REACTION-DIFFUSION EQUATIONS FOR POPULATION DYNAMICS WITH FORCED SPEED
II - CYLINDRICAL-TYPE DOMAINS

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Abstract. This work is the continuation of our previous paper [6]. There, we dealt with the reaction-diffusion equation

$$\partial_t u = \Delta u + f(x - cte, u), \qquad t > 0, \quad x \in \mathbb{R}^N,$$

where $e \in S^{N-1}$ and $c > 0$ are given and $f(x, s)$ satisfies some usual assumptions in population dynamics, together with $f_s(x, 0) < 0$ for $|x|$ large. The interest for such equation comes from an ecological model introduced in [1] describing the effects of global warming on biological species. In [6], we proved that existence and uniqueness of travelling wave solutions of the type $u(x, t) = U(x - cte)$ and the large time behaviour of solutions with arbitrary nonnegative bounded initial datum depend on the sign of the generalized principal eigenvalue in $\mathbb{R}^N$ of an associated linear operator. Here, we establish analogous results for the Neumann problem in domains which are asymptotically cylindrical, as well as for the problem in the whole space with f periodic in some space variables, orthogonal to the direction of the shift e.

The L^1 convergence of solution $u(t, x)$ as $t \to \infty$ is established next. In this paper, we also show that a bifurcation from the zero solution takes place as the principal eigenvalue crosses 0. We are able to describe the shape of solutions close to extinction thus answering a question raised by M. Mimura. These two results are new even in the framework considered in [6].

Another type of problem is obtained by adding to the previous one a term $g(x - c'te, u)$ periodic in x in the direction e. Such a model arises when considering environmental change on two different scales. Lastly, we also solve the case of an equation

$$\partial_t u = \Delta u + f(t, x - cte, u),$$

when $f(t, x, s)$ is periodic in t. This for instance represents the seasonal dependence of f. In both cases, we obtain a necessary and sufficient condition for the existence, uniqueness and stability of pulsating travelling waves, which are solutions with a profile which is periodic in time.

1. Introduction

In a recent paper [1], a model to study the impact of climate change (global warming) on the survival and dynamics of species was proposed. This model involves a reaction-diffusion equation on the real line

$$\partial_t u = \partial_{xx} u + f(x - ct, u), \qquad t > 0, \quad x \in \mathbb{R}.$$

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In our previous paper [6], we extended the results of [1] to arbitrary dimension N:

$$\partial_t u = \Delta u + f(x - cte, u), \qquad t > 0, \quad x \in \mathbb{R}^N, \tag{1}$$

with $c > 0$ and $e \in S^{N-1}$ given. The function $f(x, s) : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ considered in [6] (which is slightly more general than in [1]) satisfies some usual assumptions in population dynamics, together with

$$\limsup_{|x|\to\infty} f_s(x, 0) < 0. \tag{2}$$

In the ecological model, this assumption describes the fact that the favourable habitat is bounded. We proved in [6] that (1) admits a unique travelling wave solution, that is, a positive bounded solution of the form $U(x - cte)$, if and only if the generalized principal eigenvalue λ_1 of an associated linear elliptic operator in the whole space is negative. Then, we were able to characterize the large time behaviour of any solution u of (1) with nonnegative bounded and not identically equal to zero initial datum. We showed that

(i) if $\lambda_1 \geq 0$ then $u(t, x) \to 0$ as $t \to \infty$, uniformly in $x \in \mathbb{R}^N$;
(ii) if $\lambda_1 < 0$ then $(u(t, x) - U(x - cte)) \to 0$ as $t \to \infty$, uniformly in $x \in \mathbb{R}^N$.

We further considered the "two-speeds problem", obtained by adding a term $g(x-c'te, u)$ to the "pure shift problem" (1), with $x \mapsto g(x, s)$ periodic in the direction e. We derived analogous results to the previous ones, by replacing travelling waves with pulsating travelling waves.

Here, we deal with the same reaction-diffusion equation as in [6], but in different geometries.

We first consider the *pure shift problem* in a **straight infinite cylinder**

$$\Omega = \{(x_1, y) \in \mathbb{R} \times \mathbb{R}^{N-1} \ : \ x_1 \in \mathbb{R}, \ y \in \omega\},$$

where ω is a bounded smooth domain in $\mathbb{R}^{N-1}$, with Neumann boundary conditions:

$$\begin{cases} \partial_t u = \Delta u + f(x_1 - ct, y, u), & t > 0, \ x_1 \in \mathbb{R}, \ y \in \omega \\ \partial_\nu u(t, x_1, y) = 0, & t > 0, \ x_1 \in \mathbb{R}, \ y \in \partial\omega, \end{cases} \tag{3}$$

Henceforth, c is a **given** positive constant, ν denotes the exterior unit normal vector field to Ω and $\partial_\nu := \nu \cdot \nabla$. Next, we deal with the same problem in a **straight semi-infinite cylinder**

$$\Omega^+ = \{(x_1, y) \in \mathbb{R} \times \mathbb{R}^{N-1} \ : \ x_1 > 0, \ y \in \omega\},$$

under Dirichlet boundary condition on the "base" $\{0\} \times \overline{\omega}$:

$$\begin{cases} \partial_t u = \Delta u + f(x_1 - ct, y, u), & t > 0, \ x_1 > 0, \ y \in \omega \\ \partial_\nu u(t, x_1, y) = 0, & t > 0, \ x_1 > 0, \ y \in \partial\omega \\ u(t, 0, y) = \sigma(t, y) & t > 0, \ y \in \omega. \end{cases} \tag{4}$$

More generally, we consider an **asymptotically cylindrical** domain Ω' approaching Ω for x_1 large (in a sense we will make precise in Section 2.2):

$$\begin{cases} \partial_t u = \Delta u + f(x_1 - ct, y, u), & t > 0, \ (x_1, y) \in \Omega' \\ \partial_{\nu'} u(t, x_1, y) = 0, & t > 0, \ (x_1, y) \in \partial\Omega', \end{cases} \tag{5}$$

where ν' is the exterior unit normal vector field to Ω' and $\partial_{\nu'} := \nu' \cdot \nabla$.

We further study problem (1) when f is **lateral-periodic**, that is, $x \mapsto f(x, s)$ is periodic in some directions, orthogonal to e.

We also investigate here the behaviour of travelling wave solutions near the critical threshold. This topic was not discussed in [6]. We prove that, when c crosses a critical value c_0, a bifurcation takes place: stable travelling wave solutions U disappear and the trivial solution $u \equiv 0$ becomes stable. We characterize the shape of U near c_0. Another type of results we derive here concerns the behaviour of the solution $u(t,x)$ as $t \to \infty$ in terms of the L^1 norm. This is done for problem (3) in the straight infinite cylinder as well as for problem (1) in the whole space treated in [6].

Finally, we consider the following problem:

$$\begin{cases} \partial_t u = \Delta u + f(t, x_1 - ct, y, u), & t > 0,\ x_1 \in \mathbb{R},\ y \in \omega \\ \partial_\nu u(t, x_1, y) = 0, & t > 0,\ x_1 \in \mathbb{R},\ y \in \partial\omega, \end{cases} \tag{6}$$

with f periodic in the first variable t. This equation serves as a model for instance to describe the situation in which the climate conditions in the "normal regime" (that is, in the absence of global warming) are affected by seasonal changes. The methods used to solve (6) also apply to the *two-speeds problem*

$$\begin{cases} \partial_t u = \Delta u + f(x_1 - ct, y, u) + g(x_1 - c't, y, u), & t > 0,\ x_1 \in \mathbb{R},\ y \in \omega \\ \partial_\nu u(t, x_1, y) = 0, & t > 0,\ x_1 \in \mathbb{R},\ y \in \partial\omega, \end{cases} \tag{7}$$

with $c' \neq c$ and g periodic in the x_1 variable. The term g enables one to describe situations in which some characteristics of the habitat - such as the availability of nutrient - are affected by the climate change on a time scale different from that of the overall change. One may also consider the case in which they are not affected at all: $c' = 0$ (*mixed periodic/shift* problem). However, the case of two or more cohabiting species is not treated here. One then has to consider systems of evolution equations (see e. g. [9], [16] and [11], where segregation phenomena are also described). This extension is still open.

2. Statement of the main results

2.1. **Straight infinite cylinder.** Let us list the assumptions on the function $f(x,s)$ in the case of problem (3). We will sometimes denote the generic point $x \in \Omega$ by $(x_1, y) \in \mathbb{R} \times \omega$ and we set $\partial_1 := \frac{\partial}{\partial_{x_1}}$. We will always assume that $f(x,s)$: $\Omega \times [0, +\infty) \to \mathbb{R}$ is a Carathéodory function such that

$$\begin{cases} s \mapsto f(x,s) \text{ is locally Lipschitz continuous, uniformly for a. e. } x \in \Omega, \\ \exists\, \delta > 0 \text{ such that } s \mapsto f(x,s) \in C^1([0,\delta]),\ \text{uniformly for a. e. } x \in \Omega. \end{cases} \tag{8}$$

Moreover, we will require the following assumptions which are typical in population dynamics:

$$f(x,0) = 0 \quad \text{for a. e. } x \in \Omega, \tag{9}$$

$$\exists\, S > 0 \ \text{ such that } \ f(x,s) \leq 0 \quad \text{for } s \geq S \text{ and for a. e. } x \in \Omega, \tag{10}$$

$$\begin{cases} s \mapsto \dfrac{f(x,s)}{s} \ \text{ is nonincreasing for a. e. } x \in \Omega \\ \text{and it is strictly decreasing for a. e. } x \in D \subset \Omega,\ \text{with } |D| > 0. \end{cases} \tag{11}$$

The condition asserting that the favourable zone is bounded (as in [1], [6]) is written in the form

$$\zeta := -\lim_{r \to \infty} \sup_{\substack{|x_1| > r \\ y \in \omega}} f_s(x_1, y, 0) > 0. \tag{12}$$

A *travelling wave solution* for problem (3) is a positive bounded solution of the form $u(t,x_1,y) = U(x_1 - ct, y)$. The problem for U reads

$$(13)\qquad \begin{cases} \Delta U + c\partial_1 U + f(x,U) = 0 & \text{for a. e. } x \in \Omega \\ \partial_\nu U = 0 & \text{on } \partial\Omega \\ U > 0 \ \text{ in } \Omega \\ U \ \text{ is bounded.} \end{cases}$$

In the literature, such kind of solutions are also called *pulses*. If f satisfies (9), then the linearized operator about 0 associated with the elliptic equation in (13) is

$$\mathcal{L}w = \Delta w + c\partial_1 w + f_s(x,0)w.$$

Our main results in the pure shift case depend on the stability of the solution $w \equiv 0$ for the Neumann problem $\mathcal{L}w = 0$ in Ω, $\partial_\nu w = 0$ on $\partial\Omega$, that is, on the sign of the *generalized Neumann principal eigenvalue* $\lambda_{1,N}(-\mathcal{L},\Omega)$. For a given operator L in the form $L = \Delta + \beta(x)\cdot\nabla + \gamma(x)$, with β and γ bounded, we define the quantity $\lambda_{1,N}(-L,\Omega)$ by

(14)

$$\lambda_{1,N}(-L,\Omega) := \sup\{\lambda \in \mathbb{R} \ : \ \exists\, \phi > 0,\ (L+\lambda)\phi \le 0 \ \text{ a. e. in } \Omega,\ \partial_\nu \phi \ge 0 \text{ on } \partial\Omega\}.$$

This definition of the generalized principal eigenvalue for the Neumann problem is in the same spirit as the one in [4] for the Dirichlet boundary condition case. In (14), the function ϕ is understood to belong to $W^{2,p}((-r,r)\times\omega)$ for some $p > N$ and every $r > 0$. Thus, $\partial_\nu\phi$ has the classical meaning. We will set for brief $\lambda_{1,N} := \lambda_{1,N}(-\mathcal{L},\Omega)$.

Theorem 2.1. *Assume that (9)-(12) hold. Then, the travelling wave problem (13) admits a solution if and only if $\lambda_{1,N} < 0$. Moreover, when it exists, the solution is unique and satisfies*

$$\lim_{|x_1|\to\infty} U(x_1,y) = 0,$$

uniformly with respect to $y \in \omega$.

Theorem 2.2. *Let $u(t,x)$ be the solution of (3) with an initial condition $u(0,x) = u_0(x) \in L^\infty(\Omega)$ which is nonnegative and not identically equal to zero. Under assumptions (9)-(12) the following properties hold:*

(i) *if $\lambda_{1,N} \ge 0$ then*

$$\lim_{t\to\infty} u(t,x) = 0,$$

uniformly with respect to $x \in \Omega$;

(ii) *if $\lambda_{1,N} < 0$ then*

$$\lim_{t\to\infty} (u(t,x_1,y) - U(x_1 - ct, y)) = 0,$$

uniformly with respect to $(x_1,y) \in \Omega$, where U is the unique solution of (13).

2.2. **General cylindrical-type domains.** The large time behaviour of solutions to the pure shift problem either in the semi-infinite cylinder Ω^+, as well as in the asymptotically cylindrical domain Ω', is characterized by the sign of the generalized Neumann principal eigenvalue $\lambda_{1,N} = \lambda_{1,N}(-\mathcal{L},\Omega)$ in the *straight infinite cylinder*, as defined in (14).

In the first case, in order to give sense to problem (4), the function $f(\cdot,s)$ has to be defined in the whole straight infinite cylinder Ω. We will always require that

f satisfies (8). In (4), the function σ, which defines the Dirichlet condition at the "bottom" of the cylinder, is assumed to be of class $W^{2,\infty}(\mathbb{R}^+ \times \omega)$ and to satisfy

$$\sigma \geq 0 \text{ in } \mathbb{R}^+ \times \omega, \qquad \partial_\nu \sigma = 0 \text{ on } \mathbb{R}^+ \times \partial\omega, \qquad \forall\, y \in \omega,\ \lim_{t\to\infty} \sigma(t,y) = 0. \tag{15}$$

Here is the result for the half cylinder.

Theorem 2.3. *Let $u(t,x)$ be the solution of (4) with an initial condition $u(0,x) = u_0(x) \in L^\infty(\Omega^+)$ which is nonnegative and not identically equal to zero. Under assumptions (9)-(12), (15) the following properties hold:*

(i) *if $\lambda_{1,N} \geq 0$ then*
$$\lim_{t\to\infty} u(t,x) = 0,$$
uniformly with respect to $x \in \Omega^+$;

(ii) *if $\lambda_{1,N} < 0$ then*
$$\lim_{t\to\infty} (u(t,x_1,y) - U(x_1 - ct, y)) = 0,$$
uniformly with respect to $(x_1,y) \in \Omega^+$, where U is the unique solution of (13).

For the next result, let us now make precise what we mean by Ω' being an asymptotically cylindrical domain. We assume that Ω' is uniformly smooth and that there exists a C^2 diffeomorphism $\Psi : \mathbb{R}^N \to \mathbb{R}^N$ such that

$$\begin{cases} \exists\, h > 0, \quad \Psi([h,+\infty) \times \overline{\omega}) = \overline{\Omega}' \cap ([h,+\infty) \times \mathbb{R}^{N-1}), \\ \lim\limits_{x_1\to+\infty} \|\Psi - I\|_{W^{2,\infty}((x_1,+\infty)\times\omega)} = 0, \end{cases} \tag{16}$$

where I denotes the identity map from $\mathbb{R}^N$ into itself. We define the family of sets $(\omega'(x_1))_{x_1\in\mathbb{R}}$ in $\mathbb{R}^{N-1}$ by the equality

$$\bigcup_{x_1\in\mathbb{R}} \{x_1\} \times \omega'(x_1) = \Omega'.$$

Note that by (16) the $\omega'(x_1)$ are (uniformly) smooth, bounded and connected for x_1 large enough. In order to make sense of (5), the function $f(\cdot,s)$ has to be defined in the set

$$\widetilde{\Omega} := \bigcup_{x_1\in\mathbb{R}} (-\infty, x_1] \times \omega'(x_1).$$

Clearly, one has that $\Omega \subset \widetilde{\Omega}$. Besides the regularity assumptions (8) on f, where Ω is replaced by $\widetilde{\Omega}$, we further require that f and $f_s(x,0)$ are Hölder continuous[1] in x:

$$\exists\, \alpha \in (0,1), \quad \forall\, s > 0, \quad f(\cdot,s), f_s(\cdot,0) \in C^\alpha(\widetilde{\Omega}). \tag{17}$$

In this setting, hypotheses (9)-(12) are understood to hold with Ω replaced by $\widetilde{\Omega}$, except for the condition $D \subset \Omega$ in (11) which is unchanged.

Theorem 2.4. *Let $u(t,x)$ be the solution of (5) with an initial condition $u(0,x) = u_0(x) \in L^\infty(\Omega')$ which is nonnegative and not identically equal to zero. Under assumptions (9)-(12), (17) the following properties hold:*

[1] which is also understood to imply that they are bounded. Precisely, for $k \in \mathbb{N}$ and $\alpha \in (0,1)$, $C^{k+\alpha}(\mathcal{O})$ denotes the space of functions $\phi \in C^k(\mathcal{O})$ whose derivatives up to order k are bounded and uniformly Hölder continuous with exponent α in $\mathcal{O}$.

(i) *if* $\lambda_{1,N} \geq 0$ *then*

$$\lim_{t\to\infty} u(t,x) = 0,$$

uniformly with respect to $x \in \Omega'$*;*

(ii) *if* $\lambda_{1,N} < 0$ *then*

$$\lim_{t\to\infty} (u(t,x_1,y) - U(x_1 - ct, y)) = 0,$$

uniformly with respect to $(x_1, y) \in \Omega' \cap \Omega$*, where* U *is the unique solution of* (13)*. In addition,*

$$\lim_{t\to\infty} u(t,x_1,y) = 0,$$

uniformly with respect to $x_1 \leq \gamma t,\ y \in \omega'(x_1)$*, for any* $\gamma < c$*.*

Remark 1. Since by Theorem 2.1 $\lim_{|x_1|\to\infty} U(x_1,y) = 0$ uniformly in $y \in \omega$, the two limits in the statement (ii) of Theorem 2.4 are not contradictory. More generally, the second one ($\lim_{t\to\infty} u(t,x_1,y) = 0$) actually holds uniformly with respect to $x_1 \leq \gamma(t),\ y \in \omega'(x_1)$, for any function γ such that $\lim_{t\to\infty}(\gamma(t) - ct) = -\infty$.

Remark 2. Another way to state Theorem 2.4 part (ii) is by extending the unique solution U of (13) to a function $U \in W^{1,\infty}(\mathbb{R}^N)$ satisfying

$$\forall\ x_1 \in \mathbb{R}, \quad \|U(x_1,\cdot)\|_{L^\infty(\mathbb{R}^{N-1})} = \|U(x_1,\cdot)\|_{L^\infty(\omega)}.$$

Then, since $\lim_{|x_1|\to\infty} U(x_1,y) = 0$ uniformly in $y \in \omega$ and u and U are uniformly continuous, applying Theorem 2.4 part (ii) with, for instance, $\gamma = c/2$ we see that

$$\lim_{t\to\infty} (u(t,x_1,y) - U(x_1 - ct, y)) = 0,$$

uniformly with respect to $(x_1,y) \in \Omega'$.

Actually, the results of Theorem 2.4 hold under more general boundary conditions than those considered in (5). In fact, it is only needed that they coincide with Neumann boundary conditions for x_1 large (and that they imply the existence of a unique solution of the evolution problem for any given initial datum, as well as the validity of the comparison principle). Since Ω^+ is a particular case of asymptotically cylindrical domain (with $\Psi \equiv 1$ and $h = 0$), Theorem 2.3 is actually contained in Theorem 2.4. However, we treat it separately because the proof is much simpler.

2.3. **Lateral-periodic conditions.** In the last case considered for the pure shift problem, we deal with problem (1) with $c > 0$ and $e \in S^{N-1}$ given and with f periodic in the last P variables, $1 \leq P \leq N-1$. That is, there exist P positive constants $l_1, \cdots, l_P$ such that

$$\forall\ i \in \{1,\cdots,P\},\ s \in \mathbb{R}, \quad f(x + l_i e_{N-P+i}, s) = f(x,s) \ \text{ for a. e. } x \in \mathbb{R}^N, \tag{18}$$

where $\{e_1,\cdots,e_N\}$ denotes the canonical basis of $\mathbb{R}^N$. We assume that the shift direction $e \in S^{N-1}$ is orthogonal to the directions in which f is periodic: $e \cdot e_i = 0$ for $i = N-P+1,\cdots,N$. We set $M := N - P$ and we will sometimes denote the generic point $x \in \mathbb{R}^N$ by $x = (z,y) \in \mathbb{R}^M \times \mathbb{R}^P$, in order to distinguish the periodic directions y from the others. Henceforth, we say that a function $\phi : \mathbb{R}^N \to \mathbb{R}$ is lateral-periodic (with period $(l_1,\cdots,l_P)$) if $\phi(x + l_i e_{M+i}) = \phi(x)$ for $i = 1,\cdots,P$ and a. e. $x \in \mathbb{R}^N$.

Besides the regularity assumptions (8) (with Ω now replaced by $\mathbb{R}^N$) we require that f satisfies

$$f(x,0) = 0 \quad \text{for a. e. } x \in \mathbb{R}^N, \tag{19}$$

(20) $$\exists\, S>0 \ \text{ such that } \ f(x,s)\le 0 \quad \text{for } s\ge S \text{ and for a. e. } x\in\mathbb{R}^N,$$

(21) $$\begin{cases} s\mapsto \dfrac{f(x,s)}{s} \ \text{ is nonincreasing for a. e. } x\in\mathbb{R}^N \\ \text{and it is strictly decreasing for a. e. } x\in D\subset\mathbb{R}^N, \ \text{with } |D|>0, \end{cases}$$

(22) $$\zeta := -\lim_{r\to\infty}\ \sup_{\substack{|z|>r\\ y\in\mathbb{R}^P}} f_s(z,y,0)>0.$$

The problem for travelling wave solutions $u(t,x)=U(x-cte)$ reads

(23) $$\begin{cases} \Delta U + ce\cdot\nabla U + f(x,U)=0 \quad \text{a. e. in } \mathbb{R}^N \\ U>0 \quad \text{in } \mathbb{R}^N \\ U \quad \text{is bounded.} \end{cases}$$

The associated linearized operator $\mathcal{L}$ about 0 is the same as before but in $\mathbb{R}^N$. We consider the *generalized principal eigenvalue* of a linear elliptic operator $-L$ in a domain $\mathcal{O}\subset\mathbb{R}^N$, as defined in [4]:

(24)
$$\lambda_1(-L,\mathcal{O}) := \sup\{\lambda\in\mathbb{R}:\ \exists\,\phi\in W^{2,N}_{loc}(\mathcal{O}),\ \phi>0 \text{ and } (L+\lambda)\phi\le 0 \ \text{ a. e. in } \mathcal{O}\}.$$

In the sequel, we will set $\lambda_1 := \lambda_1(-\mathcal{L},\mathbb{R}^N)$. We now state our main results for the lateral periodic (pure shift) problem.

Theorem 2.5. *Assume that (18)-(22) hold. Then, problem (23) admits a solution if and only if $\lambda_1<0$. Moreover, when it exists, the solution is unique, lateral-periodic and satisfies*

$$\lim_{|z|\to\infty} U(z,y)=0,$$

uniformly with respect to $y\in\mathbb{R}^P$.

Theorem 2.6. *Let $u(t,x)$ be the solution of (1) with an initial condition $u(0,x)=u_0(x)\in L^\infty(\mathbb{R}^N)$ which is nonnegative and not identically equal to zero. Under assumptions (18)-(22) the following properties hold:*

(i) *if $\lambda_1\ge 0$ then*

$$\lim_{t\to\infty} u(t,x)=0,$$

uniformly with respect to $x\in\mathbb{R}^N$;

(ii) *if $\lambda_1<0$ then*

$$\lim_{t\to\infty}(u(t,z,y)-U((z,y)-cte))=0,$$

globally uniformly with respect to $z\in\mathbb{R}^M$ and locally uniformly with respect to $y\in\mathbb{R}^P$, where U is the unique solution of (23). If, in addition, u_0 is either lateral-periodic or satisfies

(25) $$\forall\, r>0, \qquad \inf_{\substack{|z|<r\\ y\in\mathbb{R}^P}} u_0(z,y)>0,$$

then the previous limit holds globally uniformly also with respect to $y\in\mathbb{R}^P$.

It is easy to see that, in general, the convergence of $u(t,z,y)$ to $U((z,y)-cte)$ is not uniform globally with respect to y. For instance, if the initial datum u_0 has compact support, then, for all fixed $t>0, u(t,x)\to 0$ as $|x|\to\infty$.

2.4. **Behaviour near critical value.** The next result is to answer a question that was raised by Professor Mimura to one of the authors regarding the behaviour of the solutions near the extinction limit. We show here that a simple bifurcation takes place when the generalized principal eigenvalue becomes nonnegative (or, in other terms, when the speed c crosses a critical value c_0). For simplicity, we only state the result in the case of pure shift problem (3) in the straight infinite cylinder, but it also holds in the whole space case (1), either under the hypotheses of the lateral periodic framework, as well as under condition (2) considered in [6].

We assume that f and Ω in (3) are such that $c_0 > 0$, where c_0 is the critical speed defined in Section 3.2, i. e. that $\lambda_{1,N} < 0$ when $c = 0$. Below, for any $0 < c < c_0$, U^c denotes the unique (stable) solution of (13) given by Theorem 2.1.

Theorem 2.7. *Assume that (9)-(12) hold. Then, the following properties hold:*

(i)
$$\lim_{c\to c_0^-} U^c(x) = 0,$$
uniformly with respect to $x \in \Omega$;

(ii)
$$\lim_{c\to c_0^-} \frac{U^c(x)}{\|U^c\|_{L^\infty(\Omega)}} = \varphi(x),$$
uniformly with respect to $x \in \Omega$, where φ is the unique positive solution of

$$\begin{cases} \Delta\varphi + c_0\partial_1\varphi + f_s(x,0)\varphi = 0 & \text{a. e. in } \Omega \\ \partial_\nu\varphi = 0 & \text{on } \partial\Omega \\ \|\varphi\|_{L^\infty(\Omega)} = 1. \end{cases} \tag{26}$$

It should be noted that the uniqueness of the solution to (26) is a remarkable property which does not hold in general for positive solutions of linear equations in unbounded domains.

2.5. L^1 **convergence.** We still consider the case of straight infinite cylinder. Starting from the pointwise convergence of the solution $u(t,x)$ of (3) as $t\to\infty$, we are able to show that the convergence also holds in $L^1(\Omega)$. This is interesting from the point of view of biological models, as $\|u(t,\cdot)\|_{L^1(\Omega)}$ represents the total population at time t.

Theorem 2.8. *Consider problem (3) in the straight infinite cylinder Ω. The convergences in Theorem 2.2 also hold in the L^1 sense, provided the initial datum u_0 belongs to $L^1(\Omega)$.*

An analogous result holds true for the pure shift problem in the whole space considered in [6] which we now state.

Theorem 2.9. *Let $u(t,x)$ be the solution of (1) in all of space with an initial condition $u(0,x) = u_0(x) \in L^\infty(\mathbb{R}^N)\cap L^1(\mathbb{R}^N)$ which is nonnegative and not identically equal to zero. Under assumptions (19)-(21) and (2) the following properties hold:*

(i) *if $\lambda_1 \geq 0$ then*
$$\lim_{t\to\infty} \|u(t,\cdot)\|_{L^1(\mathbb{R}^N)} = 0;$$

(ii) *if $\lambda_1 < 0$ then*
$$\lim_{t\to\infty} \|u(t,\cdot) - U(\cdot - cte)\|_{L^1(\mathbb{R}^N)} = 0,$$
where U is the unique solution of (23).

2.6. **Seasonal dependence.** We consider problem (6) with $f(t,x,s) : \mathbb{R}\times\Omega\times[0,+\infty)\to\mathbb{R}$ periodic in t, with period $T>0$:

$$\forall\, t\in\mathbb{R},\ x\in\Omega,\ s\geq 0,\quad f(t+T,x,s)=f(t,x,s). \tag{27}$$

As in the case of asymptotically cylindrical domains, besides conditions (8), which are now required uniformly in $t\in\mathbb{R}$, we need some Hölder continuity assumptions on f for some $\alpha\in(0,1)$:

$$\forall\, s>0,\quad f(\cdot,\cdot,s), f_s(\cdot,\cdot,0)\in C^{\frac{\alpha}{2},\alpha}_{t,x}(\mathbb{R}\times\Omega),$$

where $C^{\frac{\alpha}{2},\alpha}_{t,x}(\mathcal{I}\times\mathcal{O})$, with $\mathcal{I}\subset\mathbb{R}$ and $\mathcal{O}\subset\mathbb{R}^N$, denotes the space of functions $\phi(t,x)$ such that $\phi(\cdot,x)\in C^{\frac{\alpha}{2}}(\mathcal{I})$ and $\phi(t,\cdot)\in C^{\alpha}(\mathcal{O})$ uniformly with respect to $x\in\mathcal{O}$ and $t\in\mathcal{I}$ respectively. The other assumptions on f are:

$$f(t,x,0)=0\quad\text{for } t\in\mathbb{R},\ x\in\Omega, \tag{28}$$

$$\exists\, S>0\ \text{ such that }\ \forall\, t\in\mathbb{R},\ x\in\Omega,\ s\geq S,\quad f(t,x,s)\leq 0, \tag{29}$$

$$\begin{cases} s\mapsto \dfrac{f(t,x,s)}{s}\ \text{ is nonincreasing for } t\in\mathbb{R},\ x\in\Omega \\ \text{and it is strictly decreasing for some } t\in\mathbb{R},\ x\in\Omega. \end{cases} \tag{30}$$

The analogue of condition (12) is required uniformly in t, that is,

$$\lim_{r\to\infty}\ \sup_{\substack{t\in\mathbb{R}\\ |x_1|>r\\ y\in\omega}} f_s(t,x_1,y,0)<0. \tag{31}$$

The notion of travelling wave is replaced in this framework by that of *pulsating travelling wave*, that is, a solution u to (6) such that $U(t,x_1,y):=u(t,x_1+ct,y)$ is periodic in t with period T. Thus, U satisfies

$$\begin{cases} \partial_t U=\Delta U+c\partial_1 U+f(t,x,U), & t\in\mathbb{R},\ x\in\Omega \\ \partial_\nu U(t,x)=0, & t\in\mathbb{R},\ x\in\partial\Omega \\ U>0\quad\text{in } \mathbb{R}\times\Omega & \\ U\ \text{ is bounded} & \\ U\ \text{ is } T\text{-periodic in } t, & \end{cases} \tag{32}$$

where U is extended by periodicity for $t<0$. We denote by $\mathcal{P}$ the linearized operator about the steady state $w\equiv 0$ associated with the parabolic equation in (32):

$$\mathcal{P}w=\partial_t w-\Delta w-\partial_1 w-f_s(t,x,0)w.$$

By analogy to (24), we define the *generalized T-periodic Neumann principal eigenvalue* of the parabolic operator $\mathcal{P}$ in $\mathbb{R}\times\Omega$ in the following way:

(33)

$$\mu_{1,N}:=\sup\{\mu\in\mathbb{R}\ :\ \exists\,\phi\in C^{1,2}_{t,x}(\mathbb{R}\times(-r,r)\times\overline{\omega}),\ \forall\, r>0,\ \phi \text{ is } T\text{-periodic in } t,$$
$$\phi>0 \text{ and } (\mathcal{P}-\mu)\phi\geq 0\ \text{ in } \mathbb{R}\times\Omega,\ \partial_\nu\phi\geq 0\ \text{ on } \mathbb{R}\times\partial\Omega\}.$$

Theorem 2.10. *Assume that (27)-(31) hold. Then problem (32) admits a solution if and only if $\mu_{1,N}<0$. Moreover, when it exists, the solution is unique and satisfies*

$$\lim_{|x_1|\to\infty} U(t,x_1,y)=0,$$

uniformly with respect to $t\in\mathbb{R}$ and $y\in\omega$.

Theorem 2.11. *Let $u(t,x)$ be the solution of (6) with an initial condition $u(0,x) = u_0(x) \in L^\infty(\Omega)$ which is nonnegative and not identically equal to zero. Under assumptions (28)-(31) the following properties hold:*

(i) *if $\mu_{1,N} \geq 0$ then*
$$\lim_{t\to\infty} u(t,x) = 0,$$
uniformly with respect to $x \in \Omega$;

(i) *if $\mu_{1,N} < 0$ then*
$$\lim_{t\to\infty} (u(t,x_1,y) - U(t,x_1 - ct, y)) = 0,$$
uniformly with respect to $(x_1,y) \in \Omega$, where U is the unique solution of (32).

One is also led to (32) by considering the two speeds problem (7), with c and c' given, $c \neq c'$ and $g(x_1,y,s)$ periodic in x_1, with period $l > 0$:
$$\forall\ (x_1,y) \in \Omega,\ s \geq 0, \quad g(x_1 + l, y, s) = g(x_1,y,s).$$
Indeed, if u is a solution of (7) then $\tilde{u}(t,x_1,y) := u(t,x_1 + ct, y)$ satisfies
$$\partial_t \tilde{u} = \Delta\tilde{u} + c\partial_1\tilde{u} + h(t,x,\tilde{u}), \quad t \in \mathbb{R},\ x \in \Omega,$$
where the function
$$h(t,x_1,y,s) := f(x_1,y,s) + g(x_1 + (c - c')t, y, s)$$
is $l/(c-c')$-periodic in t. As a consequence, the problem of pulsating travelling wave solutions u to (7) such that $U(t,x_1,y) := u(t,x_1 + ct, y)$ is $l/(c-c')$-periodic in t is given by (32) with f replaced by h and $T = l/(c-c')$. Furthermore, as the transformation $\tilde{u}(t,x_1,y) := u(t,x_1 + ct,y)$ reduces (6) and (7) to the same kind of problem, Theorem 2.11 holds with (6) replaced by (7), f by h and $T = l/(c-c')$.

3. The pure shift problem: Straight infinite cylinder

Let us recall the notation used in this framework:
$$\Omega = \mathbb{R} \times \omega,$$
$$\mathcal{L}w = \Delta w + c\partial_1 w + f_s(x_1,y,0)w,$$
$$\zeta = -\lim_{r\to\infty} \sup_{\substack{|x_1|>r \\ y\in\omega}} f_s(x_1,y,0),$$
$$\lambda_{1,N} = \lambda_{1,N}(-\mathcal{L},\Omega).$$
We further denote
$$\forall\ r > 0, \quad \Omega_r := (-r,r) \times \omega.$$

To prove the existence and uniqueness of travelling wave solutions to (3), Theorem 2.1, we use the same method as in [6]. The only difference is that here we take into account the Neumann boundary conditions in the definition of the generalized principal eigenvalue $\lambda_{1,N}$. This leads us to consider eigenvalue problems in the finite cylinders $(-r,r) \times \omega$, with mixed Dirichlet-Neumann boundary conditions, for which we need some regularity results up to the corners $\{\pm r\} \times \partial\omega$ presented in the appendix. Properties of this eigenvalue are described in Section 3.1. Next, we reduce the elliptic equation in (13) to an equation with self-adjoint linear term via a Liouville transformation. This will allow us to define the critical speed c_0 as well as to derive the exponential decay of solutions to (13). Using this result we prove a comparison principle for (13) which yields the uniqueness and the necessary condition for the existence of travelling wave solutions. The sufficient condition will be seen to follow from the properties of $\lambda_{1,N}$ and a sub and supersolution argument. Thanks to Theorem 2.1, we will derive a result about entire solutions to (3) which is useful in completing the proof of Theorem 2.2.

3.1. **Properties of $\lambda_{1,N}$.** We derive some results concerning the generalized Neumann principal eigenvalue $\lambda_{1,N}(-L,\Omega)$ that will be needed in the sequel. Here, L is an operator of the type

$$Lw := \Delta w + \beta(x)\cdot\nabla w + \gamma(x)w,$$

with $\beta = (\beta_1,\cdots,\beta_N)$ and γ bounded.

We first introduce the principal eigenvalues in the finite cylinders Ω_r, with Neumann boundary conditions on the "sides" $(-r,r)\times\partial\omega$ and Dirichlet boundary conditions on the "bases" $\{\pm r\}\times\omega$. The existence of such eigenvalues follows from the Krein-Rutman theory, as for the principal eigenvalues in bounded smooth domains with either Dirichlet or Neumann boundary conditions. Some technical difficulties arise due to the non-smoothness of Ω_r on the "corners" $\{\pm r\}\times\partial\omega$. This problem can be handled by extending the solutions outside Ω_r by reflection. Since such an argument is quite classical and technical, we postpone the proof of the next result to Appendix A.

Theorem 3.1. *For any $r>0$ there exists a unique real number $\lambda(r)$ such that the eigenvalue problem*

$$\begin{cases} -L\varphi_r = \lambda(r)\varphi_r & a.\ e.\ in\ \Omega_r \\ \partial_\nu\varphi_r = 0 & on\ (-r,r)\times\partial\omega \\ \varphi_r = 0 & on\ \{\pm r\}\times\overline{\omega} \end{cases}$$

admits a positive solution $\varphi_r\in W^{2,p}(\Omega_r)$, for any $p>1$. Moreover, φ_r is unique up to a multiplicative constant.

The quantity $\lambda(r)$ and the function φ_r in the previous theorem are respectively called principal eigenvalue and eigenfunction of $-L$ in Ω_r (with mixed Dirichlet/Neumann boundary conditions).

Proposition 1. *The function $\lambda(r):\mathbb{R}^+\to\mathbb{R}$ of principal eigenvalues of $-L$ in Ω_r is decreasing and satisfies*

$$\lim_{r\to\infty}\lambda(r) = \lambda_{1,N}(-L,\Omega).$$

Furthermore, there exists a generalized Neumann principal eigenfunction of $-L$ in Ω, that is, a positive function $\varphi\in W^{2,p}(\Omega_r)$, for any $p>1$ and $r>0$, such that

$$(34)\qquad \begin{cases} -L\varphi = \lambda_{1,N}(-L,\Omega)\varphi & a.\ e.\ in\ \Omega \\ \partial_\nu\varphi = 0 & on\ \partial\Omega. \end{cases}$$

Proof. Let $0<r_1<r_2$ and assume, by way of contradiction, that $\lambda_{1,N}(r_1)\le\lambda_{1,N}(r_2)$. Consider the associated principal eigenfunctions φ_{r_1} and φ_{r_2} of $-L$ in Ω_{r_1} and Ω_{r_2} respectively. Note that the Hopf lemma yields $\varphi_{r_2}>0$ on $(-r_2,r_2)\times\partial\omega$. Set

$$k := \max_{\overline{\Omega}_{r_1}}\frac{\varphi_{r_1}}{\varphi_{r_2}}.$$

Clearly, $k > 0$ and the function $w := k\varphi_{r_2} - \varphi_{r_1}$ is nonnegative, vanishes at some point $x_0 \in \overline{\Omega}_{r_1}$ and satisfies

$$(L + \lambda_{1,N}(r_2))w \le 0 \qquad \text{a. e. in } \Omega_{r_1}.$$

Since $\varphi_{r_1} = 0$ on $\{\pm r_1\} \times \overline{\omega}$, the point x_0 must belong to $(-r_1, r_1) \times \overline{\omega}$. If $x_0 \in \Omega_{r_1}$ then the strong maximum principle yields $w \equiv 0$, which is impossible. As a consequence, it is necessarily the case that $x_0 \in (-r_1, r_1) \times \partial\omega$. But this leads to another contradiction in view of Hopf's lemma:

$$0 > \partial_\nu w(x_0) = k\partial_\nu \varphi_{r_2}(x_0) - \partial_\nu \varphi_{r_1}(x_0) = 0.$$

Hence, the function $\lambda(r) : \mathbb{R}^+ \to \mathbb{R}$ is decreasing. Let us show that the quantity $\lambda_{1,N}(-L, \Omega)$ is well defined and satisfies

$$\forall\, r > 0, \quad \lambda_{1,N}(-L, \Omega) \le \lambda(r). \tag{35}$$

Taking $\phi \equiv 1$ in (14) shows that $\lambda_{1,N}(-L, \Omega) \ge -\sup_\Omega \gamma$. If (35) does not hold then there exists $R > 0$ such that $\lambda(R) < \lambda_{1,N}(-L, \Omega)$. By definition (14), we can find a constant $\lambda > \lambda(R)$ and a positive function $\phi \in W^{2,N+1}(\Omega_r)$, for any $r > 0$, such that

$$\begin{cases} (L + \lambda)\phi \le 0 & \text{a. e. in } \Omega \\ \partial_\nu \phi \ge 0 & \text{on } \partial\Omega. \end{cases}$$

A contradiction follows by arguing as before, with φ_{r_1} and φ_{r_2} replaced by φ_R and ϕ respectively. Consequently,

$$\tilde{\lambda} := \lim_{r\to\infty} \lambda(r) \ge \lambda_{1,N}(-L, \Omega).$$

To prove equality, consider the sequence of generalized principal eigenfunctions $(\varphi_n)_{n\in\mathbb{N}}$, normalized by $\varphi_n(x_0) = 1$, where x_0 is fixed, say in Ω_1. Extending by reflection the functions φ_n to larger cylinders, as done in Appendix A, and using the Harnack inequality, we see that, for $m \in \mathbb{N}$, the $(\varphi_n)_{n>m}$ are uniformly bounded in Ω_m. Hence, by standard elliptic estimates and embedding theorems, there exists a subsequence $(\varphi_{n_k})_{k\in\mathbb{N}}$ converging in $C^1(\overline{\Omega}_\rho)$ and weakly in $W^{2,p}(\Omega_\rho)$, for any $\rho > 0$ and $p > 1$, to some nonnegative function φ satisfying

$$\begin{cases} -L\varphi = \tilde{\lambda}\varphi & \text{a. e. in } \Omega \\ \partial_\nu \varphi = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $\varphi(x_0) = 1$, the strong maximum principle yields $\varphi > 0$ in Ω. Thus, taking $\phi = \varphi$ in (14) we get $\lambda_{1,N}(-L, \Omega) \ge \tilde{\lambda}$, which concludes the proof. □

In what follows, $\lambda(r)$ and φ_r will always denote respectively the principal eigenvalue and eigenfunction of $-\mathcal{L}$ in Ω_r. We will further denote by φ a generalized Neumann principal eigenfunction of $-\mathcal{L}$ in Ω, given by Proposition 1.

3.2. **Definition of the critical speed c_0.** Through the Liouville transformation $V(x_1, y) := U(x_1, y)e^{\frac{c}{2}x_1}$, problem (13) reduces to

$$\begin{cases} \Delta V + f(x_1, y, V(x_1, y)e^{-\frac{c}{2}x_1})e^{\frac{c}{2}x_1} - \dfrac{c^2}{4}V = 0 & \text{for a. e. } (x_1, y) \in \Omega \\ \partial_\nu V = 0 & \text{on } \partial\Omega \\ V > 0 \quad \text{in } \Omega & \\ V(x_1, y)e^{-\frac{c}{2}x_1} \quad \text{bounded.} & \end{cases} \tag{36}$$

The associated linearized operator about $V \equiv 0$ is

$$\tilde{\mathcal{L}}w := \Delta w + (f_s(x, 0) - c^2/4)w.$$

Since $\tilde{\mathcal{L}}\phi = (\mathcal{L}(\phi e^{-\frac{c}{2}x_1}))e^{\frac{c}{2}x_1}$ for any function ϕ, an immediate consequence of definition (14) is that $\lambda_{1,N}(-\tilde{\mathcal{L}},\Omega) = \lambda_{1,N}$.

In order to define the critical speed c_0, we introduce the linear operator

$$\mathcal{L}_0 u := \Delta u + f_s(x,0)u$$

and we set $\lambda_0 := \lambda_{1,N}(-\mathcal{L}_0,\Omega)$.

Proposition 2. *Define the critical speed as*

$$c_0 := \begin{cases} 2\sqrt{-\lambda_0} & \text{if } \lambda_0 < 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then $\lambda_{1,N} < 0$ iff $0 < c < c_0$.

Proof. This simply follows from the fact that

$$\lambda_{1,N} = \lambda_{1,N}(-\tilde{\mathcal{L}},\Omega) = \lambda_{1,N}(-\mathcal{L}_0 + c^2/4,\Omega) = \lambda_0 + c^2/4.$$

□

3.3. **Exponential decay of travelling waves.** Owing to the results of Section 3.1, the exponential decay of solutions to (13) follows essentially as in [6]. However, for the sake of completeness, we include the proofs here.

Lemma 3.2. *Let $V \in W^{2,p}(\Omega_r)$, for some $p > N$ and every $r > 0$, be a positive function such that $\partial_\nu V \le 0$ on $\partial\Omega$. Assume that for some $\gamma > 0$, V satisfies*

$$\sup_{(x_1,y)\in\Omega} V(x_1,y)e^{-\sqrt{\gamma}|x_1|} < \infty, \qquad \liminf_{|x_1|\to\infty} \frac{\Delta V(x_1,y)}{V(x_1,y)} > \gamma,$$

uniformly in $y \in \omega$. Then,

$$\lim_{|x_1|\to\infty} V(x_1,y)e^{\sqrt{\gamma}|x_1|} = 0,$$

uniformly in $y \in \omega$.

Proof. By the hypotheses on V, there exist $\varepsilon, R > 0$ such that $\Delta V \ge (\gamma+\varepsilon)V$ a. e. in $\Omega\backslash\overline{\Omega}_R$. Set $\kappa := \sup_{(x_1,y)\in\Omega} V(x_1,y)e^{-\sqrt{\gamma}|x_1|}$. For $a > 0$ let $\vartheta_a : [R, R+a] \to \mathbb{R}$ be the solution to

$$\begin{cases} \vartheta'' = (\gamma+\varepsilon)\vartheta & \text{in } (R, R+a) \\ \vartheta(R) = \kappa e^{\sqrt{\gamma}R} \\ \vartheta(R+a) = \kappa e^{\sqrt{\gamma}(R+a)}. \end{cases}$$

Hence, $\vartheta_a(\rho) = A_a e^{-\sqrt{\gamma+\varepsilon}\,\rho} + B_a e^{\sqrt{\gamma+\varepsilon}\,\rho}$, with

$$A_a = \kappa e^{(\sqrt{\gamma}+\sqrt{\gamma+\varepsilon})R}\left(1 - \frac{e^{\sqrt{\gamma}a} - e^{-\sqrt{\gamma+\varepsilon}a}}{e^{\sqrt{\gamma+\varepsilon}a} - e^{-\sqrt{\gamma+\varepsilon}a}}\right),$$

$$B_a = \kappa e^{(\sqrt{\gamma}-\sqrt{\gamma+\varepsilon})R}\frac{e^{\sqrt{\gamma}a} - e^{-\sqrt{\gamma+\varepsilon}a}}{e^{\sqrt{\gamma+\varepsilon}a} - e^{-\sqrt{\gamma+\varepsilon}a}}.$$

The function $\theta_a(x_1,y) := \vartheta_a(|x_1|)$ satisfies

$$\begin{cases} \Delta\theta_a(x_1,y) = (\gamma+\varepsilon)\theta_a(x_1,y) & \text{for } R < |x_1| < R+a,\ y \in \omega \\ \partial_\nu\theta_a(x_1,y) = 0 & \text{for } R < |x_1| < R+a,\ y \in \partial\omega. \end{cases}$$

Since V is a subsolution of the above problem and $V \le \theta_a$ on $\{\pm R, \pm(R+a)\}\times\omega$, the comparison principle yields $V \le \theta_a$ in $\Omega_{R+a}\backslash\overline{\Omega}_R$, for any $a > 0$. Therefore, for $|x_1| > R$ and $y \in \omega$ we get

$$V(x_1,y) \le \lim_{a\to\infty} \theta_a(x_1,y) = \kappa e^{(\sqrt{\gamma}+\sqrt{\gamma+\varepsilon})R} e^{-\sqrt{\gamma+\varepsilon}\,|x_1|},$$

which concludes the proof. □

Proposition 3. *Let U be a solution of (13) and assume that (9), (11), (12) hold. Then, there exist two constants $h, \beta > 0$ such that*

$$\forall\,(x_1, y) \in \Omega, \quad U(x_1, y) \leq h e^{-\beta |x_1|}.$$

Proof. The function $V(x_1, y) := U(x_1, y) e^{\frac{c}{2} x_1}$ is a solution of (36). Hence,

$$\frac{\Delta V(x_1, y)}{V(x_1, y)} = -\xi(x_1, y) + \frac{c^2}{4} \quad \text{for a. e. } (x_1, y) \in \Omega,$$

where

$$\xi(x_1, y) := \frac{f(x_1, y, U(x_1, y))}{U(x_1, y)},$$

which belongs to $L^\infty(\Omega)$ because $f(x, \cdot)$ vanishes at 0 and is locally Lipschitz continuous. Moreover, $\xi(x_1, y) \leq f_s(x_1, y, 0)$ due to (11). Consider a constant $\gamma \in (c^2/4, \zeta + c^2/4)$, where ζ is the positive constant in (12). We see that

$$\lim_{r \to \infty} \inf_{\substack{|x_1| > r \\ y \in \omega}} \frac{\Delta V(x_1, y)}{V(x_1, y)} \geq - \lim_{r \to \infty} \sup_{\substack{|x_1| > r \\ y \in \omega}} f_s(x_1, y, 0) + \frac{c^2}{4} = \zeta + \frac{c^2}{4} > \gamma.$$

On the other hand, $V(x_1, y) e^{-\sqrt{\gamma}|x_1|} \leq V(x_1, y) e^{-\frac{c}{2} x_1}$ which is bounded on Ω. Therefore, by Lemma 3.2 there exists a positive constant C such that

$$\forall\,(x_1, y) \in \Omega, \quad U(x_1, y) = V(x_1, y) e^{-\frac{c}{2} x_1} \leq C e^{-\sqrt{\gamma}\,|x_1| - \frac{c}{2} x_1} \leq C e^{-(\sqrt{\gamma} - \frac{c}{2})|x_1|}.$$

□

3.4. **Comparison principle.** The following is a comparison principle which contains, as a particular case, the uniqueness of solutions to (13) vanishing at infinity.

Theorem 3.3. *Assume that (9), (11), (12) hold. Let $\underline{U}, \overline{U} \in W^{2,p}(\Omega_r)$, for some $p > N$ and every $r > 0$, be two nonnegative functions satisfying*

$$\begin{cases} -\Delta \underline{U} - c\partial_1 \underline{U} \leq f(x, \underline{U}) & \text{for a. e. } x \in \Omega \\ \partial_\nu \underline{U} \leq 0 & \text{on } \partial\Omega, \end{cases}$$

$$\begin{cases} -\Delta \overline{U} - c\partial_1 \overline{U} \geq f(x, \overline{U}) & \text{for a. e. } x \in \Omega \\ \partial_\nu \overline{U} \geq 0 & \text{on } \partial\Omega, \end{cases}$$

$$\overline{U} > 0 \ \text{ in } \Omega, \qquad \lim_{|x_1| \to \infty} \underline{U}(x_1, y) = 0 \ \text{ uniformly in } y \in \omega.$$

Then $\underline{U} \leq \overline{U}$ in Ω.

Proof. For any $\varepsilon > 0$ define the set

$$K_\varepsilon := \{k > 0 \ : \ k\overline{U} \geq \underline{U} - \varepsilon \text{ in } \overline{\Omega}\}.$$

Since by hypothesis there exists $R(\varepsilon) > 0$ such that

$$\forall\,|x_1| \geq R(\varepsilon),\ y \in \omega, \quad \underline{U}(x_1, y) - \varepsilon \leq 0, \tag{37}$$

and $\overline{U} > 0$ in $\overline{\Omega}$ by Hopf's lemma, the set K_ε is nonempty. For $\varepsilon > 0$ set $k(\varepsilon) := \inf K_\varepsilon$. Clearly, the function $k : \mathbb{R}^+ \to \mathbb{R}$ is nonincreasing. Let us assume, by way of contradiction, that

$$k^* := \lim_{\varepsilon \to 0^+} k(\varepsilon) > 1$$

(with, possibly, $k^*=\infty$). For any $0<\varepsilon<\sup_\Omega \underline{U}$ we see that $k(\varepsilon)>0$, $k(\varepsilon)\overline{U}-\underline{U}+\varepsilon\geq 0$ in $\overline{\Omega}$ and there exists a sequence $((x_{1,n}^\varepsilon,y_n^\varepsilon))_{n\in\mathbb{N}}$ in $\overline{\Omega}$ such that

$$\left(k(\varepsilon)-\frac{1}{n}\right)\overline{U}(x_{1,n}^\varepsilon,y_n^\varepsilon)<\underline{U}(x_{1,n}^\varepsilon,y_n^\varepsilon)-\varepsilon.$$

From (37) it follows that, for fixed $\varepsilon>0$, $(x_{1,n}^\varepsilon,y_n^\varepsilon)\in\Omega_{R(\varepsilon)}$ for n large enough and then, up to subsequences, $(x_{1,n}^\varepsilon,y_n^\varepsilon)$ converges to some $(x_1(\varepsilon),y(\varepsilon))\in\overline{\Omega}_{R(\varepsilon)}$ as n goes to infinity. Hence, $k(\varepsilon)\overline{U}(x_1(\varepsilon),y(\varepsilon))\leq\underline{U}(x_1(\varepsilon),y(\varepsilon))-\varepsilon$. Consequently, for any $\varepsilon>0$ we have the following:

$$k(\varepsilon)\overline{U}-\underline{U}+\varepsilon\geq 0\ \text{ in }\overline{\Omega},\qquad (k(\varepsilon)\overline{U}-\underline{U}+\varepsilon)(x_1(\varepsilon),y(\varepsilon))=0. \tag{38}$$

We consider separately two different situations.

Case 1: $\liminf\limits_{\varepsilon\to 0^+}|x_1(\varepsilon)|<\infty$.

Then, there exists a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^+$ such that

$$\lim_{n\to\infty}\varepsilon_n=0,\qquad \xi:=\lim_{n\to\infty}x_1(\varepsilon_n)\in\mathbb{R},\qquad \eta:=\lim_{n\to\infty}y(\varepsilon_n)\in\overline{\omega}.$$

From (38) it follows that $k^*<\infty$ and that the function $W:=k^*\overline{U}-\underline{U}$ is nonnegative and vanishes at (ξ,η). Also, since $k^*>1$, condition (11) yields

$$-\Delta W-c\partial_1 W\geq k^*f(x,\overline{U})-f(x,\underline{U})\geq f(x,k^*\overline{U})-f(x,\underline{U})\quad\text{a. e. in }\Omega,$$

with strict inequality a. e. in D. Therefore, thanks to the Lipschitz continuity of f in the second variable, W is a supersolution of a linear elliptic equation in Ω. Since W is nonnegative in Ω, vanishes at (ξ,η) and $\partial_\nu W=0$ on $\partial\Omega$, the strong maximum principle and the Hopf lemma yield $W\equiv 0$. This is a contradiction because W is a strict supersolution in D.

Case 2: $\lim\limits_{\varepsilon\to 0^+}|x_1(\varepsilon)|=\infty$.

For $\varepsilon>0$ set $W^\varepsilon:=k(\varepsilon)\overline{U}-\underline{U}+\varepsilon$. By (38) we have that $W^\varepsilon\geq 0$ and $W^\varepsilon(x_1(\varepsilon),y(\varepsilon))=0$. Furthermore, for $\varepsilon>0$ small enough $k(\varepsilon)>1$ and then, for a. e. $x\in\Omega$,

$$\begin{aligned}-\Delta W^\varepsilon-c\partial_1 W^\varepsilon&\geq k(\varepsilon)f(x,\overline{U})-f(x,\underline{U})\\&\geq f(x,k(\varepsilon)\overline{U})-f(x,\underline{U}).\end{aligned}\tag{39}$$

Since $|x_1(\varepsilon)|\to\infty$, by (12) we can take $\varepsilon,\delta>0$ small enough in such a way that $f_s(x,0)<0$ for $x\in\mathcal{O}:=B_\delta\cap\Omega$, where B_δ is the ball of radius δ about $(x_1(\varepsilon),y(\varepsilon))$. Moreover, up to chosing a smaller δ if need be, we can assume that $\underline{U}>k(\varepsilon)\overline{U}$ in $\mathcal{O}$. Using (9) and (11) we derive, for $x\in\mathcal{O}$,

$$\begin{aligned}f(x,k(\varepsilon)\overline{U})-f(x,\underline{U})&\geq f(x,k(\varepsilon)\overline{U})-\frac{f(x,k(\varepsilon)\overline{U})}{k(\varepsilon)\overline{U}}\underline{U}\\&=\frac{f(x,k(\varepsilon)\overline{U})}{k(\varepsilon)\overline{U}}(k(\varepsilon)\overline{U}-\underline{U})\\&\geq f_s(x,0)(k(\varepsilon)\overline{U}-\underline{U})\\&>0.\end{aligned}$$

Thus, in view of (39), $(x_1(\varepsilon),y(\varepsilon))$ cannot be an interior minimum for W^ε. Then, $(x_1(\varepsilon),y(\varepsilon))\in\partial\Omega$ and by Hopf's lemma in $\mathcal{O}$ one has $\partial_\nu W^\varepsilon((x_1(\varepsilon),y(\varepsilon))<0$, which contradicts the assumption.

We have shown that $k^* := \lim_{\varepsilon\to 0^+} k(\varepsilon) \leq 1$. Consequently, from (38) we finally get

$$\underline{U} \leq \lim_{\varepsilon\to 0^+} (k(\varepsilon)\overline{U} + \varepsilon) \leq \overline{U} \quad \text{in } \overline{\Omega}.$$

□

3.5. **Existence and uniqueness of travelling waves.**

Proof of Theorem 2.1. Case 1: $\lambda_{1,N} < 0$.
We proceed exactly as in [2]. By Proposition 1 there exists $R > 0$ such that $\lambda(R) < 0$. Define the function

$$\underline{U}(x) := \begin{cases} \kappa\varphi_R(x) & x \in \Omega_R \\ 0 & \text{otherwise,} \end{cases}$$

where $\kappa > 0$ will be chosen appropriately small later. We see that $\partial_\nu \underline{U} = 0$ on $\partial\Omega$ and that

$$-\Delta(\kappa\varphi_R) - c\partial_1(\kappa\varphi_R) = (f_s(x,0) + \lambda(R))\kappa\varphi_R \quad \text{a. e. in } \Omega_R.$$

Hence, since $f(x,0) = 0$ by (9) and $s \mapsto f(x,s) \in C^1([0,\delta])$, for κ small enough $\underline{U}$ satisfies $-\Delta\underline{U} - c\partial_1\underline{U} \leq f(x,\underline{U})$ a. e. in Ω_R. One can then readily check that $\underline{U} \in W^{1,\infty}(\Omega)$ is a (weak) subsolution of

$$\begin{cases} \Delta U + c\partial_1 U + f(x,U) = 0 & \text{a. e. in } \Omega \\ \partial_\nu U = 0 & \text{on } \partial\Omega. \end{cases} \tag{40}$$

On the other hand, the function $\overline{U}(x) \equiv S$ - where S is the constant in (10) - is a supersolution to (40). Also, we can chose κ small enough in such a way that $\underline{U} \leq \overline{U}$. Consequently, using a classical iterative scheme (see e. g. [3]) we can find a function $U \in W^{2,p}(\Omega_r)$, for any $p > 1$ and $r > 0$, satisfying (40) and $\underline{U} \leq U \leq \overline{U}$ in Ω. The strong maximum principle implies that U is strictly positive and then it solves (13).

Case 2: $\lambda_{1,N} \geq 0$.
Assume by contradiction that (13) admits a solution U. Let φ be a generalized Neumann principal eigenfunction of $-\mathcal{L}$ in Ω (cf. Proposition 1), normalized in such a way that $0 < \varphi(x_0) < U(x_0)$, for some $x_0 \in \Omega$. Then, φ satisfies $\partial_\nu\varphi = 0$ on $\partial\Omega$ and, by (9), (11),

$$-\Delta\varphi - c\partial_1\varphi = (f_s(x_1,y,0) + \lambda_{1,N})\varphi \geq f(x_1,y,\varphi) \quad \text{a. e. in } \Omega.$$

Therefore, since by Proposition 3 $\lim_{|x_1|\to\infty} U(x_1,y) = 0$ uniformly in $y \in \omega$, we can apply Theorem 3.3 with $\underline{U} = U$ and $\overline{U} = \varphi$ and infer that $U \leq \varphi$: contradiction.

The uniqueness result immediately follows from Proposition 3 and Theorem 3.3.

□

3.6. **Large time behaviour.**

We will make use of a result concerning entire solutions (that is, solutions for all $t \in \mathbb{R}$) of the evolution problem associated with (13):

$$\begin{cases} \partial_t u^* = \Delta u^* + c\partial_1 u^* + f(x,u^*), & t \in \mathbb{R},\ x \in \Omega \\ \partial_\nu u^* = 0, & t \in \mathbb{R},\ x \in \partial\Omega. \end{cases} \tag{41}$$

Lemma 3.4. *Let u^* be a nonnegative bounded solution of (41). Under assumptions (9)-(12) the following properties hold:*

(i) *if $\lambda_{1,N} \geq 0$ then $u^* \equiv 0$;*

(ii) *if* $\lambda_{1,N} < 0$ *and there exist a sequence* $(t_n)_{n\in\mathbb{N}}$ *in* $\mathbb{R}$ *and a point* $x_0 \in \overline{\Omega}$ *such that*

$$\lim_{n\to\infty} t_n = -\infty, \qquad \liminf_{n\to\infty} u^*(t_n, x_0) > 0, \tag{42}$$

then $u^*(t,x) \equiv U(x)$*, where* U *is the unique solution of* (13)*.*

Proof. Let S be the positive constant in (10). Set

$$S^* := \max\{S, \|u^*\|_{L^\infty(\mathbb{R}\times\Omega)}\}$$

and let w be the solution to (41) for $t > 0$, with initial condition $w(0,x) = S^*$. Since the constant function S^* is a stationary supersolution to (41), the parabolic comparison principle implies that w is nonincreasing in t (and it is nonnegative). Consequently, as $t \to +\infty$, $w(t,x)$ converges pointwise in $x \in \Omega$ to a function $W(x)$. Using standard parabolic estimates up to the boundary, together with compact injection results, one sees that this convergence is actually uniform in Ω_ρ, for any $\rho > 0$, and that W solves (40). For any $h \in \mathbb{R}$ the function $w_h(t,x) := w(t+h,x)$ is a solution to (41) in $(-h,+\infty)\times\Omega$ satisfying $w_h(-h,x) = S^* \geq u^*(-h,x)$. Thus, again the parabolic comparison principle yields $w_h \geq u^*$ in $(-h,+\infty)\times\Omega$. Therefore,

$$\forall\, t \in \mathbb{R},\ x \in \Omega, \quad u^*(t,x) \leq \lim_{h\to+\infty} w_h(t,x) = W(x). \tag{43}$$

Let us consider separately the two different cases.

(i) $\lambda_{1,N} \geq 0$.

Due to Theorem 2.1, the function W cannot be strictly positive in Ω. Thus, W vanishes somewhere in Ω and then the elliptic strong maximum principle yields $W \equiv 0$. The statement then follows from (43).

(ii) $\lambda_{1,N} < 0$ and (42) holds for some $(t_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ and $x_0 \in \overline{\Omega}$.

We claim that condition (42) yields

$$\forall\, r > 0, \quad \liminf_{\substack{n\to\infty \\ x\in\Omega_r}} u^*(t_n, x) > 0. \tag{44}$$

Let us postpone for a moment the proof of (44). By Proposition 1, there exists $R > 0$ such that $\lambda(R) < 0$. Consider the same function $\underline{U}$ as in the proof of Theorem 2.1:

$$\underline{U}(x) := \begin{cases} \kappa\varphi_R(x) & x \in \Omega_R \\ 0 & \text{otherwise,} \end{cases}$$

We know that, for κ small enough, $\underline{U}$ is a subsolution to (40). Moreover, owing to (44), κ can be chosen in such a way that $\underline{U}(x) \leq u^*(t_n,x)$ for n large enough and $x \in \Omega$. Let v be the solution to (41) for $t > 0$, with initial condition $v(0,x) = \underline{U}(x)$. By comparison, we know that the function v is nondecreasing in t and it is bounded from above by S^*. Then, as t goes to infinity, $v(t,x)$ converges locally uniformly to the unique solution U to (13) (the strict positivity follows from the elliptic strong maximum principle). For n large enough the function $v_n(t,x) := v(t-t_n,x)$ satisfies

$$\forall\, x \in \Omega, \quad v_n(t_n,x) = \underline{U}(x) \leq u^*(t_n,x).$$

Hence, the parabolic comparison principle yields

$$\forall\, t \in \mathbb{R},\ x \in \Omega, \quad u^*(t,x) \geq \lim_{n\to\infty} v_n(t,x) = U(x).$$

Combining the above inequality with (43) we obtain

$$\forall\, t \in \mathbb{R},\ x \in \Omega, \quad U(x) \leq u^*(t,x) \leq W(x).$$

This shows that W is positive and then it is a solution to (13). The uniqueness result of Theorem 2.1 then yields $u^* \equiv U$.

To conclude the proof, it only remains to show (44). Assume by contradiction that there exists $r > 0$ such that the inequality does not hold. Then, there exists a sequence $((x_1^n, y^n))_{n\in\mathbb{N}}$ in Ω_r such that (up to subsequences)

$$\lim_{n\to\infty} u^*(t_n, x_1^n, y^n) = 0.$$

It is not restrictive to assume that (x_1^n, y^n) converges to some $(\xi, \eta) \in \overline{\Omega}_r$ as n goes to infinity. Parabolic estimates and embedding theorems imply that the sequence of functions $u_n^*(t, x_1, y) := u^*(t+t_n, x_1, y)$ converges (up to subsequences) in $(-\rho, \rho) \times \Omega_\rho$, for any $\rho > 0$, to a nonnegative solution u_∞^* of (41) satisfying $u_\infty^*(0, \xi, \eta) = 0$. If u_∞^* was smooth then the parabolic strong maximum principle and Hopf's lemma would imply $u_\infty^*(t, x) = 0$, for $t \leq 0$ and $x \in \overline{\Omega}$, which is impossible because $u_\infty^*(0, x_0) > 0$ by (42). To handle the case where u_∞^* is only a weak solution of (41), one can extend u_∞^* to a nonnegative solution of a parabolic equation in $\mathbb{R}\times\mathbb{R}\times\tilde{\omega}$, with $\omega \subset\subset \tilde{\omega}$, as done in the appendix. Hence, one gets a contradiction by applying the strong maximum principle. □

Proof of Theorem 2.2. Set $S' := \max\{S, \|u_0\|_{L^\infty(\Omega)}\}$, where S is the positive constant in (10). Since the constant functions 0 and S' are a sub and a supersolution of (3), with initial datum respectively below and above u_0, standard theory of semilinear parabolic equations yields the existence of a unique (weak) solution u to (1) with initial condition $u(0, x) = u_0(x)$ (see e. g. [14], [15]). Moreover, u satisfies $0 \leq u \leq S'$ in $\mathbb{R}^+ \times \overline{\Omega}$. By extending $u(t, \cdot)$ to a larger cylinder $(\mathbb{R}\times\tilde{\omega}) \supset\supset \Omega$ by reflection (see Appendix A) and applying the parabolic strong maximum principle, we find that $u(t, x) > 0$ for $t > 0$ and $x \in \overline{\Omega}$. Define $\tilde{u}(t, x_1, y) := u(t, x_1 + ct, y)$. Then, $\tilde{u}$ satisfies $0 < \tilde{u} \leq S'$ in $\mathbb{R}^+ \times \overline{\Omega}$ and solves

$$\left\{\begin{array}{ll} \partial_t\tilde{u} = \Delta\tilde{u} + c\partial_1\tilde{u} + f(x, \tilde{u}), & t > 0,\ x \in \Omega \\ \partial_\nu\tilde{u}(t, x) = 0, & t > 0,\ x \in \partial\Omega, \end{array}\right. \tag{45}$$

with initial condition $\tilde{u}(0, x) = u_0(x)$. The rest of the proof is divided into two parts.

Step 1: the function $\tilde{u}$ satisfies

$$\lim_{t\to\infty} \tilde{u}(t, x) = U(x) \quad \text{uniformly in } x \in \Omega_r,\ \forall\, r > 0, \tag{46}$$

where $U \equiv 0$ if $\lambda_{1,N} \geq 0$, while U is the unique solution to (13) if $\lambda_{1,N} < 0$.

Let $(t_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ satisfying $\lim_{k\to\infty} t_k = +\infty$. Then, parabolic estimates and embedding theorems imply that (up to subsequences) the functions $\tilde{u}(t + t_k, x)$ converge as $k \to \infty$, uniformly in $(-\rho, \rho) \times \Omega_\rho$, for any $\rho > 0$, to some function $u^*(t, x)$ which is a nonnegative bounded solution to (41). If $\lambda_{1,N} \geq 0$ then $u^* \equiv 0$ by Lemma 3.4. Therefore, owing to the arbitrariness of the sequence $(t_k)_{k\in\mathbb{N}}$, (46) holds in this case. Consider now the case $\lambda_{1,N} < 0$. Set $x_0 := (0, y_0)$, where y_0 is an arbitrary point in ω. Let us show that hypothesis (42) in Lemma 3.4 holds for any sequence $(t_n)_{n\in\mathbb{N}}$ tending to $-\infty$. By Proposition 1, there exists $R > 0$ such that $\lambda(R) < 0$. Arguing as in the proof of Theorem 2.1, we can choose $\kappa > 0$ small enough in such a way that the function $\underline{U} := \kappa\varphi_R$ satisfies $\underline{U}(x) \leq \tilde{u}(1, x)$ and is a subsolution to the elliptic equation of (13) in Ω_R. Hence, $(t, x) \mapsto \underline{U}(x)$ is a subsolution to (45) in $\mathbb{R}\times\Omega_R$ and satisfies $\underline{U}(\pm R, y) = 0 \leq \tilde{u}(t, \pm R, y)$ for $t > 0,\ y \in \omega$. The parabolic comparison principle yields $\underline{U}(x) \leq \tilde{u}(t+1, x)$ for $t > 0$

and $x \in \Omega_R$. As a consequence,

$$\inf_{t\in\mathbb{R}} u^*(t,x_0) \geq \underline{U}(x_0) > 0,$$

We can then apply Lemma 3.4 and derive $u^* \equiv U$. Thus, (46) holds.

Step 2: conclusion of the proof.

Assume, by way of contradiction, that $\lim_{t\to\infty} \tilde{u}(t,x) = U(x)$ does not hold uniformly in $x \in \Omega$, either in the case (i) with $U \equiv 0$, or in the case (ii) with U unique solution to (13) (given by Theorem 2.1). Hence, there exist $\varepsilon > 0$, $(t^n)_{n\in\mathbb{N}}$ in $\mathbb{R}^+$ and $((x_1^n, y^n))_{n\in\mathbb{N}}$ in Ω such that

$$\lim_{n\to\infty} t^n = \infty, \qquad \forall\, n \in \mathbb{N}, \quad |\tilde{u}(t^n, x_1^n, y^n) - U(x_1^n, y^n)| \geq \varepsilon.$$

It is not restrictive to assume that y^n converges to some $\eta \in \overline{\omega}$. We know from step 1 that $\lim_{n\to\infty} |x_1^n| = \infty$. Then, $\lim_{n\to\infty} U(x_1^n, y^n) = 0$ in both cases (i) and (ii). We then get

$$\liminf_{n\to\infty} \tilde{u}(t^n, x_1^n, y^n) \geq \varepsilon.$$

Using standard parabolic estimates and compact injection theorems, we find that, as n goes to infinity and up to subsequences, $\tilde{u}(t+t^n, x_1 + x_1^n, y)$ converges to a function $\tilde{u}_\infty(t,x_1,y)$ uniformly in $(-\rho,\rho)\times\Omega_\rho$, for any $\rho > 0$. The function $\tilde{u}_\infty$ satisfies $\tilde{u}_\infty(0,0,\eta) \geq \varepsilon$ and, by (9), (11) and (12),

$$\begin{cases} \partial_t \tilde{u}_\infty \leq \Delta \tilde{u}_\infty + c\partial_1 \tilde{u}_\infty - \zeta \tilde{u}_\infty, & t \in \mathbb{R},\ x \in \Omega \\ \partial_\nu \tilde{u}_\infty = 0, & t \in \mathbb{R},\ x \in \partial\Omega. \end{cases} \tag{47}$$

For any $h \geq 0$ define the function $\theta_h(t,x) := S' e^{-\zeta(t+h)}$. It satisfies $\partial_t \theta_h = -\zeta\theta_h$ in $\mathbb{R}\times\Omega$, $\partial_\nu \theta_h = 0$ on $\mathbb{R}\times\partial\Omega$ and $\theta_h(-h,x) = S' \geq \tilde{u}_\infty(-h,x)$. Therefore, for any $h \geq 0$, the parabolic maximum principle yields $\tilde{u}_\infty \leq \theta_h$ in $(-h,+\infty)\times\overline{\Omega}$. Consequently,

$$\tilde{u}_\infty(0,0,\eta) \leq \lim_{h\to+\infty} \theta_h(0,0,\eta) = 0,$$

which is a contradiction. Since $u(t,x_1,y) = \tilde{u}(t, x_1 - ct, y)$, the proof is concluded. □

Remark 3. The results of Theorems 2.1 and 2.2 also hold if one considers Dirichlet boundary condition $u(t,x) = 0$ on $\mathbb{R}^+ \times \partial\Omega$ in (3). In this case, the existence, uniqueness and stability of travelling waves depend on the sign of the generalized principal eigenvalue $\lambda_1(-\mathcal{L},\Omega)$ defined by (24). The proofs are easier than in the Neumann case considered here. In particular, one can consider an increasing sequence of bounded smooth domains converging to Ω instead of the Ω_r. This avoids any difficulty due to the lack of smoothness of the boundary in the definition of the principal eigenvalues. Robin boundary conditions are also allowed.

4. Large time behaviour in general cylindrical-type domains

In this section, we use the same notation as in Section 3:

$$\Omega = \mathbb{R}\times\omega,$$
$$\mathcal{L}w = \Delta w + c\partial_1 w + f_s(x_1,y,0)w,$$
$$\zeta = -\lim_{r\to\infty} \sup_{\substack{|x_1|>r \\ y\in\omega}} f_s(x_1,y,0),$$
$$\lambda_{1,N} = \lambda_{1,N}(-\mathcal{L},\Omega),$$
$$\forall\, r > 0, \quad \Omega_r = (-r,r)\times\omega.$$

The basic idea to prove Theorems 2.3 and 2.4 is to show that, as $\tau \to \infty$, the function $\tilde{u}(t+\tau, x)$ (where $\tilde{u}(t, x_1, y) := u(t, x_1 + ct, y)$) converges locally uniformly (up to subsequences) to an entire solution $u^*(t, x)$ in the *straight infinite cylinder* Ω. Thus, owing to Lemma 3.4, the convergence results of statements (i) and (ii) hold locally uniformly provided u^* satisfies (42). In the case of semi-infinite cylinder, condition (42) is derived by comparing $\tilde{u}$ with the principal eigenfunction φ_R of $-\mathcal{L}$ in Ω_R, as done in the proof of Theorem 2.2. The case of asymptotically cylindrical domain is actually much more delicate, because $\tilde{u}$ and φ_R do not satisfy the same boundary conditions and therefore cannot be compared. We overcome this difficulty by replacing φ_R with a suitable "generalized" strict subsolution which is compactly supported. Then, we can conclude using the fact that, essentially, the problem satisfied by $\tilde{u}$ "approaches" locally uniformly the Neumann problem in the straight cylinder as $t \to \infty$.

4.1. **Straight semi-infinite cylinder.** We start by considering here problem (4) which is set in a straight semi-infinite cylinder $\Omega^+ = \mathbb{R}^+ \times \omega$.

Proof of Theorem 2.3. Set

$$S' := \max\{S, \|u_0\|_{L^\infty(\Omega^+)}, \|\sigma\|_{L^\infty(\mathbb{R}^+\times\omega)}\},$$

where S is the constant in (10). Since 0 and S' are respectively a sub and a supersolution of (4), the same arguments as in the proof of Theorem 2.2 show that the unique solution u to (4) with initial condition $u(0, x) = u_0(x)$ satisfies $0 < u \leq S'$ in $\mathbb{R}^+ \times \mathbb{R}^+ \times \overline{\omega}$. The function defined by $\tilde{u}(t, x_1, y) := u(t, x_1 + ct, y)$ satisfies the following equation and boundary conditions:

$$\begin{cases} \partial_t \tilde{u} = \Delta \tilde{u} + c\partial_1 \tilde{u} + f(x_1, y, \tilde{u}), & t > 0,\ x_1 > -ct,\ y \in \omega \\ \partial_\nu \tilde{u}(t, x_1, y) = 0, & t > 0,\ x_1 > -ct,\ y \in \partial\omega \\ \tilde{u}(t, -ct, y) = \sigma(t, y), & t > 0,\ y \in \omega, \end{cases} \tag{48}$$

with initial condition $\tilde{u}(0, x) = u_0(x)$ for $x \in \Omega^+$. For the rest of the proof, U denotes the unique solution to (13) if $\lambda_{1,N} < 0$, while $U \equiv 0$ if $\lambda_{1,N} \geq 0$. We first derive the local convergence of $\tilde{u}$ to U.

Step 1: the function $\tilde{u}$ satisfies (46).
Let $(t_k)_{k\in\mathbb{N}}$ be a sequence such that $\lim_{k\to\infty} t_k = +\infty$. By standard arguments we see that, as $k \to \infty$ and up to subsequences, the functions $\tilde{u}(t + t_k, x)$ converge locally uniformly in $\mathbb{R} \times \overline{\Omega}$ to a solution u^* of (41). Owing to Lemma 3.4, we only need to show that if $\lambda_1 < 0$, then (42) holds. By Proposition 1, there exists $R > 0$ such that $\lambda(R) < 0$. As we have seen in the proof of Theorem 2.1, for $\kappa > 0$ small enough the function

$$\underline{U}(x) := \kappa\varphi_R(x)$$

is a subsolution to (40) in Ω_R. Set $t_R := R/c + 1$. The function $\tilde{u}$ is well defined and strictly positive in $[t_R, +\infty) \times \overline{\Omega}_R$. Hence, up to decreasing κ if need be, we can assume that $\underline{U}(x) \leq \tilde{u}(t_R, x)$ for $x \in \overline{\Omega}_R$. Since $(t, x) \mapsto \underline{U}(x)$ is a subsolution to (48) in $\mathbb{R} \times \Omega_R$ and

$$\forall\, t > t_R,\ y \in \omega, \quad \underline{U}(\pm R, y) = 0 < \tilde{u}(t, \pm R, y),$$

the comparison principle yields $\underline{U}(x) \leq \tilde{u}(t, x)$ for $t > t_R,\ x \in \Omega_R$. Therefore, for any $x_0 \in \Omega_R$,

$$\inf_{t\in\mathbb{R}} u^*(t, x_0) \geq \underline{U}(x_0) > 0,$$

that is, (42) holds for any sequence $(t_n)_{n\in\mathbb{N}}$ tending to $-\infty$.

Step 2: conclusion of the proof.
Argue by contradiction and assume that there exist $\varepsilon > 0$ and some sequences $(t^n)_{n\in\mathbb{N}}$ in $\mathbb{R}^+$ and $((x_1^n, y^n))_{n\in\mathbb{N}}$ in Ω^+ such that

$$\lim_{n\to\infty} t^n = \infty, \qquad \forall\, n \in \mathbb{N}, \quad |u(t^n, x_1^n, y^n) - U(x_1^n - ct^n, y^n)| \geq \varepsilon.$$

We may assume that y^n converges to some $\eta \in \overline{\omega}$. By step 1 we know that the sequence $(x_1^n - ct^n)_{n\in\mathbb{N}}$ cannot be bounded. Since $U(\cdot, y)$ vanishes at infinity, we get in particular that

$$\liminf_{n\to\infty} u(t^n, x_1^n, y^n) \geq \varepsilon, \tag{49}$$

whatever the sign of $\lambda_{1,N}$ is. Suppose for a moment that $(x_1^n)_{n\in\mathbb{N}}$ is unbounded. Then, by parabolic estimates and embedding theorems, the functions $u_n(t, x_1, y) := u(t + t^n, x_1 + x_1^n, y)$ converge, as $n \to \infty$ and up to subsequences, uniformly in $(-\rho, \rho) \times \Omega_\rho$, for any $\rho > 0$, to a nonnegative function u_∞ satisfying

$$\begin{cases} \partial_t u_\infty \leq \Delta u_\infty - \zeta u_\infty, & t \in \mathbb{R},\ x_1 \in \mathbb{R},\ y \in \omega \\ \partial_\nu u_\infty = 0, & t \in \mathbb{R},\ x_1 \in \mathbb{R},\ y \in \partial\omega, \end{cases} \tag{50}$$

and, by (49), $u_\infty(0, 0, \eta) \geq \varepsilon$. We then get a contradiction by comparing u_∞ with $\theta_h(t, x) := S' e^{-\zeta(t+h)}$ in $(-h, +\infty) \times \Omega$ and letting h go to infinity, as done at the end of the proof of Theorem 2.2.

It remains to consider the case when $(x_1^n)_{n\in\mathbb{N}}$ is bounded. For $n \in \mathbb{N}$ define $u_n(t, x_1, y) := u(t+t^n, x_1, y)$. Using L^p estimates up to the boundary for u, $\partial_t u$, Δu (which hold good here owing to the compatibility condition $\partial_\nu \sigma = 0$ on $\mathbb{R}^+ \times \partial\omega$, see e. g. [14], [15]) we infer that (a subsequence of) $(u_n)_{n\in\mathbb{N}}$ converges uniformly in $(-\rho, \rho) \times (0, \rho) \times \omega$, for any $\rho > 0$, to a function u_∞ satisfying (50) for $x_1 > 0$, together with $u_\infty(t, 0, y) = 0$ for $t \in \mathbb{R}$, $y \in \omega$. Moreover, (49) yields $u_\infty(0, \xi, \eta) \geq \varepsilon$, where ξ is the limit of a subsequence of $(x_1^n)_{n\in\mathbb{N}}$. A contradiction follows exactly as before, by comparison with the functions $\theta_h(t, x) := S' e^{-\zeta(t+h)}$. □

Remark 4. If σ does not converge to zero as $t \to \infty$ then in Theorem 2.3 the convergences only hold "far away" from the base $\{0\} \times \omega$, that is, uniformly in $(\gamma(t), +\infty) \times \omega$, for any function γ such that $\gamma \to +\infty$ as $t \to \infty$. Let us also point out that the results in Theorem 2.3 hold under different boundary conditions on $\{0\} \times \omega$, such as Neumann condition $\partial_1 u = 0$ or Robin condition $\beta_0(t, y)u - \beta_1(t, y)\partial_1 u = 0$, with

$$\beta_0, \beta_1 \geq 0, \qquad \beta_0 + \beta_1 > 0.$$

4.2. **Asymptotically cylindrical domain.** As in the case of the straight cylinder that we considered in the previous section, the large time behaviour of u rests on proving that $\tilde{u}(t, x) := u(t, x_1 + ct, y)$ does not converge to 0 as $t \to \infty$ when $\lambda_{1,N} < 0$. With respects to the straight cylinder, the difficulty here is that the condition $\lambda_{1,N}$ allows one to construct a compactly supported stationary subsolution of the Neumann problem in the straight cylinder, but not in the time-dependent domain where $\tilde{u}$ is defined. Thus, the proof becomes technically more involved. Let us sketch our strategy to prove this result. Through the mapping Ψ we can transform $\tilde{u}$ into a function $\tilde{v}$ solution of an oblique derivative problem with a modified operator but in the straight cylinder. The transformed problem converges, in some sense, to the Neumann problem (45) as $t \to \infty$. Thus, for t large enough, it is possible to derive a positive lower bound for $\tilde{v}$ by the same comparison argument as in the previous sections, provided that (45) admits some kind of compactly supported stationary *strict* subsolution. Actually, we construct a generalized strict subsolution V in the sense of [3]: V is the supremum of two strict subsolutions. The precise properties of V are stated in the next lemma, which is proved at the end of the section.

In the sequel, we will make use of the following fact, which is a consequence of (16):

$$\lim_{t\to\infty} \nu'(\Psi(x_1+ct,y)) = \nu(x_1,y), \tag{51}$$

locally uniformly with respect to $(x_1,y)\in\partial\Omega$. Note that the right hand side does not depend on x_1.

Lemma 4.1. *If $\lambda_{1,N}<0$, there exist a bounded piecewise smooth domain $\mathcal{O}\subset\Omega$, a constant $\kappa>0$, two functions $V_1,V_2\in W^{1,\infty}(\mathcal{O})$ and two open sets $\mathcal{O}_1,\mathcal{O}_2$ such that $\mathcal{O}_1\cup\mathcal{O}_2=\mathcal{O}$,*

$$\forall\, x\in\mathcal{O},\quad V(x):=\max(V_1(x),V_2(x))>0,\qquad V=0\ \text{ on }\ \partial\mathcal{O}\cap\Omega,$$

and, for $\sigma\in\{1,2\}$, $V_\sigma\in C^2(\mathcal{O}_\sigma)\cap C^1(\overline{\mathcal{O}}_\sigma)$,

$$x\in\mathcal{O},\ V_\sigma(x)=V(x)\ \Rightarrow\ x\in\mathcal{O}_\sigma,$$

$$\begin{cases} -\mathcal{L}V_\sigma \le -\kappa\Big(\displaystyle\sum_{i,j=1}^N|\partial_{ij}V_\sigma| + \sum_{i=1}^N|\partial_i V_\sigma| + V_\sigma\Big) & \text{in } \mathcal{O}_\sigma\\ \partial_\nu V_\sigma<0 & \text{on } \partial\mathcal{O}_\sigma\cap\partial\Omega.\end{cases}$$

Proof of Theorem 2.4. As usual, the existence of a unique solution to (5) with initial datum u_0 follows from standard parabolic theory. Moreover, $0<u\le S'$ in $\mathbb{R}^+\times\Omega'$, where $S':=\max\{S,\|u_0\|_{L^\infty(\Omega')}\}$. The function $\tilde u(t,x_1,y):=u(t,x_1+ct,y)$ satisfies

$$\begin{cases} \partial_t\tilde u=\Delta\tilde u+c\partial_1\tilde u+f(x_1,y,\tilde u), & t>0,\ x_1\in\mathbb{R},\ y\in\omega'(x_1+ct)\\ \nabla\tilde u(t,x_1,y)\cdot\nu'(x_1+ct,y)=0, & t>0,\ x_1\in\mathbb{R},\ y\in\partial\omega'(x_1+ct),\end{cases} \tag{52}$$

together with the initial condition $\tilde u(0,x)=u_0(x)$.

Step 1: the function $\tilde u$ satisfies

$$\lim_{t\to\infty}\tilde u(t,x)=U(x),\quad \text{locally uniformly in } x\in\Omega, \tag{53}$$

where $U\equiv 0$ if $\lambda_{1,N}\ge 0$, while U is the unique solution of (13) if $\lambda_{1,N}<0$.

Let $(t_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ such that $\lim_{k\to\infty}t_k=+\infty$. From parabolic estimates it follows that the functions $\tilde u(t+t_k,x)$ converge as $k\to\infty$ (up to subsequences) locally uniformly in $\mathbb{R}\times\Omega$ to some function $u^*(t,x)$ which is a nonnegative bounded solution of the parabolic equation in (41). Moreover, using (51) and estimates up to the boundary of Ω', one can check that u^* satisfies also the boundary condition of (41). Hence, if $\lambda_{1,N}\ge0$, Lemma 3.4 yields $u^*\equiv0$, that is, (53) holds. In the case $\lambda_{1,N}<0$, we want to show that (42) holds. To do this, we consider the domains $\mathcal{O}_1,\ \mathcal{O}_2,\ \mathcal{O}$, the constant κ and the functions $V_1,\ V_2,\ V$ given by Lemma 4.1. We set $e_1:=(1,0,\cdots,0)\in\mathbb{R}^N$. By (16) there exists $t_0>0$ such that

$$\forall\, t\ge t_0,\quad \Psi(\mathcal{O}+\{cte_1\})\subset\Omega',\quad \Psi((\partial\mathcal{O}\cap\partial\Omega)+\{cte_1\})\subset\partial\Omega'.$$

We introduce the function $\tilde v(t,x):[t_0,+\infty)\times\mathcal{O}\to\mathbb{R}$ defined by

$$\tilde v(t,x):=\tilde u(t,\Psi(x+cte_1)-cte_1).$$

One can check that $\tilde v$ solves a problem of the type

$$\begin{cases}\partial_t\tilde v=\operatorname{tr}(A(t,x)H\tilde v)+b(t,x)\cdot\nabla\tilde v+f(\Psi(x+cte_1)-cte_1,\tilde v), & t>t_0,\ x\in\mathcal{O}\\ \beta(t,x)\cdot\nabla\tilde v=0, \qquad t>t_0,\ x\in\partial\mathcal{O}\cap\partial\Omega,\end{cases}$$

($H\tilde{v}$ denoting the Hessian matrix of $\tilde{v}$ in the x variables) where the matrix field A and the vector fields b, β depend on the Jacobian matrix and the vector Laplacian of Ψ^{-1} at the point $\Psi(x+cte_1)$. Moreover, the following limits hold

$$\lim_{t\to\infty}\|A(t,\cdot)-I\|_{L^\infty(\mathcal{O})}=0,\qquad \lim_{t\to\infty}\|b(t,\cdot)-ce_1\|_{L^\infty(\mathcal{O})}=0,$$

and, thanks to (51),

$$\lim_{t\to\infty}\|\beta(t,\cdot)-\nu\|_{L^\infty(\partial\mathcal{O}\cap\partial\Omega)}=0.$$

Take $t_1>t_0$ large enough in such a way that, for $\sigma\in\{1,2\}$, the following inequalities hold in $(t_1,+\infty)\times\mathcal{O}_\sigma$:

$$\begin{aligned}-\mathrm{tr}(A\,HV_\sigma)-b\cdot\nabla V_\sigma &\le -\mathcal{L}V_\sigma+f_s(x,0)V_\sigma+\kappa\Big(\sum_{i,j=1}^N|\partial_{ij}V_\sigma|+\sum_{i=1}^N|\partial_iV_\sigma|\Big)\\ &\le (f_s(x,0)-\kappa)V_\sigma\\ &< \Big(f_s(\Psi(x+cte_1)-cte_1,0)-\frac{\kappa}{2}\Big)V_\sigma.\end{aligned}$$

Here, the last inequality is a consequence of (16) and the uniform continuity of $f_s(x,0)$. Moreover, up to increasing t_1, it is seen that

$$\forall\, t>t_1,\ x\in\partial\mathcal{O}_\sigma\cap\partial\Omega,\quad \beta(t,x)\cdot\nabla V_\sigma<0. \tag{54}$$

Therefore, as $f(x,0)=0$ and $s\mapsto f_s(x,s)\in C^1([0,\delta])$, uniformly in x, there exists $k_\sigma>0$ such that for any $k\in(0,k_\sigma]$ the function kV_σ is a strict subsolution of the problem solved by $\tilde{v}$ in $(t_1,+\infty)\times\mathcal{O}_\sigma$. Let $\tau>t_1$ be such that the matrix field $A(t,x)$ is uniformly elliptic for $t>\tau$ and $x\in\mathcal{O}$ and the vector field $\beta(t,x)$ points outside Ω for $t>\tau$ and $x\in\partial\mathcal{O}\cap\partial\Omega$. Let $k<\min(k_1,k_2)$ be such that the function $\underline{U}:=kV$ satisfies

$$\forall\, x\in\overline{\mathcal{O}},\quad \underline{U}(x)<\tilde{v}(\tau,x).$$

Assume by contradiction that $\tilde{v}(t,x)<\underline{U}(x)$ for some $t>\tau$ and $x\in\mathcal{O}$. Thus, there exists a first contact point $(\underline{t},\underline{x})\in(\tau,+\infty)\times\overline{\mathcal{O}}$ between $\underline{U}$ and $\tilde{v}$, i. e:

$$\forall\, t\in[\tau,\underline{t}),\ x\in\overline{\mathcal{O}},\quad \underline{U}(x)<\tilde{v}(t,x),\qquad \underline{U}(\underline{x})=\tilde{v}(\underline{t},\underline{x}).$$

Therefore, there exists $\sigma\in\{1,2\}$ such that $(\underline{t},\underline{x})$ is the first contact point between kV_σ and $\tilde{v}$ in $(\tau,+\infty)\times(\mathcal{O}_\sigma\cup(\partial\mathcal{O}_\sigma\cap\partial\mathcal{O}))$. If $\underline{x}\in\mathcal{O}_\sigma$ then we get a contradiction by the parabolic strong maximum principle. Hence, since $kV_\sigma\le\underline{U}=0<\tilde{v}$ on $\overline{\partial\mathcal{O}\cap\Omega}$, it follows that $\underline{x}\in\partial\mathcal{O}_\sigma\cap\partial\Omega$. Moreover, as $\underline{x}\notin\overline{\partial\mathcal{O}\cap\Omega}$, we can find a neighbourhood of $\underline{x}$ where $\mathcal{O}$ coincides with Ω. In particular, the vector $-\beta(\underline{t},\underline{x})$ points inside $\mathcal{O}$ and then $-\beta(\underline{t},\underline{x})\cdot(\nabla\tilde{v}(\underline{t},\underline{x})-k\nabla V_\sigma(\underline{x}))\ge0$ because $\underline{x}$ is a minimum point of $\tilde{v}(\underline{t},\cdot)-kV_\sigma$ in $\overline{\mathcal{O}}$. This contradicts (54).

Step 2: for any sequences $(t^n)_{n\in\mathbb{N}}$ in $\mathbb{R}^+$ and $((x_1^n,y^n))_{n\in\mathbb{N}}$ in $\Omega\cap\Omega'$ we have:

$$\lim_{n\to\infty}t^n=\lim_{n\to\infty}x_1^n=+\infty\quad\Rightarrow\quad\lim_{n\to\infty}(u(t^n,x_1^n,y^n)-U(x_1^n-ct^n,y^n))=0.$$

Assume by contradiction that the above property does not hold for some $(t^n)_{n\in\mathbb{N}}$ and $((x_1^n,y^n))_{n\in\mathbb{N}}$. Hence, setting $\xi^n:=x_1^n-ct^n$ we get

$$\limsup_{n\to\infty}|\tilde{u}(t^n,\xi^n,y^n)-U(\xi^n,y^n)|>0.$$

Suppose for a moment that $(\xi^n)_{n\in\mathbb{N}}$ is bounded. Then, using the uniform continuity of $\tilde{u}$ and U one can find another sequence $(\eta^n)_{n\in\mathbb{N}}$ in ω such that

$$((\xi^n,\eta^n))_{n\in\mathbb{N}}\subset K\subset\subset\Omega,\qquad \limsup_{n\to\infty}|\tilde{u}(t^n,\xi^n,\eta^n)-U(\xi^n,\eta^n)|>0.$$

This contradicts (53). The case of $(\xi^n)_{n\in\mathbb{N}}$ unbounded can be handled exactly as in the second step of the proof of Theorem 2.3.

Step 3: for any $\rho>0$ the following property holds:

$$\lim_{t\to\infty} u(t,x_1,y)=0 \quad \text{uniformly in } x_1\leq\rho,\ y\in\omega'(x_1). \tag{55}$$

For $\tau>0$ define

$$m(\tau):=\sup_{\substack{t>\tau\\ y\in\omega'(\frac{c}{2}\tau)}} u(t,\frac{c}{2}\tau,y).$$

The uniform continuity of u and step 2 imply:

$$\lim_{\tau\to+\infty} m(\tau)=\lim_{\tau\to+\infty}\sup_{\substack{t>\tau\\ y\in\omega'(\frac{c}{2}\tau)\cap\omega}} u(t,\frac{c}{2}\tau,y)=\lim_{\tau\to+\infty}\sup_{\substack{t>\tau\\ y\in\omega'(\frac{c}{2}\tau)\cap\omega}} U(\frac{c}{2}\tau-ct,y)=0,$$

where the last equality holds true because $U(\cdot,y)$ vanishes at infinity. Consider the functions

$$\theta_\tau(t,x):=\|u\|_{L^\infty(\mathbb{R}^+\times\Omega')}e^{\frac{\zeta}{2}(\tau-t)}.$$

By the hypotheses on f, for τ large enough the θ_τ are supersolutions of (5) in the set $t>\tau,\ x_1<\frac{c}{2}\tau$ and $y\in\omega'(x_1)$. Moreover, $\theta_\tau(\tau,x)\geq u(\tau,x)$ for $x\in\Omega'$ and, setting

$$\varsigma(\tau):=\frac{2}{\zeta}\ln\frac{\|u\|_{L^\infty(\mathbb{R}^+\times\Omega')}}{m(\tau)},$$

$$\forall\, t\in[\tau,\tau+\varsigma(\tau)],\ y\in\omega'(\frac{c}{2}\tau),\quad \theta_\tau(t,\frac{c}{2}\tau,y)\geq m(\tau)\geq u(t,\frac{c}{2}\tau,y).$$

Therefore, the comparison principle yields

$$\forall\, t\in[\tau,\tau+\varsigma(\tau)],\ x_1\leq\frac{c}{2}\tau,\ y\in\omega'(x_1),\quad u(t,x_1,y)\leq\theta_\tau(t,x_1,y).$$

Since $\varsigma(\tau)$ goes to $+\infty$ as $\tau\to+\infty$, for any $h\in\mathbb{N}$ we can find $\tau_h\geq 2\rho/c$ such that $\varsigma(\tau)>h$ for $\tau\geq\tau_h$. Consequently, for $t>\tau_h+h$, taking $\tau=t-h$ we see that $t\in[\tau,\tau+\varsigma(\tau)]$ and $\rho\leq c\tau/2$, which implies:

$$\forall\, x_1\leq\rho,\ y\in\omega'(x_1),\quad u(t,x_1,y)\leq\theta_\tau(t,x_1,y)\leq\|u\|_{L^\infty(\mathbb{R}^+\times\Omega')}e^{-\frac{\zeta}{2}h}.$$

Property (55) then follows from the arbitrary charachter of h.

Step 4: conclusion of the proof.

Note that if $\lambda_{1,N}\geq 0$ then condition (16) and the uniform continuity of u imply that the result of step 2 holds even if we drop the assumption $(x_1^n,y^n)\in\Omega$. Therefore, Theorem 2.4 part (i) follows from steps 2 and 3. Assume by contradiction that statement (ii) does not hold. Then, there exist $\varepsilon>0$, $(t^n)_{n\in\mathbb{N}}$ in $\mathbb{R}^+$, $((x_1^n,y^n))_{n\in\mathbb{N}}$ in Ω' such that $\lim_{n\to\infty}t^n=\infty$ and either

$$\forall\, n\in\mathbb{N},\quad y^n\in\omega,\quad |u(t^n,x_1^n,y^n)-U(x_1^n-ct^n,y^n)|\geq\varepsilon,$$

or

$$\exists\,\gamma<c,\quad \forall\, n\in\mathbb{N},\quad x_1^n<\gamma t^n,\quad u(t^n,x_1^n,y^n)\geq\varepsilon.$$

The first case is ruled out because the sequence $(x_1^n)_{n\in\mathbb{N}}$ is not bounded from above - by step 3 and the last statement of Theorem 2.1 - nor unbounded from above - by step 2. In the second case, step 3 implies that $x_1^n\to+\infty$ as $n\to\infty$. Hence, owing to the uniform continuity of u, we can assume without loss of generality that $(x_1^n,y^n)\in\Omega'\cap\Omega$ for n large enough. As a consequence, since $x_1^n-ct^n\to-\infty$ as $n\to\infty$, we derive

$$\liminf_{n\to\infty}(u(t^n,x_1^n,y^n)-U(x_1^n-ct^n,y^n))=\liminf_{n\to\infty}u(t^n,x_1^n,y^n)\geq\varepsilon,$$

which is in contradiction with step 2. □

We now turn to the proof of Lemma 4.1. Let is first describe the ideas before giving the technical details of the construction. We first define the function V_1 as the principal eigenfunction of $-\mathcal{L}$ in some bounded smooth domain $\widetilde{\mathcal{O}}$ under boundary condition of Robin type. The advantage of taking Robin boundary conditions is that we obtain a function with negative normal derivative, which is useful for comparison purposes. Using the fact that $\lambda_{1,N} < 0$, we are able to choose $\widetilde{\mathcal{O}}$ and the boundary condition in such a way that the associated principal eigenvalue λ is strictly negative. Hence, in the set where V_1 is bounded away from zero, we can take κ small enough such that

$$-\mathcal{L}V_1 = \lambda V_1 \leq -\kappa\Big(\sum_{i,j=1}^{N} |\partial_{ij}V_1| + \sum_{i=1}^{N} |\partial_i V_1| + V_1\Big).$$

The above inequality may fail when V_1 approaches 0, and this is why we introduce the function V_2. We want V_2 to be positive in a bounded domain $\mathcal{O} \supset \widetilde{\mathcal{O}}$, to vanish on $\partial\mathcal{O}\cap\Omega$ and to satisfy the above inequality together with $\partial_\nu V_2 < 0$ at least in the set where it is small. The differential inequality is obviously fulfilled by taking a function of exponential type. The boundary condition is less easy to obtain because it implies that at the "corners " $\overline{\partial\mathcal{O}\cap\Omega}\cap\partial\Omega$ the vector field $-\nu$ has to point inside $\mathcal{O}$ (hence, we cannot take $\mathcal{O} = \Omega_r$ for some $r > 0$). This is achieved by taking $\mathcal{O}$ to be the straight cylinder truncated by two "caps" - see Figure 1 - obtained as the graph of a function ξ satisfying $\xi = 0$ and $\partial_{\nu_\omega}\xi < 0$ on $\partial\omega$ (∂_{ν_ω} denoting the exterior normal derivative to ω). A simple way to find such a function ξ is by solving the Dirichlet problem $-\Delta\xi = 1$ in ω, $\xi = 0$ on $\partial\omega$. The functions that will be used to define V_1 and V_2 are constructed in Lemma 4.2 and Lemma 4.3 respectively.

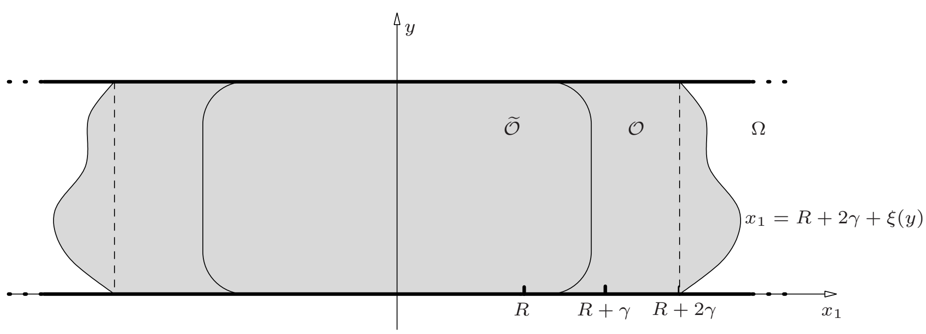

FIGURE 1. construction of $\mathcal{O}$

Lemma 4.2. *If $\lambda_{1,N} < 0$ then for any $\gamma > 0$ there exist two constants $R, h > 0$, a smooth domain $\widetilde{\mathcal{O}}$ and a function $\phi \in C^2(\overline{\widetilde{\mathcal{O}}})$ such that $\Omega_R \subset \widetilde{\mathcal{O}} \subset \Omega_{R+\gamma}$, $\phi > 0$ in $\widetilde{\mathcal{O}} \cup \overline{\Omega}_R$ and*

$$\begin{cases} -\mathcal{L}\phi = -h\phi & \text{in } \widetilde{\mathcal{O}} \\ \partial_\nu\phi < 0 & \text{on } \partial\widetilde{\mathcal{O}}\cap\partial\Omega \\ \phi = 0 & \text{on } \partial\widetilde{\mathcal{O}}\backslash\partial\Omega. \end{cases}$$

Proof. Fix $\gamma > 0$. By Proposition 1 there exists $R > 0$ such that $\lambda(R) < 0$. Let $\widetilde{\mathcal{O}}$ be a smooth domain satisfying $\Omega_{R+\gamma/2} \subset \widetilde{\mathcal{O}} \subset \Omega_{R+\gamma}$. Consider a function $\vartheta \in C^\infty(\mathbb{R})$ such that

$$0 \leq \vartheta \leq 1 \ \text{ in } \ \mathbb{R}, \qquad \vartheta(x_1) = 0 \ \text{ for } \ |x_1| \leq R, \qquad \vartheta(x_1) = 1 \ \text{ for } \ |x_1| \geq R + \gamma/2.$$

For any constant $\varepsilon \geq 0$ let λ^ε and ϕ^ε be respectively the principal eigenvalue and eigenfunction of $-\mathcal{L}$ in $\widetilde{\mathcal{O}}$ under the Robin boundary condition $(1 - \vartheta(x_1))\partial_\nu \phi^\varepsilon + (\vartheta(x_1) + \varepsilon)\phi^\varepsilon = 0$ on $\partial\widetilde{\mathcal{O}}$ and normalized by $\|\phi^\varepsilon\|_{L^\infty(\widetilde{\mathcal{O}})} = 1$. That is, $\phi^\varepsilon > 0$ in $\widetilde{\mathcal{O}}$ and

$$\begin{cases} -\mathcal{L}\phi^\varepsilon = \lambda^\varepsilon \phi^\varepsilon & \text{in } \widetilde{\mathcal{O}} \\ (1 - \vartheta(x_1))\partial_\nu \phi^\varepsilon + (\vartheta(x_1) + \varepsilon)\phi^\varepsilon = 0 & \text{on } \partial\widetilde{\mathcal{O}}. \end{cases}$$

Note that the above boundary condition is well defined and is of Robin type because, if $\vartheta(x_1) < 1$ for some $(x_1, y) \in \partial\widetilde{\mathcal{O}}$, then $|x_1| < R + \gamma/2$ and consequently $(x_1, y) \in \partial\Omega$ and $\nu(x_1, y)$ coincides with the outer normal to $\widetilde{\mathcal{O}}$. The existence of such eigenvalues and eigenfunctions follow in a standard way from the Krein-Rutman theory (because $\mathcal{L}$ and $\widetilde{\mathcal{O}}$ are smooth). We claim that $\lambda^\varepsilon < 0$ for ε small enough. To prove this, we show that $\lambda^\varepsilon \to \lambda^0$ as $\varepsilon \to 0^+$ with $\lambda_0 < \lambda(R)$. Assume by contradiction that there exists $\varepsilon \geq 0$ such that $\lambda^\varepsilon \leq \lambda(R+2\gamma)$. By Hopf's lemma the eigenfunction $\varphi_{R+2\gamma}$ associated with $\lambda(R + 2\gamma)$ is strictly positive in $\overline{\widetilde{\mathcal{O}}}$. Define

$$k := \max_{\overline{\widetilde{\mathcal{O}}}} \frac{\phi^\varepsilon}{\varphi_{R+2\gamma}}.$$

The function $w := k\varphi_{R+2\gamma} - \phi^\varepsilon$ vanishes at some point $x^* = (x_1^*, y^*) \in \overline{\widetilde{\mathcal{O}}}$ and satisfies $w \geq 0$ and $(\mathcal{L} + \lambda^\varepsilon)w \leq 0$ in $\widetilde{\mathcal{O}}$. Moreover, for any $(x_1, y) \in \partial\widetilde{\mathcal{O}}$ such that $\vartheta(x_1) = 1$ we see that

$$w(x_1, y) = k\varphi_{R+2\gamma}(x_1, y) > 0.$$

Hence, the strong maximum principle implies that $x^* \in \partial\widetilde{\mathcal{O}}$ and $\vartheta(x_1^*) < 1$. As a consequence,

$$\partial_\nu w(x_1^*, y^*) = -\partial_\nu \phi^\varepsilon(x_1^*, y^*) = \frac{\vartheta(x_1^*) + \varepsilon}{1 - \vartheta(x_1^*)} \, \phi^\varepsilon(x_1^*, y^*) \geq 0,$$

which is in contradiction with the Hopf lemma. Therefore, the λ^ε are bounded from below by $\lambda(R + 2\gamma)$. A direct application of the strong maximum principle shows that they are bounded from above by the Dirichlet principal eigenvalue of $-\mathcal{L}$ in any domain $A \subset\subset \widetilde{\mathcal{O}}$. Hence, from any positive sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ converging to 0 one can extract a subsequence $(\varepsilon_{n_k})_{k\in\mathbb{N}}$ such that $(\lambda^{\varepsilon_{n_k}})_{k\in\mathbb{N}}$ converges to some $\lambda^* \in \mathbb{R}$. Using Schauder's estimates up to the boundary and the Arzela Ascoli theorem we see that (up to subsequences) the $\phi^{\varepsilon_{n_k}}$ converge as $k \to \infty$ in $C^2(\overline{\widetilde{\mathcal{O}}})$ to a non-negative nontrivial solution ϕ^* of

$$\begin{cases} -\mathcal{L}\phi^* = \lambda^* \phi^* & \text{in } \widetilde{\mathcal{O}} \\ (1 - \vartheta(x_1))\partial_\nu \phi^* + \vartheta(x_1)\phi^* = 0 & \text{on } \partial\widetilde{\mathcal{O}}. \end{cases}$$

Thus, $\phi^* > 0$ in $\widetilde{\mathcal{O}}$ by the strong maximum principle and then the uniqueness of the principal eigenvalue of $-\mathcal{L}$ in $\widetilde{\mathcal{O}}$ under Robin boundary condition yields $\lambda^* = \lambda^0$. This shows that the λ^ε converge to λ^0 as $\varepsilon \to 0^+$. To check that $\lambda^0 < \lambda(R)$ one uses the same contradictory argument as before: suppose that $\lambda^0 \geq \lambda(R)$ and set $w := k\phi^0 - \varphi_R$, with

$$k := \max_{\overline{\Omega}_R} \frac{\varphi_R}{\phi^0}.$$

Note that $\partial_\nu \phi^0 = 0$ on $[-R, R] \times \partial\omega$ and then $\phi^0 > 0$ in $\overline{\Omega}_R$ by Hopf's lemma. The points where w vanishes do not lie neither on $\{\pm R\} \times \overline{\omega}$, because $\varphi_R = 0$ there, nor in Ω_R due to the strong maximum principle. Neither do they lie on $(-R, R) \times \partial\omega$ due to Hopf's lemma. This yields a contradiction and the claim is then proved. Thus, we can chose $\varepsilon > 0$ small enough in such a way that the function $\phi := \phi^\varepsilon$ satisfies $-\mathcal{L}\phi = -h\phi$ in $\widetilde{\mathcal{O}}$, where $h := -\lambda^\varepsilon > 0$. The Hopf lemma implies that $\phi > 0$ in $\overline{\Omega}_R$. Hence, it only remains to check that ϕ satisfies the desired boundary conditions. The negativity of $\partial_\nu \phi(x_1, y)$ for $(x_1, y) \in \partial\widetilde{\mathcal{O}} \cap \partial\Omega$ follows from the Hopf lemma, if $\phi(x_1, y) = 0$, and from equality

$$(1 - \vartheta(x_1))\partial_\nu \phi + (\vartheta(x_1) + \varepsilon)\phi = 0,$$

if $\phi(x_1, y) > 0$. If $(x_1, y) \in \partial\widetilde{\mathcal{O}} \backslash \partial\Omega$ then, necessarily, $|x_1| \geq R + \gamma/2$. Consequently, $\vartheta(x_1) = 1$ and then $\phi(x_1, y) = 0$. □

Lemma 4.3. *There exist two functions $\xi \in C^2(\overline{\omega})$, $\chi \in C^2(\overline{\Omega})$ and a positive constant ε satisfying*

$$\xi > 0 \quad \text{in } \omega, \qquad \xi = 0 \ \text{ and } \ \partial_{\nu_\omega} \xi < 0 \quad \text{on } \partial\omega,$$

$$\chi(\xi(y), y) = 0 \quad \text{for } y \in \omega, \qquad \chi(x_1, y) > 0 \quad \text{for } x_1 < \xi(y),\ y \in \overline{\omega},$$

$$\begin{cases} -\Delta\chi + c|\partial_1\chi| + \|f_s(\cdot, 0)\|_{L^\infty(\Omega)}\chi \leq -\varepsilon\Big(\displaystyle\sum_{i,j=1}^N |\partial_{ij}\chi| + \sum_{i=1}^N |\partial_i\chi| + \chi\Big) & \text{in } \Omega \\ \partial_\nu \chi(0, y) < 0 \quad \text{for } y \in \partial\omega. \end{cases}$$

Proof. Consider the solution $\xi \in C^2(\overline{\omega})$ of the Dirichlet problem

$$\begin{cases} -\Delta\xi = 1 & \text{in } \omega \\ \xi = 0 & \text{on } \partial\omega. \end{cases}$$

The weak and strong maximum principle imply that $\xi > 0$ in ω and the Hopf lemma that $\partial_{\nu_\omega}\xi < 0$ on $\partial\omega$. Consider a constant $\beta \geq 1$ large enough to have $-\beta^2 + \beta + c\beta + \|f_s(\cdot, 0)\|_{L^\infty(\Omega)} < 0$. Then, define the function $\chi : \overline{\Omega} \to \mathbb{R}$ by $\chi(x_1, y) := e^{\beta(\xi(y) - x_1)} - 1$. By computation,

$$-\Delta\chi + c|\partial_1\chi| + \|f_s(\cdot, 0)\|_{L^\infty(\Omega)}\chi \leq (-\beta^2 + \beta + c\beta + \|f_s(\cdot, 0)\|_{L^\infty(\Omega)})e^{\beta(\xi(y) - x_1)}$$

Since there exists a positive constant C such that

$$\forall\ (x_1, y) \in \Omega, \quad \sum_{i,j=1}^N |\partial_{ij}\chi| + \sum_{i=1}^N |\partial_i\chi| + \chi \leq C\beta^2 e^{\beta(\xi(y) - x_1)},$$

we can choose $\varepsilon > 0$ in such a way that

$$-\Delta\chi + c|\partial_1\chi| + \|f_s(\cdot, 0)\|_{L^\infty(\Omega)}\chi \leq -\varepsilon\Big(\sum_{i,j=1}^N |\partial_{ij}\chi| + \sum_{i=1}^N |\partial_i\chi| + \chi\Big) \quad \text{in } \Omega.$$

Furthermore,

$$\forall\ y \in \partial\omega, \quad \partial_\nu\chi(0, y) = \beta\partial_{\nu_\omega}\xi(y) < 0.$$

□

Proof of Lemma 4.1. Consider the functions ξ, χ and the constant ε given by Lemma 4.3. There exists $\gamma > 0$ such that $\partial_\nu \chi < 0$ on $[-2\gamma, 0] \times \partial\omega$. Let R, h, $\widetilde{\mathcal{O}}$ and ϕ be the constants, the domain and the function given by Lemma 4.2 associated with γ. We define $\mathcal{O}$ in the following way:

$$\mathcal{O} := \{(x_1, y) \in \Omega \ : \ |x_1| < R + 2\gamma + \xi(y)\}.$$

Take $k > 0$ small enough in such a way that

$$k \max_{y\in\overline{\omega}} \chi(-2\gamma, y) < \min_{\overline{\Omega}_R} \phi.$$

Then, we define $V_1 := \phi$ in $\widetilde{\mathcal{O}}$, extended by 0 outside $\widetilde{\mathcal{O}}$,

$$V_2(x_1, y) := \begin{cases} k\chi(-2\gamma, y) & \text{if } (x_1, y) \in \overline{\Omega}_R \\ k\chi(|x_1| - R - 2\gamma, y) & \text{if } (x_1, y) \in \overline{\mathcal{O}}\backslash\overline{\Omega}_R. \end{cases}$$

$$\mathcal{O}_1 := \{x \in \mathcal{O} \ : \ V_1(x) > \frac{1}{2} V_2(x)\}, \qquad \mathcal{O}_2 := \mathcal{O}\backslash\overline{\Omega}_R.$$

Note that $V_2 > 0$ in $\mathcal{O}$ and $\partial_\nu V_2 < 0$ on $\partial\mathcal{O}_2 \cap \partial\Omega$. Moreover, since $\mathcal{O}_1 \subset \widetilde{\mathcal{O}}$ and V_2 is bounded from below away from zero in $\widetilde{\mathcal{O}} \subset \Omega_{R+\gamma}$, it follows that $\inf_{\mathcal{O}_1} V_1 > 0$. Thus, it holds true in $\mathcal{O}_1$ that

$$-\mathcal{L}V_1 \leq -hV_1 \leq -h \inf_{\mathcal{O}_1} V_1 < 0.$$

It is then possible to find a positive constant $\kappa < \varepsilon$ such that

$$-\mathcal{L}V_1 \leq -\kappa(\sum_{i,j=1}^{N} |\partial_{ij} V_1| + \sum_{i=1}^{N} |\partial_i V_1| + V_1) \quad \text{in } \mathcal{O}_1.$$

The proof is thereby complete. □

5. The lateral-periodic case

Henceforth, for every $Q \in \mathbb{N}$ and $r > 0$, B_r^Q stands for the ball in $\mathbb{R}^Q$ centred at the origin with radius r, and $B_r := B_r^N$. Other notations used in this section are:

$$\mathcal{L}w = \Delta w + ce \cdot \nabla w + f_s(x, 0)w,$$

$$\lambda_1 = \lambda_1(-\mathcal{L}, \mathbb{R}^N),$$

$$\zeta = -\lim_{r\to\infty} \sup_{\substack{|z|>r \\ y\in\mathbb{R}^P}} f_s(z, y, 0),$$

$$\forall\, r > 0, \quad \mathcal{O}_r := B_r^M \times \mathbb{R}^P.$$

In Section 5.1, we introduce the *lateral-periodic principal eigenvalues* $\lambda_{1,l}(r)$ of an elliptic operator $-L$ in the domains $\mathcal{O}_r$, under Dirichlet boundary condition on $\partial\mathcal{O}_r$ and periodicity condition in the last P variables. Then, we show that as $r \to \infty$ the $\lambda_{1,l}(r)$ converge to a quantity that we call $\lambda_{1,l}(-L, \mathbb{R}^N)$.

Let us explain why we need to consider both λ_1 and $\lambda_{1,l} := \lambda_{1,l}(-\mathcal{L}, \mathbb{R}^N)$. The negativity of $\lambda_{1,l}$ yields the existence of a lateral-periodic subsolution $\underline{V}$ to (23) which is as small as we want. This function allows one to prove the existence of a travelling wave, but not to derive the large time behaviour of solutions u to (1), because we cannot put $\underline{V}$ below $u(1, x)$. Instead, the subsolution $\underline{U}$ one can construct when $\lambda_1 < 0$ is compactly supported and then we can put it below $u(1, x)$ and derive Theorem 2.6 part (ii). For similar reasons, we use λ_1 instead of $\lambda_{1,l}$ to prove the uniqueness of travelling wave solutions. On the other hand, we make use of the lateral periodic principal eigenfunction χ associated with $\lambda_{1,l}$ to derive the nonexistence result for travelling waves when $\lambda_{1,l} \geq 0$, because it satisfies the needed property $\inf_{\mathcal{O}_r} \chi > 0$ for any $r > 0$, while the principal eigenfunction associated with λ_1 does not. Thus, a crucial point to prove our main results consists in showing that λ_1 and $\lambda_{1,l}$ have the same sign. Actually, using a general result for self-adjoint operators quoted from [7], we will show that they coincide.

5.1. **The lateral-periodic principal eigenvalue.** Here, L denotes an elliptic operator of the form

$$Lw := \partial_i(a_{ij}(x)\partial_i w) + \beta_i(x)\partial_i w + \gamma(x)w,$$

where $(a_{ij})_{ij}$ is an elliptic and symmetric matrix field with Lipschitz continuous entries and β_i, γ are bounded. We further require that a_{ij}, β_i, γ are lateral-periodic, that is, they are periodic in the last P variables, with the same period $(l_1, \cdots, l_P)$. We remark that, through a regularizing argument, one can prove that the results of this section hold for more general elliptic operators in non-divergence form.

First of all, we reclaim some properties of λ_1. A basic result of [4] is that if $\mathcal{O}$ is bounded and smooth then $\lambda_1(-L, \mathcal{O})$ coincides with the Dirichlet principal eigenvalue of $-L$ in $\mathcal{O}$, that is, the unique real number λ such that the problem

$$\begin{cases} -L\phi = \lambda\phi & \text{a. e. in } \mathcal{O} \\ \phi = 0 & \text{on } \partial\mathcal{O} \end{cases}$$

admits a positive solution ϕ (called Dirichlet principal eigenfunction, which is unique up to a multiplicative constant). Another result we will use is

Proposition 4 ([4] and Proposition 4.2 in [2]). *Let $\mathcal{O}$ be a general domain in $\mathbb{R}^N$ and $(\mathcal{O}^n)_{n\in\mathbb{N}}$ be a sequence of domains such that*

$$\mathcal{O}^n \subset \mathcal{O}^{n+1}, \qquad \bigcup_{n\in\mathbb{N}} \mathcal{O}^n = \mathcal{O}.$$

Then, $\lambda_1(-L, \mathcal{O}^n) \searrow \lambda_1(-L, \mathcal{O})$ as $n \to \infty$.

Next, we consider the eigenvalue problem with mixed Dirichlet/periodic conditions.

Theorem 5.1. *For any $r > 0$ there exists a unique number $\lambda_{1,l}(r)$ such that the eigenvalue problem*

$$\begin{cases} -L\chi_r = \lambda_{1,l}(r)\chi_r & \text{a. e. in } \mathcal{O}_r \\ \chi_r = 0 & \text{on } \partial\mathcal{O}_r \\ \chi_r \text{ is lateral-periodic} & \end{cases}$$

admits a positive solution. We call $\lambda_{1,l}(r)$ and χ_r (which is unique up to a multiplicative constant) respectively the lateral-periodic principal eigenvalue *and* eigenfunction *of $-L$ in $\mathcal{O}_r$.*

Proof. Define the Banach space

$$X_r = \{\phi \in C^1(\overline{\mathcal{O}}_r) \ : \ \phi = 0 \text{ on } \partial\mathcal{O}_r, \ \phi \text{ is lateral-periodic}\},$$

equipped with the $W^{1,\infty}(\mathcal{O}_r)$ norm. Set $\mathcal{M} := L - d$, with d large enough such that the associated bilinear form is coercive on the space of lateral-periodic functions $\phi \in H^1(B_r^M \times (0, l_1) \times \cdots \times (0, l_P))$ satisfying $\phi = 0$ in $H^{1/2}(\partial B_r^M \times (0, l_1) \times \cdots \times (0, l_P))$. Then, the result follows from the Krein-Rutman theorem (as in the proof of Theorem 3.1 in the appendix, but now we do not have the problem of non-smoothness of the boundary). □

Proposition 5. *The map $r \mapsto \lambda_{1,l}(r)$ is decreasing and, as r goes to infinity, $\lambda_{1,l}(r)$ converges to a quantity that we call $\lambda_{1,l}(-L,\mathbb{R}^N)$.*

Furthermore, there exists a lateral-periodic principal eigenfunction associated with $\lambda_{1,l}(-L,\mathbb{R}^N)$, that is, a lateral-periodic positive function χ such that

$$-L\chi = \lambda_{1,l}(-L,\mathbb{R}^N)\chi \quad \text{a. e. in } \mathbb{R}^N.$$

Proof. We follow the same arguments as in the proof of Proposition 5 in [2]. Let $0 < r_1 < r_2$. Owing to the lateral-periodicity of the principal eigenfunctions χ_{r_1} and χ_{r_2}, there exists $k > 0$ such that $k\chi_{r_2}$ touches from above χ_{r_1} at some point in $\mathcal{O}_{r_1}$. If $\lambda_{1,l}(r_1) \leq \lambda_{1,l}(r_2)$ then the function $w := k\chi_{r_2} - \chi_{r_1}$ satisfies

$$-Lw \geq \lambda_{1,l}(r_1)w \quad \text{a. e. in } \mathcal{O}_{r_1}.$$

Thus, the strong maximum principle yields $w \equiv 0$, which is impossible. Again by the strong maximum principle, we immediately see that $\lambda_{1,l}(r) > -\sup_{\mathcal{O}_r}\gamma$, for any $r > 0$. Hence, the quantity

$$\lambda_{1,l}(-L,\mathbb{R}^N) := \lim_{r\to\infty}\lambda_{1,l}(r)$$

is a well defined real number.

Let us show the existence of a lateral-periodic principal eigenfunction associated with $\lambda_{1,l}(-L,\mathbb{R}^N)$. By Harnack's inequality, the family $(\chi_r)_{r>0}$, normalized by $\chi_r(0) = 1$, is uniformly bounded in any compact subset of $\mathbb{R}^N$. Then, interior elliptic estimates and embedding theorems imply that, up to subsequences, the χ_r converge as $r \to \infty$, locally uniformly in $\mathbb{R}^N$, to a function χ satisfying $-L\chi = \lambda_{1,l}(-L,\mathbb{R}^N)\chi$ a. e. in $\mathbb{R}^N$. Moreover, χ is lateral-periodic, satisfies $\chi(0) = 1$ and it is strictly positive by the strong maximum principle. □

As for the Neumann principal eigenfunction in Proposition 1, the function χ is not unique a priori.

In order to compare λ_1 and $\lambda_{1,l}$, we consider another notion of generalized principal eigenvalue of $-L$ in a domain $\mathcal{O}$:

(56)
$$\lambda_1'(-L,\mathcal{O}) := \inf\{\lambda \;:\; \exists\, \phi \in W^{2,N}_{loc}(\mathcal{O}) \cap L^\infty(\mathcal{O}),\ \phi > 0 \text{ and } -(L+\lambda)\phi \leq 0 \text{ in } \mathcal{O},$$
$$\phi \in W^{1,\infty}(\mathcal{O}\cap B_r),\ \forall\, r > 0, \text{ and } \phi = 0 \text{ on } \partial\mathcal{O} \text{ if } \partial\mathcal{O} \neq \emptyset\}$$

We quote from [7] the following result about self-adjoint operators:

Theorem 5.2 ([7])**.** *If $\mathcal{O}$ is smooth and L is self-adjoint (i. e. $b_i \equiv 0$) then $\lambda_1'(-L,\mathcal{O}) = \lambda_1(-L,\mathcal{O})$.*

Proposition 6. *If L is self-adjoint then $\lambda_{1,l}(-L,\mathbb{R}^N) = \lambda_1(-L,\mathbb{R}^N)$.*

Proof. Let $r > 0$. Taking $\phi = \chi_r$ in (24) and (56) we see that

$$\lambda_1'(-L,\mathcal{O}_r) \leq \lambda_{1,l}(r) \leq \lambda_1(-L,\mathcal{O}_r).$$

Hence, Theorem 5.2 yields $\lambda_{1,l}(r) = \lambda_1(-L,\mathcal{O}_r)$. The statement then follows from Propositions 4 and 5. □

Remark 5. If L is not self-adjoint equality need not hold between $\lambda_{1,l}(-L,\mathbb{R}^N)$ and $\lambda_1(-L,\mathbb{R}^N)$. Indeed, consider the lateral-periodic (with $P = 1$) operator

$$Lu(z,y) = \Delta u(z,y) + 2\partial_y u(z,y), \qquad (z,y) \in \mathbb{R}\times\mathbb{R}.$$

For any $r > 0$, the function $\chi_r(z,y) := \cos(\frac{\pi}{2r}z)$ is the lateral-periodic principal eigenfunction of both $-L$ and $-\Delta$ in $\mathcal{O}_r$, with eigenvalue $\frac{\pi^2}{4r^2}$. Therefore, $\lambda_{1,l}(-L,\mathbb{R}^2) = \lambda_{1,l}(-\Delta,\mathbb{R}^2)$. On the other hand, for any $\phi \in W^{2,N}_{loc}(\mathbb{R}^2)$, it holds true that $\Delta(\phi e^y) = (L\phi + \phi)e^y$. Hence, by the definition (24) we see that $\lambda_1(-L,\mathbb{R}^2) = \lambda_1(-\Delta,\mathbb{R}^2) + 1$. Proposition 6 then yields

$$\lambda_{1,l}(-L,\mathbb{R}^2) = \lambda_{1,l}(-\Delta,\mathbb{R}^2) = \lambda_1(-\Delta,\mathbb{R}^2) = \lambda_1(-L,\mathbb{R}^2) - 1.$$

From now on, $\lambda_{1,l}(r)$ and χ_r will always denote the lateral-periodic principal eigenvalue and eigenfunction of $-\mathcal{L}$ in $\mathcal{O}_r$. We further set $\lambda_{1,l} := \lambda_{1,l}(-\mathcal{L},\mathbb{R}^N)$ and we denote by χ an associated lateral-periodic principal eigenfunctions (cf. Proposition 5). In order to show that $\lambda_{1,l} = \lambda_1$, we make the usual Liouville transformation which reduces (23) to a problem whose linearized operator is self-adjoint. Then, we apply Proposition 6.

Proposition 7. *If f satisfies (18) then $\lambda_{1,l} = \lambda_1$.*

Proof. For any $\phi \in W^{2,N}_{loc}(\mathbb{R}^N)$ the following property holds:

$$(\mathcal{L}(\phi e^{-\frac{c}{2}x\cdot e}))e^{\frac{c}{2}x\cdot e} = \tilde{\mathcal{L}}\phi,$$

where $\tilde{\mathcal{L}}w := \Delta w + (f_s(x,0) - c^2/4)w$. It follows that the operators $\tilde{\mathcal{L}}$ and $\mathcal{L}$ have the same lateral-periodic principal eigenvalues $\lambda_{1,l}(r)$ in Ω_r, for $r > 0$, and, by definition (24), that $\lambda_1(-\tilde{\mathcal{L}},\mathbb{R}^N) = \lambda_1$. Propositions 5 and 6 then yield

$$\lambda_1 = \lambda_1(-\tilde{\mathcal{L}},\mathbb{R}^N) = \lambda_{1,l}(-\tilde{\mathcal{L}},\mathbb{R}^N) = \lim_{r\to\infty}\lambda_{1,l}(r) = \lambda_{1,l}.$$

□

5.2. **Travelling wave solutions.** Arguing as in Section 3.3, one can show that solutions $U(z,y)$ to (1) decay exponentially in z. Now, the Liouville transformation reducing (61) to a problem with self-adjoint linearized operator is $V(x) := U(x)e^{\frac{c}{2}x\cdot e}$. We omit the proofs of the next two results because they are essentially the same as those of Lemma 3.2 and Proposition 3 respectively.

Lemma 5.3. *Let $V \in W^{2,N}_{loc}(\mathbb{R}^N)$ be a positive function satisfying, for some $\gamma > 0$,*

$$\sup_{(z,y)\in\mathbb{R}^N} V(z,y)e^{-\sqrt{\gamma}|z|} < \infty, \qquad \liminf_{|z|\to\infty}\frac{\Delta V(z,y)}{V(z,y)} > \gamma,$$

uniformly in $y \in \mathbb{R}^P$. Then,

$$\lim_{|z|\to\infty} V(z,y)e^{\sqrt{\gamma}|z|} = 0,$$

uniformly in $y \in \mathbb{R}^P$.

Proposition 8. *Let U be a solution of (23) and assume that (18), (19), (21), (22) hold. Then, there exist two constants $h,\beta > 0$ such that*

$$\forall\,(z,y) \in \mathbb{R}^N, \quad U(z,y) \le he^{-\beta|z|}.$$

We can now derive the comparison principle.

Theorem 5.4. *Assume that (18), (19), (21), (22) hold. Let $\underline{U},\overline{U} \in W^{2,N}_{loc}(\mathbb{R}^N)$ be two nonnegative functions satisfying*

$$-\Delta\underline{U} - ce\cdot\nabla\underline{U} \le f(x,\underline{U}), \qquad -\Delta\overline{U} - ce\cdot\nabla\overline{U} \ge f(x,\overline{U}), \qquad \textit{for a. e. } x \in \mathbb{R}^N,$$

$$\forall\, r > 0, \quad \inf_{\mathcal{O}_r}\overline{U} > 0, \qquad \lim_{|z|\to\infty}\underline{U}(z,y) = 0 \textit{ uniformly in } y \in \mathbb{R}^P$$

and for any $\rho > 0$ *there exists* $C_\rho > 0$ *such that*

$$\forall\, y_0 \in \mathbb{R}^P, \quad \|\underline{U}\|_{W^{2,N}(B_\rho(0,y_0))} + \|\overline{U}\|_{W^{2,N}(B_\rho(0,y_0))} \leq C_\rho, \tag{57}$$

Then $\underline{U} \leq \overline{U}$ *in* $\mathbb{R}^N$.

Proof. First note that, by the embedding theorem, condition (57) yields $\underline{U}, \overline{U} \in C^0(\mathbb{R}^N) \cap L^\infty(\mathcal{O}_r)$, for any $r > 0$. For $\varepsilon > 0$ define

$$k(\varepsilon) := \inf\{k > 0 \ : \ k\overline{U} \geq \underline{U} - \varepsilon \text{ in } \mathbb{R}^N\}$$

(the above set is nonempty by the hypotheses on $\underline{U}$ and $\overline{U}$). Clearly, $\varepsilon \mapsto k(\varepsilon)$ is nonincreasing. Furthermore, for $\varepsilon \in (0, \sup \underline{U})$, the function $W^\varepsilon := k(\varepsilon)\overline{U} - \underline{U} + \varepsilon$ is nonnegative and there exist a bounded sequence $(z_n^\varepsilon)_{n\in\mathbb{N}}$ in $\mathbb{R}^M$ and a sequence $(y_n^\varepsilon)_{n\in\mathbb{N}}$ in $\mathbb{R}^P$ such that

$$\lim_{n\to\infty} W^\varepsilon(z_n^\varepsilon, y_n^\varepsilon) = 0.$$

We use the lateral periodicity of f and condition (57) to reduce to the case where the minimizing sequence is bounded: let $(q_n^\varepsilon)_{n\in\mathbb{N}}$ be the sequence in $\mathbb{Z}l_1 \times \cdots \times \mathbb{Z}l_P$ such that $\eta_n^\varepsilon := y_n^\varepsilon - q_n^\varepsilon$ belongs to $[0, l_1) \times \cdots \times [0, l_P)$. For $n \in \mathbb{N}$ define $\underline{U}_n(z, y) := \underline{U}(z, y + q_n)$ and $\overline{U}_n(z, y) := \overline{U}(z, y + q_n)$. As f is lateral-periodic, these functions satisfy the same differential inequalities as $\underline{U}$ and $\overline{U}$ respectively. By (57), as $n \to \infty$ and up to subsequences, $\underline{U}_n \to \underline{U}_\infty$ and $\overline{U}_n \to \overline{U}_\infty$ locally uniformly in $\mathbb{R}^N$, where $\underline{U}_\infty$ and $\overline{U}_\infty$ satisfy the same hypotheses as $\underline{U}$ and $\overline{U}$ respectively. Therefore, denoting $(z(\varepsilon), y(\varepsilon))$ the limit of (a subsequence of) $((z_n^\varepsilon, \eta_n^\varepsilon))_{n\in\mathbb{N}}$, we find that the function $W_\infty^\varepsilon := k(\varepsilon)\overline{U}_\infty - \underline{U}_\infty + \varepsilon$ is nonnegative and vanishes at $(z(\varepsilon), y(\varepsilon))$. Note that $y(\varepsilon)$ are bounded with respect to ε. The result then follows exactly as in the proof of Theorem 3.3. □

Proof of Theorem 2.5.

Step 1: existence.

If $\lambda_1 < 0$ then by Proposition 4 there exists $R > 0$ large enough such that $\lambda_1(-\mathcal{L}, B_R) < 0$. We recall that, as B_R is bounded and smooth, $\lambda_1(-\mathcal{L}, B_R)$ coincides with the Dirichlet principal eigenvalue of $-\mathcal{L}$ in B_R. That is, there exists a function ϕ_R which is positive in B_R and satisfies

$$\begin{cases} -\mathcal{L}\phi_R = \lambda_1(-\mathcal{L}, B_R)\phi_R & \text{a. e. in } B_R \\ \phi_R = 0 & \text{on } \partial B_R \end{cases}$$

For $\kappa \in \mathbb{R}$ and for a. e. $x \in B_R$ we see that

$$-\Delta(\kappa\phi_R) - ce\cdot\nabla(\kappa\phi_R) - f(x, \kappa\phi_R) = (f_s(x, 0) + \lambda_1(-\mathcal{L}, B_R))\kappa\phi_R - f(x, \kappa\phi_R).$$

Then, owing to the C^1 regularity of $f(x, \cdot)$, there exists $\kappa_0 > 0$ such that for any $0 < \kappa \leq \kappa_0$ the function $\kappa\phi_R$ is a subsolution to

$$\Delta U + ce\cdot\nabla U + f(x, U) = 0 \qquad \text{a. e. in } B_R. \tag{58}$$

Hence, the function $\underline{U}$ equal to $\kappa_0\phi_R$ in B_R and extended by 0 outside B_R is a generalized subsolution of the elliptic equation in (23). Since by (20) the function

$$\overline{U} := \max\{S, \kappa_0\|\phi_R\|_{L^\infty(B_R)}\}$$

is a supersolution of the same equation, a standard iterative method implies the existence of a solution $\underline{U} \leq U \leq \overline{U}$. The function U is strictly positive by the strong maximum principle and then it solves (23). Assume by contradiction that $\lambda_1 \geq 0$ and (23) admits a solution U. Let χ be a lateral-periodic principal eigenfunction associated with $\lambda_{1,l}$ (cf. Proposition 5) normalized in such a way that $\chi(0) < U(0)$.

If we show that the hypotheses of Theorem 5.4 are satisfied by $\underline{U} := U$ and $\overline{U} := \chi$, we would get the following contradiction: $U \leq \chi$. Proposition 8 yields

$$\lim_{|z|\to\infty} U(z,y) = 0 \quad \text{uniformly in } y \in \mathbb{R}^P,$$

while $\lambda_{1,l} \geq 0$ and condition (21) imply that χ is a supersolution of (23). The other hypotheses are immediate to check.

Step 2: uniqueness.
It follows from the comparison principle, Theorem 5.4, provided that we show that any solution U to (23) satisfies the hypotheses on both $\underline{U}$ and $\overline{U}$ there. All conditions are immediate to check (the decay of $U(z,y)$ with respect to z is given by Proposition 8), except the following one:

$$\forall\, r > 0, \quad \inf_{\mathcal{O}_r} U > 0 \tag{59}$$

(note indeed that we do not assume a priory that U is lateral-periodic). The existence result implies that if (23) admits a solution U then $\lambda_1 < 0$. In order to prove that U satisfies (59), fix $r > 0$ and consider the same constants R, κ_0 and function ϕ_R as in the first step. It is not restrictive to assume that $B_R \supset \overline{B}_r^M \times [0,l_1] \times \cdots \times [0,l_P]$. For any $q \in \mathbb{Z}l_1 \times \cdots \times \mathbb{Z}l_P$ define

$$\kappa(q) := \inf_{(z,y)\in B_R} \frac{U(z,y+q)}{\phi_R(z,y)}.$$

Hence, $\kappa(q)\phi_R(z,y) \leq U(z,y+q)$ for $(z,y) \in B_R$ and, as $\phi_R = 0$ on ∂B_R, there exists $(z_q,y_q) \in B_R$ such that $\kappa(q)\phi_R(z_q,y_q) = U(z_q,y_q+q)$. If $\kappa(q) \leq \kappa_0$ for some $q \in \mathbb{Z}l_1 \times \cdots \times \mathbb{Z}l_P$, then $U(z,y+q)$ and $\kappa(q)\phi_R(z,y)$ would be respectively a solution and a subsolution of (58) and then they would coincide in B_R by the strong maximum principle. This is impossible because $\phi_R = 0$ on ∂B_R. Therefore,

$$\forall\, q \in \mathbb{Z}l_1 \times \cdots \times \mathbb{Z}l_P,\ (z,y) \in B_R, \quad U(z,y+q) \geq \kappa(q)\phi_R(z,y) > \kappa_0\phi_R(z,y).$$

Since ϕ_R has a positive minimum on $\overline{B}_r^M \times [0,l_1] \times \cdots \times [0,l_P] \subset B_R$, (59) follows. The lateral-periodicity of the solution to (23) follows from the uniqueness result. □

5.3. **Large time behaviour.** Once we have proved Theorem 2.5, Theorem 2.6 follows essentially from the same ideas as Theorem 2.2. Thus, we will skip some details.

Proof of Theorem 2.6. The function $\tilde{u}(t,x) := u(t,x+cte)$ satisfies

$$0 < \tilde{u} \leq S' := \max\{S, \|u_0\|_{L^\infty(\Omega)}\} \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^N,$$

where S is the positive constant in (20), and solves

$$\partial_t \tilde{u} = \Delta\tilde{u} + ce\cdot\nabla\tilde{u} + f(x,\tilde{u}), \qquad t > 0, \quad x \in \mathbb{R}^N, \tag{60}$$

with initial condition $\tilde{u}(0,x) = u_0(x)$. Let w be the solution to (60) with initial condition $w(0,x) = S'$. The comparison principle implies that w satisfies $\tilde{u} \leq w \leq S'$, is nonincreasing in t and, as $t \to \infty$, converges locally uniformly in $\mathbb{R}^N$ to a nonnegative bounded solution W of

$$\Delta U + ce\cdot\nabla U + f(x,U) = 0 \qquad \text{a. e. in } \mathbb{R}^N. \tag{61}$$

Since $w(t,x)$ is lateral-periodic in x by uniqueness, it follows that W is lateral-periodic too and that

$$\forall\, r > 0, \quad \lim_{t\to\infty} \sup_{x\in\mathcal{O}_r} \big(\tilde{u}(t,x) - W(x)\big) \leq \lim_{t\to\infty} \sup_{x\in\mathcal{O}_r} \big(w(t,x) - W(x)\big) = 0. \tag{62}$$

Step 1: the function $\tilde{u}$ satisfies

$$\lim_{\min(t,|z|)\to\infty} \tilde{u}(t,z,y) = 0 \quad \text{uniformly in } y \in \mathbb{R}^P.$$

As $\tilde{u} \leq w$, it is sufficient to show that the above property is satisfied by w. The advantage is that w is lateral-periodic. Suppose that there exist $\varepsilon > 0$, $(t_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^+$ and $((z_n, y_n))_{n\in\mathbb{N}}$ in $\mathbb{R}^N$ such that

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} |z_n| = \infty, \qquad \forall\, n \in \mathbb{N}, \quad w(t_n, z_n, y_n) > \varepsilon.$$

It is not restrictive to assume that $(y_n)_{n\in\mathbb{N}}$ is bounded. Thus, we get a contradiction by arguing as in the step 2 of the proof of Theorem 2.2.

Step 2: conclusion of the proof.
In the case $\lambda_1 \geq 0$, the function W can not be strictly positive by Theorem 2.5. Hence, the strong maximum principle yields $W \equiv 0$ and then statement (i) follows from (62) and step 1. Consider the case $\lambda_1 < 0$. We know that, for R large enough and κ small enough, the function $\kappa\phi_R$ is a subsolution to (58) (see the proof of Theorem 2.5 above). Hence, for κ small the function

$$\underline{U}(x) := \begin{cases} \kappa\phi_R(x) & x \in B_R \\ 0 & \text{otherwise.} \end{cases}$$

is a subsolution of (61) and satisfies $\underline{U}(x) \leq \tilde{u}(1,x)$ in $\mathbb{R}^N$. Let v be the solution to (60) with initial condition $v(0,x) = \underline{U}(x)$. Then, $0 \leq v(t,x) \leq \tilde{u}(t+1,x)$, $v(t,x)$ is nondecreasing in t and, as $t \to \infty$, converges locally uniformly in $\mathbb{R}^N$ to a nonnegative bounded solution V of (61) satisfying $\underline{U} \leq V \leq W$. Therefore, the strong maximum principle yields $0 < V \leq W$ and then both V and W coincide with the unique solution U to (23). By (62) we then infer that, as $t \to \infty$, $\tilde{u}(t,x)$ converges to U locally uniformly in $x \in \mathbb{R}^N$. Assume by contradiction that there exist $\varepsilon > 0$, $(t_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^+$ and $(z_n, y_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^N$ such that $(y_n)_{n\in\mathbb{N}}$ is bounded,

$$\lim_{n\to\infty} t_n = \infty, \qquad \forall\, n \in \mathbb{N}, \quad |\tilde{u}(t_n, z_n, y_n) - U(z_n, y_n)| \geq \varepsilon.$$

Owing to the local uniform convergence of $\tilde{u}$, we necessarily have that the sequence $(z_n)_{n\in\mathbb{N}}$ diverges. Hence, step 1 and Proposition 8 yield a contradiction. It only remains to show that if u_0 is either lateral-periodic or it satisfies (25) then

$$\lim_{t\to\infty} \tilde{u}(t,x) = U(x) \quad \text{uniformly in } x \in \mathbb{R}^N. \tag{63}$$

By Propositions 5 and 7 there exists $\rho > 0$ such that $\lambda_{1,l}(\rho) < 0$ (we recall that $\lambda_{1,l}(\rho)$ denotes the lateral-periodic principal eigenvalue of $-\mathcal{L}$ in $\mathcal{O}_\rho$, and χ_ρ the associated eigenfunction). With usual arguments, one sees that the function

$$\underline{\tilde{U}}(x) := \begin{cases} \kappa\chi_\rho(x) & x \in \mathcal{O}_\rho \\ 0 & \text{otherwise.} \end{cases}$$

is a subsolution to (61) for κ small enough. Moreover, if (25) holds then we can chose κ in such a way that $\underline{\tilde{U}} \leq u_0$. On the other hand, if u_0 is lateral-periodic then $\tilde{u}(t,x)$ is lateral-periodic in x and then, as it is positive for $t > 0$, $\underline{\tilde{U}}(x) \leq \tilde{u}(1,x)$ for κ small enough. In the first case we define $\tilde{v}$ as the solution to (60) satisfying $\tilde{v}(0,x) = \underline{\tilde{U}}(x)$, while in the second as the solution to (60) for $t > 1$ satisfying $\tilde{v}(1,x) = \underline{\tilde{U}}(x)$. In both cases, the maximum principle implies that $\tilde{v}(t,x) \leq \tilde{u}(t,x)$ for $t \geq 1$, $x \in \mathbb{R}^N$ and that $\tilde{v}$ is nondecreasing in t and lateral-periodic in x. Then, as $t \to \infty$ it converges to the unique solution $U \equiv W$ to (23) uniformly in $\mathcal{O}_r$, for any $r > 0$. Therefore, (62) yields

$$U(x) = \lim_{t\to\infty} \tilde{v}(t,x) \leq \lim_{t\to\infty} \tilde{u}(t,x) \leq \lim_{t\to\infty} w(t,x) = U(x),$$

uniformly in $x \in \mathcal{O}_r$, for any $r > 0$. Step 1 and the decay of U then imply (63). □

6. Behaviour near critical value

Bifurcation results of the type of Theorem 2.7 have been proved by Crandall and Rabinowitz [8] in very general frameworks. However, we will not make use of the abstract result of [8], but rather give a direct proof. Indeed, to check that its hypotheses are satisfied in our case requires essentially the same work as the direct derivation of Theorem 2.7 which we give here.

In order to prove statement (ii) of Theorem 2.7 we make use of the fact that the generalized Neumann principal eigenvalue $\lambda_{1,N}$ is simple when $c = c_0$ (i. e. when $\lambda_{1,N} = 0$). This type of property, which follows directly from the Krein-Rutman theory in the case of the principal eigenvalue of an operator in a bounded smooth domain, is not true in general for unbounded domains. Thus, this part is rather delicate. It holds here because of the additional property that the zero order term of $\mathcal{L}$ is negative at infinity, cf. condition (12). We prove this result in [7] by first showing that there exists a generalized principal eigenvalue which vanishes at infinity and then using a comparison result of the same type as Theorem 3.3 here.

Theorem 6.1 ([7])**.** *Let L be the operator defined by $Lw = \Delta w + \beta(x)\cdot\nabla w + \gamma(x)w$, with $\beta, \gamma \in L^\infty(\Omega)$. If*

$$\lambda_{1,N}(-L,\Omega) < -\lim_{r\to\infty} \sup_{\substack{|x_1|>r \\ y\in\omega}} \gamma(x_1,y),$$

then the generalized Neumann principal eigenfunction of $-L$ in Ω (i. e. positive solution of (34)) is unique up to a positive multiplicative constant.

The reader is referred to [7] for the details of the proof.

Proof of Theorem 2.7. (i) Assume by contradiction that there exist $\varepsilon > 0$ and two sequences $(c_n)_{n\in\mathbb{N}}$ in $(0, c_0)$ and $((x_1^n, y^n))_{n\in\mathbb{N}}$ in Ω such that

$$\lim_{n\to\infty} c_n = c_0, \qquad U^{c_n}(x_1^n, y^n) \geq \varepsilon.$$

We know that $0 < U^{c_n} \leq S$, where the second inequality - with S given by (10) - follows from Theorem 3.3. By elliptic estimates and embedding theorems (a subsequence of) the sequence $(U^{c_n})_{n\in\mathbb{N}}$ converges uniformly in Ω_r, for any $r > 0$, to a nonnegative bounded solution U^* of

$$\begin{cases} \Delta U^* + c_0\partial_1 U^* + f(x, U^*) = 0 & \text{a. e. in } \Omega \\ \partial_\nu U^* = 0 & \text{on } \partial\Omega \end{cases}$$

Since U^* is not strictly positive by Proposition 2 and Theorem 2.1, the strong maximum principle yields $U^* \equiv 0$. Hence, the sequence $(x_1^n)_{n\in\mathbb{N}}$ has to be divergent. It is not restrictive to assume that

$$\forall\, n \in \mathbb{N}, \quad U^{c_n}(x_1^n, y^n) \geq \|U^{c_n}\|_{L^\infty(\Omega)} - \frac{1}{n}.$$

Define $U_n(x_1,y) := U^{c_n}(x_1+x_1^n,y)$. By (9), (11) and (12), the U_n converge (up to subsequences) uniformly in Ω_r, for any $r>0$, to a nonnegative bounded function U_∞ satisfying

$$\begin{cases} \Delta U_\infty + c_0\partial_1 U_\infty \geq \zeta U_\infty & \text{a. e. in } \Omega \\ \partial_\nu U_\infty = 0 & \text{on } \partial\Omega \end{cases}$$

Moreover, if $\eta\in\overline{\omega}$ is the limit of (a subsequence of) $(y^n)_{n\in\mathbb{N}}$, we see that

$$U_\infty(0,\eta)\geq\varepsilon, \qquad U_\infty(0,\eta)\geq\limsup_{n\to\infty}\|U_n\|_{L^\infty(\Omega)}\geq\|U_\infty\|_{L^\infty(\Omega)},$$

that is, U_∞ has a positive maximum at $(0,\eta)$. This is impossible due to the strong maximum principle and the Hopf lemma.

(ii)

The proof is divided into three sub-statements. Let $0<\underline{c}<c_0$ be such that $\underline{c}^2\geq c_0^2-\zeta$. For $c\in(\underline{c},c_0)$ set $W^c := U^c/\|U^c\|_{L^\infty(\Omega)}$.

Step 1:

$$\lim_{|x_1|\to\infty} W^c(x_1,y)=0 \quad \text{uniformly with respect to } y\in\omega \text{ and } c\in(\underline{c},c_0).$$

The function $V^c(x_1,y) := W^c(x_1,y)e^{\frac{c}{2}x_1}$ satisfies

$$\begin{cases} \Delta V^c + \dfrac{f(x_1,y,U^c(x_1,y))e^{\frac{c}{2}x_1}}{\|U^c\|_{L^\infty(\Omega)}} - \dfrac{c^2}{4}V^c = 0 & \text{for a. e. } (x_1,y)\in\Omega \\ \partial_\nu V = 0 & \text{on } \partial\Omega \\ 0<V(x_1,y)\leq e^{\frac{c}{2}x_1} & \text{in } \Omega. \end{cases}$$

We want to apply Lemma 3.2 showing that the V^c decay exponentially uniformly with respect to $c\in(\underline{c},c_0)$. Set $\gamma := \frac{\zeta+c_0^2}{4}$ and $\varepsilon := \frac{\zeta}{4}$ and let $R>0$ be such that $f_s(x,0)<-\frac{3}{4}\zeta$ for a. e. $x\in\Omega\backslash\overline{\Omega}_R$. By (9) and (11) we get

$$\text{for a. e. } x\in\Omega\backslash\overline{\Omega}_R, \quad \frac{\Delta V^c(x)}{V^c(x)}\geq -f_s(x,0)+\frac{c^2}{4}>\frac{3}{4}\zeta+\frac{c_0^2-\zeta}{4}=\gamma+\varepsilon.$$

Moreover, $V^c(x_1,y)e^{-\sqrt{\gamma}|x_1|}\leq 1$ in Ω. Hence, as we have seen in the proof of Lemma 3.2, it follows that $V^c(x_1,y)\leq e^{(\sqrt{\gamma}+\sqrt{\gamma+\varepsilon})R}e^{-\sqrt{\gamma+\varepsilon}|x_1|}$ for $|x_1|>R$. As a consequence,

$$\forall\, c\in(\underline{c},c_0),\ (x_1,y)\in\Omega, \quad W^c(x_1,y)\leq Ce^{(\frac{c}{2}-\sqrt{\gamma})|x_1|}\leq Ce^{(\frac{c_0}{2}-\sqrt{\gamma})|x_1|}.$$

Step 2: For any sequence $(c_n)_{n\in\mathbb{N}}$ in $(0,c_0)$ converging to c_0 there exists a subsequence $(c_{n_k})_{k\in\mathbb{N}}$ such that $(W^{c_{n_k}})_{k\in\mathbb{N}}$ converges uniformly in Ω to a positive solution of (26).

Let $(c_n)_{n\in\mathbb{N}}$ be a sequence in $(0,c_0)$ converging to c_0. Owing to step 1, there exists a bounded sequence $(x_n)_{n\in\mathbb{N}}$ in Ω such that $W^{c_n}(x_n)=1$. Let $(x_{n_k})_{k\in\mathbb{N}}$ be a subsequence converging to some $\xi\in\overline{\Omega}$. We set for brief $W_k := W^{c_{n_k}}$. By (i), we see that

$$\lim_{k\to\infty}\frac{\Delta W_k + c_{n_k}\partial_1 W_k}{W_k}=\lim_{k\to\infty}\frac{f(x,U^{c_{n_k}})}{U^{c_{n_k}}}=f_s(x,0),$$

uniformly in $x\in\Omega$. Thus, usual arguments imply that as $k\to\infty$ the W_k converge (up to subsequences) uniformly in Ω_r, for any $r>0$, to a positive solution φ of (26). Again by step 1, $\varphi(x_1,y)$ converges to 0 as $|x_1|\to\infty$ uniformly in $y\in\omega$ and then the W_k converge to φ uniformly in Ω.

Step 3: The eigenvalue problem (26) admits a unique positive solution.

In the previous step, we have explicitly exhibited the existence of a positive solution to (26). Let $L = \Delta + c_0\partial_1 + f_s(x,0)$. By the definition of c_0 (see Section 3.2) we know that $\lambda_{1,N}(-L,\Omega) = 0$. Thus,

$$0 = \lambda_{1,N}(-L,\Omega) < -\lim_{r\to\infty} \sup_{\substack{|x_1|>r\\ y\in\omega}} f_s(x_1,y,0),$$

and then Theorem 6.1 implies the uniqueness of the positive solution of (26). $\square$

7. L^1 convergence

We first derive the following result for linear parabolic problems.

Lemma 7.1. *Let $\xi \in L^\infty(\mathbb{R}^+\times\Omega)$ satisfy*

$$\lim_{r\to\infty} \sup_{\substack{t>0\\ |x_1|>r\\ y\in\omega}} \xi(t,x_1,y) < 0.$$

Let $w(t,x)$ be a nonnegative bounded solution of

$$\begin{cases} \partial_t w = \Delta w + c\partial_1 w + \xi(t,x)w, & t>0,\ x\in\Omega \\ \partial_\nu w(t,x) = 0, & t>0,\ x\in\partial\Omega, \end{cases} \tag{64}$$

such that $w(0,\cdot) = w_0 \in L^1(\Omega)$ and

$$\forall\, x\in\Omega, \quad \lim_{t\to\infty} w(t,x) = 0.$$

Then,

$$\lim_{t\to\infty} \|w(t,\cdot)\|_{L^1(\Omega)} = 0.$$

Proof of Lemma 7.1. By hypothesis, there exist $R,\beta>0$ such that

$$\forall\, t>0,\ x\in\Omega\backslash\Omega_R, \quad \xi(t,x)\le -\beta.$$

We set

$$P := \partial_t - \Delta - c\partial_1 + \beta, \qquad g(t,x) := (\xi(t,x)+\beta)w(t,x).$$

From the superposition principle it follows that $w = w_1 + w_2$, where $w_1,\ w_2$ satisfy

$$\begin{cases} Pw_1 = 0, & t>0,\ x\in\Omega \\ \partial_\nu w_1 = 0, & t>0,\ x\in\partial\Omega, \end{cases} \qquad \begin{cases} Pw_2 = g(t,x), & t>0,\ x\in\Omega \\ \partial_\nu w_2 = 0, & t>0,\ x\in\partial\Omega, \end{cases}$$

and $w_1(0,x) = w_0(x)$, $w_2(0,x) = 0$. The function $v_1(t,x) := w_1(t,x)e^{\beta t}$ satisfies $\partial_t v_1 = \Delta v_1 + c\partial_1 v_1$ in $\mathbb{R}^+\times\Omega$. Hence, it is easily seen that

$$\forall\, t>0, \quad \|v_1(t,\cdot)\|_{L^\infty(\Omega)} \le \|w_0\|_{L^\infty(\Omega)}, \quad \|v_1(t,\cdot)\|_{L^1(\Omega)} \le \|w_0\|_{L^1(\Omega)}$$

(a way to prove the second inequality is by applying the maximum principle to the functions $v_1^r(t,\rho) := \int_{-r}^r\int_\omega v_1(t,x_1+\rho,y)\,dy\,dx_1$, which satisfy $\partial_t v_1^r = \partial_{\rho\rho}v_1^r + c\partial_\rho v_1^r$ for $t\in\mathbb{R}^+$, $\rho\in\mathbb{R}$ and which are less than $\|w_0\|_{L^1(\Omega)}$ at time $t=0$). As a consequence,

$$\lim_{t\to\infty}\|w_1(t,\cdot)\|_{L^\infty(\Omega)} = \lim_{t\to\infty}\|w_1(t,\cdot)\|_{L^1(\Omega)} = 0.$$

Define the function

$$v(x_1) := \begin{cases} \frac{\|g\|_{L^\infty(\mathbb{R}^+\times\Omega)}}{\beta} e^{\frac{-c+\sqrt{c^2+4\beta}}{2}(x_1+R)} & \text{if } x_1 < -R \\ \frac{\|g\|_{L^\infty(\mathbb{R}^+\times\Omega)}}{\beta} & \text{if } -R\le x_1\le R \\ \frac{\|g\|_{L^\infty(\mathbb{R}^+\times\Omega)}}{\beta} e^{\frac{-c-\sqrt{c^2+4\beta}}{2}(x_1-R)} & \text{if } x_1 > R. \end{cases}$$

By computation, one sees that the constant function $\|g\|_{L^\infty(\mathbb{R}^+\times\Omega)}/\beta$ is a supersolution of the problem satisfied by w_2 and that $(t,x_1,y)\mapsto v(x_1)$ satisfies $Pv = 0 \ge g$

in $\mathbb{R}^+ \times (\Omega \backslash \overline{\Omega}_R)$. Thus, the comparison principle implies that $w_2 \leq v$ in $\mathbb{R}^+ \times \Omega$. Since $(x_1, y) \mapsto v(x_1) \in L^1(\Omega)$ and

$$\forall\, x \in \Omega, \quad \lim_{t\to\infty} w_2^+(t,x) = \lim_{t\to\infty} (w - w_1)^+(t,x) = 0,$$

the Lebesgue theorem implies that $w_2^+(t,\cdot)$ converges to 0 in $L^1(\Omega)$ as $t \to \infty$. The proof is thereby complete, because $0 \leq w = w_1 + w_2 \leq w_1 + w_2^+$. □

The L^1 convergence of u to 0 as $t \to \infty$ when $\lambda_{1,N} \geq 0$ immediately follows by applying Lemma 7.1 to the function $\tilde{u}(t, x_1, y) := u(t, x_1 + ct, y)$. When $\lambda_{1,N} < 0$, it would be natural to apply the same argument to the function $\tilde{u} - U$. This is not possible because $\tilde{u} - U$ is not nonnegative in general. For this reason, we will introduce two functions $\underline{u}$, $\overline{u}$ converging to U as $t \to \infty$ and satisfying $\underline{u} \leq \min(\tilde{u}, U)$, $\overline{u} \geq \max(\tilde{u}, U)$ and then we will apply Lemma 7.1 to $U - \underline{u}$ and $\overline{u} - U$.

Proof of Theorem 2.8. Let u be the solution of (3) with $u(0, x) = u_0(x)$. The function $\tilde{u}(t, x_1, y) := u(t, x_1 + ct, y)$ solves (45), with initial datum u_0. If $\lambda_{1,N} \geq 0$ then, owing to Theorem 2.2 part (i), we can apply Lemma 7.1 with $w = \tilde{u}$ and $\xi = f(x, \tilde{u})/\tilde{u}$ ($\leq f_s(x, 0)$ by (9) and (11)) and infer that $\tilde{u}(t, \cdot) \to 0$ in $L^1(\Omega)$ as $t \to \infty$. Assume that $\lambda_{1,N} < 0$. Let $\overline{u}$ be the solution of (45) with initial datum $\overline{u}(0, x) = \max(u_0(x), U(x))$. Applying Theorem 2.2 to the function $\overline{u}(t, x_1 - ct, y)$ we find that $\overline{u}(t, x) \to U(x)$ as $t \to \infty$, uniformly with respect to $x \in \Omega$. Moreover, the parabolic maximum principle yields

$$\forall\, t > 0,\ x \in \Omega, \quad \overline{u}(t, x) \geq \max(\tilde{u}(t, x), U(x)).$$

Hence, the function $w(t, x) := \overline{u}(t, x) - U(x)$ is a nonnegative bounded solution to (64), with

$$\xi(t, x) = \frac{f(x, \overline{u}) - f(x, U)}{\overline{u} - U},$$

and $w(0, x) \leq u_0(x) \in L^1(\Omega)$. By (9) and (11) we infer that

$$\forall\, t > 0,\ x \in \Omega, \quad \xi(t, x) \leq \frac{f(x, U)\overline{u}/U - f(x, U)}{\overline{u} - U} = \frac{f(x, U)}{U} \leq f_s(x, 0).$$

Therefore, Lemma 7.1 implies

$$\lim_{t\to\infty} \|\overline{u}(t, \cdot) - U\|_{L^1(\Omega)} = 0.$$

Let $\underline{u}$ be the solution of (45) satisfying $\underline{u}(0, x) = \min(u_0(x), U(x))$. Then,

$$\forall\, t > 0,\ x \in \Omega, \quad \underline{u}(t, x) \leq \min(\tilde{u}(t, x), U(x)).$$

Applying Lemma 7.1 with $w = U - \underline{u}$ and $\xi = (f(x, U) - f(x, \underline{u}))/(U - \underline{u})$ we get

$$\lim_{t\to\infty} \|U - \underline{u}(t, \cdot)\|_{L^1(\Omega)} = 0$$

(note that $w(0, x) \leq U(x)$, which belongs to $L^1(\Omega)$ by Proposition 3). This concludes the proof, because $\underline{u} \leq \tilde{u} \leq \overline{u}$. □

Remark 6. If the initial datum u_0 does not belong to $L^1(\Omega)$ then the convergences in Theorem 2.2 do not hold in general in the L^1 sense. As an example, the function

$$u(t, x) := \frac{1}{2e^t - 1},$$

is a solution of (3) with $f(x, s) = -s - s^2$ and initial datum $u_0 \equiv 1$. As $t \to \infty$, $u(t, \cdot)$ converges to 0 in $L^\infty(\Omega)$ but not in $L^1(\Omega)$.

Proof of Theorem 2.9. The result follows from the same ideas as before, with some minor changes that we briefly outline here. Indeed, owing to the uniform convergence of $u(t,x)$ to 0 as $t\to\infty$ given by Theorem 1.3 in [6], one can prove Theorem 2.9 by establishing an analogous result to Lemma 7.1.

In the whole space, the analogue of Lemma 7.1 is obtained by replacing Ω by $\mathbb{R}^N$ and assuming that ξ satisfies

$$\lim_{r\to\infty}\sup_{\substack{t>0\\|x|>r}}\xi(t,x)<0.$$

To prove it, one uses again the superposition principle, writing $w=w_1+w_2$, but then considers a different function v than that one introduced in the proof of Lemma 7.1:

$$v(x):=\begin{cases}\frac{\|g\|_{L^\infty(\mathbb{R}^+\times\mathbb{R}^N)}}{\beta} & \text{if } x\in B_R\\ \frac{\|g\|_{L^\infty(\mathbb{R}^+\times\mathbb{R}^N)}}{\beta}e^{\varepsilon(R-|x|)} & \text{if } x\in\mathbb{R}^N\backslash B_R,\end{cases}$$

where ε is chosen in such a way that $Pv\geq g$ in $\mathbb{R}^N\backslash\overline{B}_R$ (recall that $g(t,x)\leq 0$ for $|x|\geq R$). Hence, by comparison, $w_2\leq v$ and then the Lebesgue theorem yields $\lim_{t\to\infty}w_2^+(t,\cdot)=0$ in $L^1(\mathbb{R}^N)$. □

Let us mention that the arguments in the proofs of Theorems 2.8 and 2.9 allow one to prove that, in the lateral periodic case, the convergences of u given by Theorem 2.6 also hold in $L^1(\mathbb{R}^M\times K)$, for any $K\subset\subset\mathbb{R}^P$.

8. Seasonal dependence

We only outline the proofs of Theorems 2.10 and 2.11. Essentially, these results are obtained by using the same ideas as in Section 3 and Appendix A and following the strategy of [6] Section 3, where the two-speeds problem in the whole space is treated.

First, one shows the existence of the time periodic principal eigenvalue of $\mathcal{P}$ in the finite cylinders Ω_r, with mixed Dirichlet/Neumann boundary conditions, that is, the unique real number $\mu(r)$ such that the eigenvalue problem

$$\begin{cases}\mathcal{P}\psi=\mu(r)\psi & \text{in } \mathbb{R}\times\Omega_r\\ \partial_\nu\psi(t,x)=0 & \text{on } \mathbb{R}\times(-r,r)\times\partial\omega\\ \psi=0 & \text{on } \mathbb{R}\times\{\pm r\}\times\omega\\ \psi\quad\text{is } T\text{-periodic in } t\end{cases}$$

admits a positive solution ψ. The arguments of Appendix A, which enable one to apply the Krein-Rutman theory and find the $\mu(r)$, also work in this framework thanks to the Hölder continuity of $f_s(t,x,0)$. Then, proceeding as in the proof of Proposition 4, one shows that $\lim_{r\to\infty}\mu(r)=\mu_{1,N}$.

Next, one considers problem (6) in the coordinate system which follows the shift:

$$\begin{cases}\partial_t\tilde{u}=\Delta\tilde{u}+c\partial_1\tilde{u}+f(t,x,\tilde{u}), & t>0,\ x\in\Omega\\ \partial_\nu\tilde{u}(t,x)=0, & t>0,\ x\in\partial\Omega.\end{cases}\tag{65}$$

The following result is proved in [6] in the case $\Omega=\mathbb{R}^N$, but it also holds for general domains.

Theorem 8.1. *Assume that f satisfies (27)-(29). Let $v\in L^\infty(\mathbb{R}\times\Omega)$ be a nonnegative T-periodic in t generalized subsolution (resp. supersolution) of (65) and let $\tilde{u}$ be the solution of (65) with initial datum $\tilde{u}(0,x)=v(0,x)$. Then,*

$$\forall\, t\geq 0,\ x\in\Omega,\quad \tilde{u}(t+T,x)-\tilde{u}(t,x)\geq 0\quad (\textit{resp.} \leq 0).$$

Moreover,

$$\forall\, r > 0, \quad \lim_{t\to\infty} \|\tilde{u}(t,\cdot) - U(t,\cdot)\|_{L^\infty(\Omega_r)} = 0,$$

where U is a bounded T-periodic in t solution of (65) satisfying $U \geq v$ (resp. $U \leq v$) in $\mathbb{R}\times\Omega$.

If $\mu_{1,N} < 0$ then one can find $R > 0$ large enough such that $\mu(R) < 0$. Hence, the principal eigenfunction ψ associated with $\mu(R)$ - suitably normalized and extended by 0 in $\Omega\backslash\Omega_R$ - is a T-periodic in t subsolution to (65). Applying Theorem 8.1 with $v = \psi$ we then find a T-periodic in t solution $U \geq \psi$ to (65). Consequently, as $U > 0$ by the strong maximum principle, the sufficient condition of Theorem 2.10 for the existence of pulsating travelling waves is proved. To derive the necessary condition and the uniqueness result one proceeds as in Section 3, by establishing the exponential decay of solutions and a comparison principle analogous to Theorem 3.3 (see Proposition 9 and Theorem 3.3 in [6]).

Theorem 8.1 also allows one to prove that the convergences in Theorem 2.11 hold locally uniformly in Ω. We recall that to prove Theorem 2.2 we used the property that any solution of (45) coinciding with a subsolution or a supersolution of the stationary problem at the initial time is monotone in t. This is no longer true for (65) because the terms in the equation depend on time. However, owing to Theorem 8.1, one can derive the locally uniform convergence as in the case of (45) by considering solutions of (65) coinciding with a subsolution and a supersolution which is T-periodic in t. The uniform convergence then follows by arguing exactly as in the step 2 of the proof of Theorem 2.2.

Appendix A. Principal eigenvalue with mixed boundary conditions

Proof of Theorem 3.1. We introduce the Banach space

$$X_r = \{\phi \in C^1(\overline{\Omega}_r) \ :\ \phi = 0 \text{ on } \{\pm r\}\times\omega,\ \partial_\nu\phi = 0 \text{ on } (-r,r)\times\partial\omega\},$$

equipped with the $W^{1,\infty}(\Omega_r)$ norm. Define the operator

$$\mathcal{M}u := Lu - du,$$

with $d > \|\gamma\|_{L^\infty(\Omega_r)}$ constant such that the bilinear form $\mathcal{B} : H^1(\Omega_r)\times H^1(\Omega_r) \to \mathbb{R}$ defined by

$$\mathcal{B}(u,v) := \int_{\Omega_r} \nabla u\cdot\nabla v - (\beta(x)\cdot\nabla u)v - (\gamma(x) - d)uv$$

is coercive. From the elliptic theory of generalized solutions and the embedding theorems, it follows that for every $\phi \in X_r$ the problem

$$\text{(66)} \qquad \begin{cases} -\mathcal{M}u = \phi & \text{a. e. in } \Omega_r \\ u = 0 & \text{on } \{\pm r\}\times\omega \\ \partial_\nu u = 0 & \text{on } (-r,r)\times\partial\omega \end{cases}$$

admits a unique solution $u \in H^1(\Omega_r)\cap C^1(\overline{\Omega}_r\backslash\{\pm r\}\times\partial\omega)$. We claim that $u \in X_r$. In order to prove this, we only need to control the behaviour of u near the corners $\{\pm r\}\times\partial\omega$. We first show that $u \in C^0(\overline{\Omega}_r)$. Then, we extend u by reflection to a larger cylinder and we apply elliptic estimates up to the (smooth) boundary.

Step 1: $u \in C^0(\overline{\Omega}_r)$.
Define the function $v(x_1, y) := r^{2n} - x_1^{2n}$, where $n \in \mathbb{N}$ will be chosen later. The function v satisfies:

$$v \geq 0 \text{ in } \overline{\Omega}_r, \qquad v = 0 \text{ on } \{\pm r\}\times\omega, \qquad \partial_\nu v = 0 \text{ on } (-r,r)\times\partial\omega$$

and, for a. e. $(x_1, y) \in \Omega_r$,

$$\begin{aligned}\mathcal{M}v(x_1,y) &= x_1^{2n-2}(-2n(2n-1) - 2n\beta_1(x_1,y)x_1) + (\gamma(x_1,y) - d)v \\ &\leq 2nx_1^{2n-2}(-2n+1+\|\beta_1\|_\infty r).\end{aligned}$$

Therefore, it is possible to chose n large enough in order to have $-\mathcal{M}v \geq \|\phi\|_{L^\infty(\Omega_r)}$ in Ω_r and then the maximum principle yields $-v \leq u \leq v$. This means in particular that u can be extended by continuity to zero on $\{\pm r\} \times \partial\omega$.

Step 2: $u \in C^1(\overline{\Omega}_r)$.

Thanks to the (uniform) regularity of ω, there exists $\varepsilon > 0$ such that the domain

$$\tilde{\omega} := \{y \in \mathbb{R}^{N-1} \ : \ \mathrm{dist}(y, \omega) < \varepsilon\}$$

is smooth and every $y \in \overline{\tilde{\omega}} \backslash \omega$ has a unique projection on $\overline{\omega}$, denoted by $\pi(y)$. We set $\widetilde{\Omega}_r := (-r, r) \times \tilde{\omega}$ and call $\mathcal{R} : \overline{\widetilde{\Omega}}_r \backslash \overline{\Omega}_r \to \overline{\Omega}_r$ the reflection with respect to $[-r, r] \times \partial\omega$, that is,

$$\mathcal{R}(x_1, y) := (x_1, 2\pi(y) - y).$$

Define the function $\tilde{u} : [-r, r] \times \tilde{\omega} \to \mathbb{R}$ by

$$\tilde{u}(x) := \begin{cases} u(x) & \text{if } x \in \overline{\Omega}_r \\ u(\mathcal{R}(x)) & \text{if } x \in \overline{\widetilde{\Omega}}_r \backslash \overline{\Omega}_r \end{cases}$$

and the matrix field $\tilde{A} : (-r, r) \times \tilde{\omega} \to \mathcal{S}^N$ by

$$\tilde{A}(x) := \begin{cases} I & \text{if } x \in \overline{\Omega}_r \\ (J(x)J(x)^t)^{-1} & \text{if } x \in \widetilde{\Omega}_r \backslash \overline{\Omega}_r, \end{cases}$$

where $J(x)$ denotes the Jacobian matrix of $\mathcal{R}$ at x. By approximating the boundary of Ω with its tangent hyperplanes, one can check that

$$\forall\, x \in (-r, r) \times \partial\omega, \quad J(x)J^t(x) = I.$$

Hence, up to considering a smaller $\tilde{\omega}$ (i. e. decrease ε), we can assume that $\tilde{A}$ is uniformly Lipschitz continuous and elliptic in $\widetilde{\Omega}_r$. The function $\tilde{u}$ belongs to $H^1(\widetilde{\Omega}_r) \cap C^0(\overline{\widetilde{\Omega}}_r)$ and vanishes on $\{\pm r\} \times \tilde{\omega}$. Using the equation for u, one can check that $\tilde{u}$ is a weak solution to

$$-\mathrm{div}(\tilde{A}(x)\nabla\tilde{u}) - \tilde{\beta}(x) \cdot \nabla\tilde{u} - (\tilde{\gamma}(x) - d)\tilde{u} = \tilde{\phi} \quad \text{in } \widetilde{\Omega}_r,$$

with

$$\tilde{\beta} := \begin{cases} \beta & \text{in } \overline{\Omega}_r \\ (J^{-1})^t[\beta \circ \mathcal{R} - \mathrm{div}((J^{-1})^t)] & \text{in } \widetilde{\Omega}_r \backslash \overline{\Omega}_r, \end{cases}$$

$$\tilde{\gamma} := \begin{cases} \gamma & \text{in } \overline{\Omega}_r \\ \gamma \circ \mathcal{R} & \text{in } \widetilde{\Omega}_r \backslash \overline{\Omega}_r, \end{cases} \qquad \tilde{\phi} := \begin{cases} \phi & \text{in } \overline{\Omega}_r \\ \phi \circ \mathcal{R} & \text{in } \widetilde{\Omega}_r \backslash \overline{\Omega}_r. \end{cases}$$

Therefore, elliptic estimates up to the boundary and coercivity yield

$$\forall\, p > 1, \quad \|\tilde{u}\|_{W^{2,p}(\Omega)} \leq C\|\phi\|_{L^\infty(\Omega)}, \tag{67}$$

for some positive constant C independent of ϕ. Thus, $u \in C^1(\overline{\Omega}_r)$ by compact injection theorem.

We have shown that $u \in X_r$. The map $T : X_r \to X_r$ associating to ϕ the unique solution u of (66) is compact by (67). Using the strong maximum principle and the

Hopf lemma, one can check that it is also strictly positive, that is, $T(\mathcal{C}\backslash\{0\})$ is contained in the interior of $\mathcal{C}$, where $\mathcal{C}$ denotes the closed positive cone of nonnegative functions of X_r. It is at this stage that the $W^{1,\infty}$ norm is required for the space X_r. Then, from the Krein-Rutman theory (see [13] and [12]) it follows that T admits a unique eigenvalue λ (>0) with associated positive eigenfunction $\varphi_r \in X_r$ (unique up to a multiplicative constant). Therefore, the constant

$$\lambda(r) := \frac{1}{\lambda} - d$$

satisfies the desired property. □

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