|c\|_\infty$. In fact, taking $\phi \equiv 1$ in (1.7), we see that $\lambda'_1(-L, \mathbb{R}^N) \leq \|c\|_\infty$. For the other inequality, consider $\lambda \in \mathbb{R}$ and $\phi \in C^2(\mathbb{R}^N) \cap W^{2,\infty}(\mathbb{R}^N)$ such that: $\phi > 0$ and $-(L + \lambda)\phi \leq 0$. Let M be the supremum of ϕ and $(x_n)_{n \in \mathbb{N}}$ be a maximizing sequence for ϕ . For $n \in \mathbb{N}$, define:

$$\forall x \in \mathbb{R}^N, \quad \theta_n(x) := \phi(x) - (M - \phi(x_n))|x - x_n|^2.$$

Arguing similarly to the proof of Lemma 7.2 below, one can see that every function θ_n has a local maximum in a point $y_n \in B_1(x_n)$. Furthermore, $\lim_{n \rightarrow \infty} \phi(y_n) = \lim_{n \rightarrow \infty} \phi(x_n) = M$. We have:

$$0 \geq -(L + \lambda)\phi(y_n) \geq 2(\phi(x_n) - M)a_{ii}(y_n) + 2(\phi(x_n) - M)b_i(y_n)(y_n - x_n)_i - (c(y_n) + \lambda)\phi(y_n).$$

It follows that $\lambda \geq -\liminf_{n \rightarrow \infty} c(y_n) \geq -\|c\|_\infty$. This shows that $\lambda'_1(-L, \mathbb{R}^N) \geq -\|c\|_\infty$.

6 Self-adjoint operators in dimension $N \leq 3$

The proof of Theorem 4.1 consists in a not so immediate adaptation of the proof of Proposition 3.3. It makes use of the following observation, which holds in any dimension N .

Lemma 6.1 *Let $\phi \in C^2(\mathbb{R}^N)$ be a nonnegative function. Let $\Lambda(x)$ be the largest eigenvalue of the matrix $(\partial_{ij}\phi(x))_{ij}$ and assume that $\Lambda := \sup_{x \in \mathbb{R}^N} \Lambda(x) < +\infty$. Then we have:*

$$\forall x \in \mathbb{R}^N, \quad |\nabla\phi(x)|^2 \leq 2\Lambda\phi(x). \quad (6.15)$$

Proof. First, if $\Lambda \leq 0$ then, for every $i = 1, \dots, N$, $\partial_{ii}\phi \leq 0$. This shows that ϕ is concave in every direction x_i and then, since it is nonnegative, is constant. In particular, (6.15) holds.

Consider the case $\Lambda > 0$. The Taylor expansion of ϕ at the point $x \in \mathbb{R}^N$ gives:

$$\forall y \in \mathbb{R}^N, \quad \phi(y) = \phi(x) + \nabla\phi(x)(y-x) + \frac{1}{2}\partial_{ij}\phi(z)(y-x)_i(y-x)_j,$$

where z is a point lying on the segment connecting x and y . Hence,

$$0 \leq \phi(y) \leq \phi(x) + \nabla\phi(x)(y-x) + \frac{1}{2}\Lambda|y-x|^2.$$

If we take in particular $y = x - \nabla\phi(x)/\Lambda$ we obtain:

$$0 \leq \phi(x) - \frac{|\nabla\phi(x)|^2}{2\Lambda}$$

and the statement is proved. □

Note that, if ϕ is a positive function in $W^{2,\infty}(\mathbb{R}^N)$, then Lemma 6.1 yields that its gradient is controlled by the square root of ϕ . Actually, this is the reason why in (1.7) we take test functions in $W^{2,\infty}(\mathbb{R}^N)$.

Proof of Theorem 4.1. Let $\lambda \in \mathbb{R}^N$ be such that there exists a positive function $\phi \in C^2(\mathbb{R}^N) \cap W^{2,\infty}(\mathbb{R}^N)$ satisfying: $-(L + \lambda)\phi \leq 0$. We would like to proceed as in the proof of Proposition 3.3, with φ_p replaced by ϕ , and obtain $\lambda_1(-L, \mathbb{R}^N) \leq \lambda$. This is not possible because, in general, ϕ is not bounded from below away from zero. Lemma 6.1 allows to overcome this difficulty. Consider in fact the same type of cutoff functions $(\chi_R)_{R \geq 1}$ as in Proposition 3.3 and call λ_R the principal eigenvalue of $-L$ in B_R with Dirichlet boundary conditions. The representation formula (3.12), yields for $R \geq 1$:

$$\lambda_R \leq \frac{\int_{B_R} [a_{ij}(x)\partial_i(\phi\chi_R)\partial_j(\phi\chi_R) - c(x)\phi^2\chi_R^2]}{\int_{B_R} \phi^2\chi_R^2}.$$

Hence, since $\chi_R = 1$ on B_{R-1} , we get:

$$\lambda_R \leq \lambda - \frac{\int_{C_R} [2a_{ij}(x)(\partial_i\phi)(\partial_j\chi_R)\phi\chi_R + \partial_i(a_{ij}(x)\partial_j\chi_R)\phi^2\chi_R]}{\int_{B_R} \phi^2\chi_R^2}.$$

Our aim is to prove that, by appropriately choosing the cutoff functions $(\chi_R)_{R \geq 1}$ we get:

$$\limsup_{R \rightarrow \infty} \frac{\int_{C_R} [2a_{ij}(x)(\partial_i\phi)(\partial_j\chi_R)\phi\chi_R + \partial_i(a_{ij}(x)\partial_j\chi_R)\phi^2\chi_R]}{\int_{B_R} \phi^2\chi_R^2} \geq 0. \quad (6.16)$$

These conditions are:

$$\begin{aligned} \forall x \in B_R \setminus B_{R-1/2}, \quad \chi_R(x) &= \exp\left(\frac{1}{|x| - R}\right), \\ \forall x \in B_{R-1/2}, \quad \chi_R(x) &\geq e^{-1/2}. \end{aligned}$$

By direct computation, we see that, for $x \in B_R \setminus B_{R-1/2}$,

$$\nabla\chi_R(x) = -\frac{x}{|x|}(R - |x|)^{-2} \exp\left(\frac{1}{|x| - R}\right),$$

and

$$\partial_{ij}\chi_R(x) = \left[\left(\frac{x_i x_j}{|x|^3} - \frac{\delta_{ij}}{|x|} \right) (|x| - R)^2 + 2 \frac{x_i x_j}{|x|^2} (|x| - R) + \frac{x_i x_j}{|x|^2} \right] (|x| - R)^{-4} \exp\left(\frac{1}{|x| - R}\right).$$

Consequently, using the usual summation convention, we have:

$$\forall x \in B_R \setminus B_{R-1/2}, \quad \partial_i(a_{ij}(x)\partial_j\chi_R) \geq [\underline{a} - C(|x| - R)] (|x| - R)^{-4} \exp\left(\frac{1}{|x| - R}\right),$$

where C only depends on N and the $W^{1,\infty}$ norm of the a_{ij} (and not on R) and \underline{a} is given by (1.3). Therefore, there exists h independent of R with $0 < h \leq 1/2$ and such that $\partial_i(a_{ij}(x)\partial_j\chi_R) \geq 0$ in $B_R \setminus B_{R-h}$. Since $\chi_R > \exp(-h^{-1})$ in B_{R-h} , it is possible to chose C' large enough, independent of R , such that $\partial_i(a_{ij}(x)\partial_j\chi_R) \geq -C'\chi_R$ in B_R . On the other hand, owing to Lemma 6.1, we can find another constant $C'' > 0$, depending only on N , $\|a_{ij}\|_{L^\infty(\mathbb{R}^N)}$, $\|\phi\|_{W^{2,\infty}(\mathbb{R}^N)}$ and $\|\chi_R\|_{W^{2,\infty}(\mathbb{R}^N)}$ (which does not depend on R), such that

$$a_{ij}(x)(\partial_i\phi)(\partial_j\chi_R) \geq -C''\phi^{1/2}\chi_R^{1/2}.$$

Assume, by way of contradiction, that (6.16) does not hold. There exist then $\varepsilon > 0$ and $R_0 \geq 1$ such that, for $R \geq R_0$,

$$\begin{aligned} -\varepsilon \int_{B_R} \phi^2\chi_R^2 &\geq \int_{C_R} [2a_{ij}(x)(\partial_i\phi)(\partial_j\chi_R)\phi\chi_R + \partial_i(a_{ij}(x)\partial_j\chi_R)\phi^2\chi_R] \\ &\geq -\int_{C_R} \left(C'\chi_R^2\phi^2 + 2C''\phi^{3/2}\chi_R^{3/2} \right). \end{aligned}$$

Since ϕ and χ_R are bounded, the above inequalities yield the existence of a positive constant k such that, for $R \geq R_0$,

$$k \int_{B_R} \phi^2 \chi_R^2 \leq \int_{C_R} \phi^{3/2} \chi_R^{3/2}.$$

Notice that, since $\phi > 0$, we can choose $k > 0$ in such a way that the above inequality holds for any $R \geq 1$. Using the Hölder inequality, with $p = 4/3$ and $p' = 4$, we then obtain:

$$\forall R \geq 1, \quad \int_{B_R} \phi^2 \chi_R^2 \leq k^{-1} \left(\int_{C_R} \phi^2 \chi_R^2 \right)^{3/4} |C_R|^{1/4} \leq K^{-1} R^{\frac{N-1}{4}} \left(\int_{C_R} \phi^2 \chi_R^2 \right)^{3/4},$$

where K is another positive constant. For $n \in \mathbb{N}$ call $\alpha_n := \left(\int_{C_n} \phi^2 \chi_n^2 \right)^{3/4}$. Since for $n \in \mathbb{N}$ we have that

$$\int_{B_n} \phi^2 \chi_n^2 = \sum_{j=1}^{n-1} \int_{C_j} \phi^2 + \int_{C_n} \phi^2 \chi_n^2 \geq \sum_{j=1}^n \int_{C_j} \phi^2 \chi_j^2,$$

it follows that

$$\alpha_n \geq K n^{\frac{1-N}{4}} \sum_{j=1}^n \alpha_j^{4/3}. \quad (6.17)$$

We claim that the sequence $(\alpha_n)_{n \in \mathbb{N}}$ grows faster than any power of n . This is in contradiction with the definition of α_n , because

$$\alpha_n = \left(\int_{C_n} \phi^2 \chi_n^2 \right)^{3/4} \leq \|\phi\|_{L^\infty(\mathbb{R}^N)}^2 |C_n|^{3/4} \leq H n^{\frac{3}{4}(N-1)},$$

for some positive constant H . To prove our claim, we use (6.17) recursively. At the first step we have: $\alpha_n \geq K_0 n^{\beta_0}$, where $K_0 = K \alpha_1^{4/3}$ and $\beta_0 = \frac{1-N}{4}$. At the second step we get: $\alpha_n \geq K K_0^{4/3} n^{\frac{1-N}{4}} \sum_{j=1}^n j^{\frac{4}{3}\beta_0}$. If $\beta_0 > -3/4$ (i.e. if $N < 4$) then $\sum_{j=1}^n j^{\frac{4}{3}\beta_0} \sim n^{\frac{4}{3}\beta_0+1}$. Hence, there exists $K_1 > 0$ such that $\alpha_n \geq K_1 n^{\beta_1}$, where $\beta_1 = \frac{4}{3}\beta_0 + \frac{5-N}{4}$. Proceeding at the same way we find, after m steps, that $\alpha_n \geq K_m n^{\beta_m}$, where K_m is a positive constant and $\beta_m = \frac{4}{3}\beta_{m-1} + \frac{5-N}{4}$, provided that $\beta_0, \dots, \beta_{m-1} > -3/4$. If $\beta_{m-1} > -3/4$, we have that

$$\beta_m > \beta_{m-1} \quad \Leftrightarrow \quad \beta_{m-1} > \frac{3}{4}(N-5).$$

Since

$$\beta_0 > \frac{3}{4}(N-5) \quad \Leftrightarrow \quad N < 4,$$

it follows that for $N \leq 3$ the sequence $(\beta_m)_{m \in \mathbb{N}}$ is strictly increasing. Thus, $\lim_{m \rightarrow \infty} \beta_m = +\infty$ if $N \leq 3$, because the only fixed point of the sequence $x_m = \frac{4}{3}x_{m-1} + \frac{5-N}{4}$ is $\frac{3}{4}(N-5)$ which is lesser than β_0 . Therefore, as $n \rightarrow \infty$, α_n goes to infinity faster than any polynomial in n . \square

7 Limit periodic operators

Throughout this section, we consider limit periodic elliptic operators $-L$. According to Definition 4.2, we let either

$$L_n u = a_{ij}^n(x) \partial_{ij} u + b_i^n(x) \partial_i u + c^n(x) u$$

if $-L$ is a general operator, or

$$L_n u = \partial_i (a_{ij}^n(x) \partial_j u) + c^n(x) u$$

if $-L$ is self-adjoint. We denote with λ_n and φ_n respectively the periodic principal eigenvalue and a positive periodic principal eigenfunction of $-L_n$ in \mathbb{R}^N .

Our results make use of the following Lemma.

Lemma 7.1 *The sequence $(\lambda_n)_{n \in \mathbb{N}}$ is bounded and*

$$\lim_{n \rightarrow \infty} \left\| \frac{(L - L_n) \varphi_n}{\varphi_n} \right\|_{L^\infty(\mathbb{R}^N)} = 0.$$

Proof. We can assume, without loss of generality, that the operators $-L, -L_n$ are general elliptic. Since the operators L_n are periodic, from Proposition 3.2 and Remark 5.3 it follows that

$$-\|c_n\|_\infty \leq \lambda_1'(-L_n, \mathbb{R}^N) = \lambda_n \leq \|c_n\|_\infty.$$

Hence, the sequence $(\lambda_n)_{n \in \mathbb{N}}$ is bounded because $c^n \rightarrow c$ in $C_b^{0,\alpha}(\mathbb{R}^N)$. For all $n \in \mathbb{N}$, the functions φ_n satisfy $-(L_n + \lambda_n)\varphi_n = 0$. Then, using interior Schauder estimates, we can find a constant $C_n > 0$ such that

$$\forall x \in \mathbb{R}^N, \quad \|\varphi_n\|_{C_b^{2,\alpha}(B_1(x))} \leq C_n \|\varphi_n\|_{L^\infty(B_2(x))},$$

where the C_n are controlled by λ_n and $\|a_{ij}^n\|_{C_b^{0,\alpha}(\mathbb{R}^N)}, \|b_i^n\|_{C_b^{0,\alpha}(\mathbb{R}^N)}, \|c^n\|_{C_b^{0,\alpha}(\mathbb{R}^N)}$. We know that the λ_n are bounded in $n \in \mathbb{N}$, and the same is true for the $C_b^{0,\alpha}$ norms of a_{ij}^n, b_i^n and c^n because they converge in the $C_b^{0,\alpha}$ norm to a_{ij}, b_i and c respectively. Thus, there exists a positive constant C such that $C \geq C_n$ for every $n \in \mathbb{N}$. Moreover, applying the Harnack inequality for the operators $-(L_n + \lambda_n)$, we can find another positive constant C' , which is again independent of n (and x), such that

$$\forall x \in \mathbb{R}^N, \quad \|\varphi_n\|_{L^\infty(B_2(x))} \leq C' \varphi_n(x).$$

Therefore,

$$\begin{aligned} \sup_{x \in \mathbb{R}^N} \left| \frac{(L - L_n) \varphi_n(x)}{\varphi_n(x)} \right| &\leq \sup_{x \in \mathbb{R}^N} \frac{(\|a_{ij} - a_{ij}^n\|_\infty + \|b_i - b_i^n\|_\infty + \|c - c^n\|_\infty) \|\varphi_n\|_{C_b^{2,\alpha}(B_1(x))}}{\varphi_n(x)} \\ &\leq CC' (\|a_{ij} - a_{ij}^n\|_\infty + \|b_i - b_i^n\|_\infty + \|c - c^n\|_\infty), \end{aligned}$$

which goes to zero as n goes to infinity. \square

Proof of Theorem 4.3. For $n \in \mathbb{N}$ define:

$$H_n := \left\| \frac{(L - L_n)\varphi_n}{\varphi_n} \right\|_{L^\infty(\mathbb{R}^N)}. \quad (7.18)$$

By Lemma 7.1, we know that $\lim_{n \rightarrow \infty} H_n = 0$. Since $|(L + \lambda_n)\varphi_n| \leq H_n\varphi_n$, it follows that $(L + \lambda_n - H_n)\varphi_n \leq 0$ and $-(L + \lambda_n + H_n)\varphi_n \leq 0$. Hence, using φ_n as a test function in (1.6) and (1.7), we infer that $\lambda_1(-L, \mathbb{R}^N) \geq \lambda_n - H_n$ and $\lambda_1'(-L, \mathbb{R}^N) \leq \lambda_n + H_n$, for every $n \in \mathbb{N}$. The proof is complete because, passing to the \liminf and the \limsup as n goes to infinity in the above inequalities, we get:

$$\lambda_1'(-L, \mathbb{R}^N) \leq \liminf_{n \rightarrow \infty} \lambda_n \leq \limsup_{n \rightarrow \infty} \lambda_n \leq \lambda_1(-L, \mathbb{R}^N). \quad (7.19)$$

\square

The proof of Theorem 4.4 is divided into two parts, the first one being the next Lemma.

Lemma 7.2 *The sequence $(\lambda_n)_{n \in \mathbb{N}}$ converges to $\lambda_1'(-L, \mathbb{R}^N)$ as n goes to infinity.*

Proof. Proceeding as in the proof of Theorem 4.3, we derive (7.19). So, we only need to show that $\limsup_{n \rightarrow \infty} \lambda_n \leq \lambda_1'(-L, \mathbb{R}^N)$. To this end, consider a constant $\lambda \geq \lambda_1'(-L, \mathbb{R}^N)$ such that there exists a positive function $\phi \in C^2(\mathbb{R}^N) \cap W^{2,\infty}(\mathbb{R}^N)$ satisfying $-(L + \lambda)\phi \leq 0$. Fix $n \in \mathbb{N}$ and define $\psi_n := k_n\varphi_n - \phi$, where k_n is the positive constant (depending on n) such that $\inf \psi_n = 0$ (such a constant always exists - and it is unique - because φ_n is bounded from below away from zero and ϕ is bounded from above). From the inequalities

$$-(L + \lambda)\psi_n \geq -k_n(L + \lambda)\varphi_n = k_n(L_n - L)\varphi_n + k_n(\lambda_n - \lambda)\varphi_n$$

and defining H_n as in (7.18), we find that

$$-(L + \lambda)\psi_n \geq k_n(\lambda_n - \lambda - H_n)\varphi_n. \quad (7.20)$$

Since $\inf \psi_n = 0$, there exists a sequence $(x_m)_{m \in \mathbb{N}}$ in \mathbb{R}^N such that $\lim_{m \rightarrow \infty} \psi_n(x_m) = 0$. For $m \in \mathbb{N}$, define the functions:

$$\theta_m(x) := \psi_n(x) + \psi_n(x_m)|x - x_m|^2, \quad x \in \mathbb{R}^N.$$

Since $\theta_m(x_m) = \psi_n(x_m)$ and $\theta_m(x) \geq \psi_n(x_m)$ for $x \in \partial B_1(x_m)$, for any $m \in \mathbb{N}$, there exists $y_m \in B_1(x_m)$ point of local minimum of θ_m . Hence,

$$0 = \nabla \theta_m(y_m) = \nabla \psi_n(y_m) + 2\psi_n(x_m)(y_m - x_m)$$

and

$$0 \leq (\partial_{ij}\theta(y_m))_{ij} = (\partial_{ij}\psi_n(y_m))_{ij} + 2\psi_n(x_m)I,$$

where I denotes the $N \times N$ identity matrix. Thanks to the ellipticity of $-L$, we then get:

$$-(L + \lambda)\psi_n(y_m) \leq 2\psi_n(x_m)a_{ii}(y_m) + 2\psi_n(x_m)b_i(y_m)(y_m - x_m)_i - (c(y_m) + \lambda)\psi_n(y_m). \quad (7.21)$$

Furthermore, since

$$\theta_m(y_m) = \psi_n(y_m) + \psi_n(x_m)|y_m - x_m|^2 \leq \theta_m(x_m) = \psi_n(x_m),$$

we see that $\psi_n(y_m) \leq \psi_n(x_m)$. Consequently, taking the limit as m goes to infinity in (7.21), we derive: $\limsup_{m \rightarrow \infty} -(L + \lambda)\psi_n(y_m) \leq 0$. Therefore, by (7.20),

$$\limsup_{m \rightarrow \infty} k_n(\lambda_n - \lambda - H_n)\varphi_n(y_m) \leq 0,$$

which implies that $\lambda_n - \lambda - H_n \leq 0$ because $\inf_{\mathbb{R}^N} \varphi_n > 0$. Since by Lemma 7.1 we know that H_n goes to zero as n goes to infinity, it follows that

$$\lambda \geq \limsup_{n \rightarrow \infty} (\lambda_n - H_n) = \limsup_{n \rightarrow \infty} \lambda_n.$$

Taking the infimum over all such constants λ we finally get: $\lambda'_1(-L, \mathbb{R}^N) \geq \limsup_{n \rightarrow \infty} \lambda_n$. \square

Proof of Theorem 4.4. Owing to Theorem 4.3, it only remains to show that $\lambda_1(-L, \mathbb{R}^N) \leq \lambda'_1(-L, \mathbb{R}^N)$. To do this, we fix $R > 1$ and $n \in \mathbb{N}$ and proceed as in the proof of Proposition 3.3, replacing the test function φ_p by φ_n . We this get:

$$\begin{aligned} \lambda_1(-L, B_R) &\leq -\frac{\int_{B_R} (L(\varphi_n \chi_R)) \varphi_n \chi_R}{\int_{B_R} \varphi_n^2 \chi_R^2} = \frac{\int_{B_{R-1}} ((\lambda_n + L_n - L)\varphi_n) \varphi_n - \int_{C_R} (L(\varphi_n \chi_R)) \varphi_n \chi_R}{\int_{B_R} \varphi_n^2 \chi_R^2} \\ &= \lambda_n - \frac{\int_{B_{R-1}} ((L - L_n)\varphi_n) \varphi_n + \int_{C_R} ((L + \lambda_n)\varphi_n \chi_R) \varphi_n \chi_R}{\int_{B_R} \varphi_n^2 \chi_R^2}. \end{aligned}$$

Setting H_n as in (7.18), we get:

$$\lambda_1(-L, B_R) \leq \lambda_n + \frac{H_n \int_{B_{R-1}} \varphi_n^2 + K_n |C_R|}{\int_{B_R} \varphi_n^2 \chi_R^2},$$

where $|C_R|$ denotes the measure of the set C_R and K_n is a positive constant (independent of R because the χ_R are uniformly bounded in $C^2(\mathbb{R}^N)$). Therefore, since $\min_{\mathbb{R}^N} \varphi_n > 0$, there exists another constant $\tilde{K}_n > 0$ such that

$$\lambda_1(-L, B_R) \leq \lambda_n + H_n + \frac{\tilde{K}_n}{R}.$$

Letting R go to infinity in the above inequality and using Proposition 3.1, this shows that: $\lambda_1(-L, \mathbb{R}^N) \leq \lambda_n + H_n$. By Lemmas 7.1 and 7.2, we know that $H_n \rightarrow 0$ and $\lambda_n \rightarrow \lambda'_1(-L, \mathbb{R}^N)$, as n goes to infinity. Thus, we conclude that $\lambda_1(-L, \mathbb{R}^N) \leq \lambda'_1(-L, \mathbb{R}^N)$. \square

8 The inequality $\lambda'_1 \leq \lambda_1$ in dimension $N = 1$

In this section, we are concerned with general elliptic operators in dimension one, that is operators of the type:

$$-Lu = -a(x)u'' - b(x)u' - c(x)u, \quad x \in \mathbb{R},$$

with the usual regularity assumptions on a, b, c . The elliptic condition becomes: $\underline{a} \leq a(x) \leq \bar{a}$, for some constants $0 < \underline{a} \leq \bar{a}$.

Proof of Theorem 4.5. Fix $R > 0$ and call λ_R and φ_R the principal eigenvalue and eigenfunction respectively of $-L$ in $(-R, R)$, with Dirichlet boundary condition. Then define

$$\psi_R(x) := \frac{h}{k}e^{-k(x-R)}, \quad x \in \mathbb{R}$$

where h, k are two positive constants that will be chosen later. The function ψ_R satisfies:

$$-(L + \lambda_R)\psi_R = \left(-a(x)k + b(x) - (c(x) + \lambda_R)\frac{1}{k} \right) he^{-k(x-R)}.$$

There exists $k_0 > 0$ (independent of h) such that $-(L + \lambda_R)\psi_R < 0$ in \mathbb{R} for any choice of $k \geq k_0$. Our aim is to connect smoothly the functions φ_R and ψ_R in order to obtain a function $\phi_R \in C^2([0, +\infty)) \cap W^{2,\infty}([0, +\infty))$ satisfying $-(L + \lambda_R)\phi_R \leq 0$. To this end, we set: $g_R(x) := \eta(x - R + \delta)^3$, with $\eta, \delta > 0$ to be chosen. Since

$$-(L + \lambda_R)g_R = [-6a(x) - 3b(x)(x - R + \delta) - (c(x) + \lambda_R)(x - R + \delta)^2]\eta(x - R + \delta),$$

we can find a constant $\delta_0 > 0$ such that $-(L + \lambda_R)g_R \leq 0$ in $(R - \delta, R)$, for any choice of $0 < \delta \leq \delta_0$. Then we define:

$$\phi_R(x) := \begin{cases} \varphi_R(x), & \text{for } 0 \leq x \leq R - \delta \\ \varphi_R(x) + g_R(x), & \text{for } R - \delta < x \leq R \\ \psi_R(x), & \text{for } x > R. \end{cases} \quad (8.22)$$

It follows that, if $k \geq k_0$ and $\delta \leq \delta_0$, then the function ϕ_R satisfies $-(L + \lambda_R)\phi_R \leq 0$ in $(0, R - \delta) \cup (R - \delta, R) \cup (R, +\infty)$. In order to ensure the C^2 regularity of ϕ_R , we need to solve the following system in the variables h, k, η, δ :

$$\begin{cases} \eta\delta^3 = \frac{h}{k} \\ \varphi'_R(R) + 3\eta\delta^2 = -h \\ \varphi''_R(R) + 6\eta\delta = hk. \end{cases}$$

One can see that, if $h < -\varphi'_R(R)$ (notice that $\varphi'_R(R) < 0$ by the Hopf's lemma), the previous system becomes, after some simple algebra,

$$\begin{cases} \gamma(h) = \varphi''_R(R)\delta \\ \delta k = \frac{3h}{-\varphi'_R(R) - h} \\ \eta = \frac{hk - \varphi''_R(R)}{6\delta}, \end{cases} \quad (8.23)$$

where

$$\gamma(h) := \frac{3h^2}{-\varphi'_R(R) - h} + 2(h + \varphi'_R(R)).$$

We want to show that there exists δ small enough such that the system (8.23) admits positive solutions $\delta, h_\delta, k_\delta, \eta_\delta$ satisfying:

$$\delta \leq \delta_0, \quad h_\delta < -\varphi'_R(R), \quad k_\delta \geq k_0. \quad (8.24)$$

Let $0 < \delta_1 \leq \delta_0$ be such that $|\varphi''_R(R)|\delta_1 < -\varphi'_R(R)$. Thus, if $\delta \leq \delta_1$, the first equation of (8.23) yields: $|\gamma(h)| < -\varphi'_R(R)$. Since $\gamma(0) = 2\varphi'_R(R)$ and $\lim_{h \rightarrow -\varphi'_R(R)^-} \gamma(h) = +\infty$, there exists a constant $0 < h_1 < -\varphi'_R(R)$ such that, for any choice of $\delta \in (0, \delta_1)$, the first equation of (8.23) admits a solution $h_\delta \in [h_1, -\varphi'_R(R))$. For $\delta \in (0, \delta_1)$ and $h = h_\delta$, the second equation of (8.23) gives:

$$k_\delta = \frac{3h_\delta}{-\varphi'_R(R) - h_\delta} \delta^{-1} \geq \frac{3h_1}{-\varphi'_R(R) - h_1} \delta^{-1}. \quad (8.25)$$

Hence, for δ small enough, we have $k_\delta \geq k_0$. Finally, by the last equation of (8.23), for $\delta \in (0, \delta_1)$, we have

$$\eta_\delta = \frac{h_\delta k_\delta - \varphi''_R(R)}{6\delta} \geq \frac{h_1 k_\delta - \varphi''_R(R)}{6\delta},$$

and then, since k_δ satisfies (8.25), $\eta_\delta > 0$ for δ small enough. Therefore, there exist four positive constants h, k, η, δ solving (8.23) and satisfying (8.24). With this choices of h, k, η, δ , the function ϕ_R is in $C^2([0, +\infty)) \cap W^{2,\infty}([0, +\infty))$.

Proceeding as above, we can extend $\varphi_R(x)$ for x negative, and get a function $\phi_R \in C^2(\mathbb{R}) \cap W^{2,\infty}(\mathbb{R})$ such that $-(L + \lambda_R)\phi_R \leq 0$ in \mathbb{R} . Using ϕ_R as a test function in (1.7), we find that $\lambda'_1(-L, \mathbb{R}) \leq \lambda_R$. Thus, passing to the limit as R goes to infinity, by Proposition 3.1, we derive $\lambda'_1(-L, \mathbb{R}) \leq \lambda_1(-L, \mathbb{R})$. The proof is thereby complete. \square

Remark 8.1 Using the same type of construction as in Theorem 4.5, one can prove that the inequality $\lambda'_1(-L, \mathbb{R}^N) \leq \lambda_1(-L, \mathbb{R}^N)$ holds for any elliptic operator $-L$ which is rotationally invariant. Consider in fact an elliptic operator of type

$$-Lu = -a(|x|)\Delta u - b(|x|)\frac{x}{|x|} \cdot \nabla u - c(|x|)u, \quad \text{in } \mathbb{R}^N,$$

with $b(0) = 0$ and with the usual ellipticity and regularity assumptions on the coefficients. For $R > 0$, let λ_R and φ_R respectively denote its Dirichlet principal eigenvalue and eigenfunction in B_R . It is easy to see that, for any orthogonal matrix M , the function $\varphi_R(Mx)$ is again a Dirichlet positive eigenfunction of $-L$ in B_R . Hence, by uniqueness of the principal eigenfunction up to multiplication, it follows that $\varphi_R(x) \equiv \varphi_R(Mx)$, that is φ_R is a radial function. Since for any radial function $u = u(|x|)$ the expression of Lu reads:

$$Lu = a(|x|)u'' + \left(b(|x|) + \frac{N-1}{|x|}a(|x|) \right) u' + c(|x|)u,$$

we can proceed as in the one-dimensional case and build a radial function $\phi_R \in C^2(\mathbb{R}^N) \cap W^{2,\infty}(\mathbb{R}^N)$ such that $-(L + \lambda_R)\phi_R \leq 0$. Therefore, $\lambda'_1(-L, \mathbb{R}^N) \leq \lambda_R$ and then, passing to the limit as $R \rightarrow \infty$, we obtain the stated inequality between λ_1 and λ'_1

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