

Interlacing theorems for the zeros of some orthogonal polynomials from different sequences

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Abstract

We study the interlacing properties of the zeros of orthogonal polynomials p_n and r_m , $m = n$ or $n - 1$ where $\{p_n\}_{n=1}^{\infty}$ and $\{r_m\}_{m=1}^{\infty}$ are different sequences of orthogonal polynomials. The results obtained extend a conjecture by Askey, that the zeros of Jacobi polynomials $p_n = P_n^{(\alpha,\beta)}$ and $r_n = P_n^{(\gamma,\beta)}$ interlace when $\alpha < \gamma \leq \alpha + 2$, showing that the conjecture is true not only for Jacobi polynomials but also holds for Meixner, Meixner-Pollaczek, Krawtchouk and Hahn polynomials with continuously shifted parameters. Numerical examples are given to illustrate cases where the zeros do not separate each other.

Keywords: Orthogonal polynomials; Zeros; Interlacing of zeros; Separation of zeros; Meixner polynomials; Krawtchouk polynomials; Meixner-Pollaczek polynomials; Hahn polynomials.

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1. Introduction

It is well known that if $\{p_n\}_{n=1}^{\infty}$ is a sequence of orthogonal polynomials, the zeros of p_n are real and simple and the zeros $x_{1,n} < x_{2,n} < \dots < x_{n,n}$ of p_n and $x_{1,n-1} < x_{2,n-1} < \dots < x_{n-1,n-1}$ of p_{n-1} separate each other as follows (cf. [1, p.61])

$$x_{1,n} < x_{1,n-1} < x_{2,n} < x_{2,n-1} < \dots < x_{n-1,n} < x_{n-1,n-1} < x_{n,n}. \quad (1)$$

An important question that arises in the study of such interlacing properties is whether and when the zeros of two polynomials p_n and r_m , $m = n$ or $n - 1$, where $\{p_n\}_{n=1}^{\infty}$ and $\{r_m\}_{m=1}^{\infty}$ are different sequences of orthogonal polynomials on the same interval, separate each other. In 1989, Askey [2, p.28] conjectured that the zeros of Jacobi polynomials $p_n = P_n^{(\alpha,\beta)}$ and $r_n = P_n^{(\alpha+2,\beta)}$ interlace. He also posed the question whether the zeros of $p_n = P_n^{(\alpha,\beta)}$ and $r_n = P_n^{(\gamma,\beta)}$ interlace when $\alpha < \gamma \leq \alpha + 2$.

In [3] it was proved that the zeros of $p_n = P_n^{(\alpha,\beta)}$ and $r_n = P_n^{(\gamma,\delta)}$ interlace when $\alpha < \gamma \leq \alpha + 2$ and $\beta - 2 \leq \delta < \beta$. Furthermore, this paper also showed that the zeros of the $p_n = P_n^{(\alpha,\beta)}$ and $r_{n-1} = P_{n-1}^{(\gamma,\delta)}$ interlace when $\alpha < \gamma \leq \alpha + 2$ and $\beta < \delta \leq \beta + 2$. Results for the interlacing of the zeros of Gegenbauer polynomials, a special case of the Jacobi polynomials with $\alpha = \beta$, follow immediately but were proved separately in [4]. The zeros of Laguerre polynomials $p_n = L_n^{\alpha}$ and $r_m = L_m^{\gamma}$, where $m = n$ or $n - 1$ and $\alpha < \gamma \leq \alpha + 2$, also interlace (cf. [4]).

In this paper we aim to do a comprehensive study of the interlacing properties of the zeros of other classes of classical orthogonal polynomials, including discrete orthogonal polynomials such as Meixner, Krawtchouk and Hahn polynomials, where the parameters are shifted continuously.

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The classical orthogonal polynomials of a discrete variable are used in various problems of theoretical and mathematical physics, group representation theory, for example the relationship between generalised spherical harmonics for $SU(2)$ and Krawtchouk polynomials described in [5], as well as in representations of the three dimensional rotation group (cf. [6]). An application of the Meixner-Pollaczek polynomials to quantum mechanics is discussed in [7].

Interlacing properties can easily be derived from the following simple result that has been proved with slight variations in different contexts, for example when considering quasi-orthogonality in [8, Theorem 3] or in dealing with polynomials associated with sequences of power moment functions [9, p. 117].

Lemma 1.1. *Let p_n and p_{n-1} be polynomials with n real zeros $x_{1,n} < x_{2,n} < \dots < x_{n,n}$ and $n - 1$ real zeros $x_{1,n-1} < x_{2,n-1} < \dots < x_{n-1,n-1}$ respectively satisfying the interlacing property (1) on the (finite or infinite) interval (c, d) . Assume that a polynomial f of degree n or $n - 1$ satisfies the equation*

$$f(x) = a(x)p_n(x) + b(x)p_{n-1}(x). \quad (2)$$

If both a and b are continuous and have constant signs on (c, d) , then all the zeros of f are real and simple, the zeros of f and p_n interlace and the zeros of f and p_{n-1} interlace.

In other words we have “triple interlacing” and there are several possibilities for the arrangement of the zeros. Let

$$t_1 < t_2 < \dots < t_n \text{ be the zeros of } f.$$

Then, if f is of degree n , either

$$x_{i,n} < t_i < x_{i,n-1} \text{ for all } i = 1, \dots, n - 1 \text{ and } x_{n,n} < t_n \quad (3)$$

or

$$x_{i-1,n-1} < t_i < x_{i,n} \text{ for } i = 2, \dots, n \text{ and } t_1 < x_{1,n}. \quad (4)$$

It is possible to obtain necessary and sufficient conditions to distinguish between cases (3) and (4). The signs of $a(x)$ and $b(x)$ will play a role in the distinction. For example, when a and b are constants (depending on n), one may apply a generalisation of [8, Theorem 3] or [10, Theorem 5].

Corollary 1.2. *Let p_n and p_{n-1} be polynomials with n real zeros $x_{1,n} < x_{2,n} < \dots < x_{n,n}$ and $n - 1$ real zeros $x_{1,n-1} < x_{2,n-1} < \dots < x_{n-1,n-1}$ respectively satisfying the interlacing property (1) on the (finite or infinite) interval (c, d) and let f be a polynomial of degree n or $n - 1$ satisfying (2). If b is continuous and has constant sign on (c, d) , then all the zeros of f are real and simple and the zeros of f and p_n interlace.*

When f has degree n , there are two possibilities for the interlacing pattern. Let

$$t_1 < t_2 < \dots < t_n \text{ be the zeros of } f$$

as before. Then either

$$x_{1,n} < t_1 < x_{2,n} < t_2 < \dots < t_{n-1} < x_{n,n} < t_n \quad (5)$$

or

$$t_1 < x_{1,n} < t_2 < x_{2,n} < \dots < t_{n-1} < x_{n-1,n} < t_n < x_{n,n} \quad (6)$$

and again it is possible to obtain necessary and sufficient conditions on $a(x)$ and $b(x)$ to distinguish between the two cases.

Corollary 1.3. *Let p_n and p_{n-1} be polynomials with n real zeros $x_{1,n} < x_{2,n} < \dots < x_{n,n}$ and $n - 1$ real zeros $x_{1,n-1} < x_{2,n-1} < \dots < x_{n-1,n-1}$ respectively satisfying the interlacing property (1) on the (finite or infinite) interval (c, d) and let f be a polynomial of degree $n - 1$ satisfying (2). If a is continuous and has constant sign on (c, d) , then all the zeros of f are real and simple and the zeros of f and p_{n-1} interlace.*

It is possible to give a weaker hypothesis on a and b in Lemma 1.1, Corollary 1.2 and 1.3. The results follow if a has constant sign at each of the zeros of p_{n-1} and/or b has constant sign at each of the zeros of p_n . However, Corollary 1.3 is not true if f is a polynomial of degree n instead of $n - 1$, since from the assumptions we can only conclude that f has at least one zero between every two consecutive zeros of p_{n-1} . If f is of degree n , then there are 2 zeros left over, which may not interlace with the zeros of p_{n-1} (for example they may be both larger than $x_{n-1,n-1}$, they may be complex, or f may have 3 zeros between two consecutive zeros of p_{n-1}). Indeed, in this case it is easy to construct a counterexample for the interlacing. Set $n = 3$, $p_3(x) = x^3 - 9x$, $p_2(x) = x^2 - 4$. Then p_2 and p_3 have interlacing zeros. Moreover let $f(x) = (x - 1)(x - 4)(x - 5)$, $a(x) = 1$, and $b(x) = \frac{-10x^2 + 38x - 20}{x^2 - 4}$. Then (2) holds true and a is continuous with constant sign, but the zeros of f and p_{n-1} do not interlace.

Note that there is a close link between the expression (2) used to define f and the notion of quasi-orthogonal polynomials. If $\{p_n\}_{n=1}^{\infty}$ is a sequence of orthogonal polynomials and a and b are constants, then f is a quasi-orthogonal polynomial (with respect to the same weight) of order 1. Also, if $a(x)$ and $b(x)$ are polynomials then it follows from an idea by Shohat [11] that f is quasi-orthogonal (of some order) (cf. [8, Theorem 1]). On the other hand every quasi-orthogonal polynomial can be written in the form (2), where $a(x)$ and $b(x)$ are polynomials, using the three term recurrence relation.

In this respect it is worth noting the following result, which can be proved in the same way as Lemma 1.1.

Lemma 1.4. *Let p_n , p_{n-1} and p_{n-2} be polynomials with n , $n - 1$ and $n - 2$ real zeros respectively satisfying the interlacing property on the (finite or infinite) interval (d, e) . Assume that the leading coefficients of p_n and p_{n-2} have the same sign. Furthermore assume that a polynomial f of degree $n - 1$ or $n - 2$ satisfies the equation*

$$f(x) = a(x)p_n(x) + b(x)p_{n-1}(x) + c(x)p_{n-2}(x). \quad (7)$$

If both a and c are continuous, have constant signs on (d, e) , and $ac < 0$, then all the zeros of f are real and simple, and the zeros of f and p_{n-1} interlace.

In the following we will apply the principles in Lemma 1.1 and Corollary 1.2 to the Meixner, Meixner-Pollaczek, Krawtchouk and Hahn polynomials to obtain the interlacing property of the zeros of these polynomials with the zeros of polynomials with shifted parameter values. Our method of proof uses the generating functions of Meixner polynomials as well as the contiguous function relations of hypergeometric polynomials to obtain mixed recurrence relations that involve polynomials with different parameters. For the discrete orthogonal polynomials we make extensive use of an interesting generalisation of Markov's theorem for the monotonicity of the zeros (cf. [12]). Numerical examples will be given in cases where the zeros do not separate each other.

2. Interlacing of the zeros of Meixner polynomials with different parameters

The Meixner polynomials are defined by (cf. [12])

$$M_n(x; \beta, c) = {}_2F_1 \left(\begin{matrix} -n, & -x \\ & \beta \end{matrix} \middle| 1 - \frac{1}{c} \right). \quad (8)$$

For $\beta > 0$ and $c \in (0, 1)$ they satisfy the discrete orthogonality relation

$$\sum_{x=0}^{\infty} M_n(x; \beta, c) M_m(x; \beta, c) \frac{(\beta)_x}{x!} c^x = \frac{n!(1-c)^{-\beta}}{c^n (\beta)_n} \delta_{m,n} \quad (9)$$

where $(\beta)_x = (\beta)(\beta+1) \dots (\beta+x-1)$ is Pochhammer's symbol. The Meixner polynomials have the generating function

$$\sum_{n=0}^{\infty} \frac{(\beta)_n}{n!} M_n(x; \beta, c) t^n = \left(1 - \frac{t}{c}\right)^x (1-t)^{-x-\beta}$$

that gives rise to the identity

$$(1 - 2t + t^2) \sum_{n=0}^{\infty} \frac{(\beta + 2)_n}{n!} M_n(x; \beta + 2, c) t^n = \sum_{n=0}^{\infty} \frac{(\beta)_n}{n!} M_n(x; \beta, c) t^n.$$

Equating like powers of t we obtain

$$\begin{aligned} \frac{(\beta)_n}{n!} M_n(x; \beta, c) &= \frac{(\beta + 2)_n}{n!} M_n(x; \beta + 2, c) \\ &- 2 \frac{(\beta + 2)_{n-1}}{(n-1)!} M_{n-1}(x; \beta + 2, c) + \frac{(\beta + 2)_{n-2}}{(n-2)!} M_{n-2}(x; \beta + 2, c) \end{aligned}$$

and from this

$$\begin{aligned} \beta(\beta + 1)M_n(x; \beta, c) &= (\beta + n)(\beta + n + 1)M_n(x; \beta + 2, c) \\ &- 2n(\beta + n)M_{n-1}(x; \beta + 2, c) + n(n-1)M_{n-2}(x; \beta + 2, c). \end{aligned} \tag{10}$$

Shifting β to $\beta + 2$, n to $n - 1$ in the recurrence relation for Meixner polynomials (cf. [13])

$$[(c-1)x + (n + (n + \beta)c)]M_n(x; \beta, c) = c(n + \beta)M_{n+1}(x, \beta, c) + nM_{n-1}(x; \beta, c)$$

and substituting into (10), we have

$$\begin{aligned} &\beta(\beta + 1)M_n(x; \beta, c) \\ &= (\beta + n - nc)(\beta + n + 1)M_n(x; \beta + 2, c) + \\ &\quad n(\beta(c-2) + (x + n + 1)(c-1))M_{n-1}(x; \beta + 2, c). \end{aligned}$$

The coefficient of $M_n(x; \beta + 2, c)$ clearly has constant sign on $(0, \infty)$. The coefficient of $M_{n-1}(x; \beta + 2, c)$ does not change sign on $(0, \infty)$ since the zeros of the function $n(\beta(c-2) + (x + n + 1)(c-1))$ occur when

$$x = -\left(n + 1 + \frac{\beta(c-2)}{c-1}\right) < 0 \text{ for all } \beta > 0 \text{ and } 0 < c < 1.$$

We can now apply Lemma 1.1. Since the zeros increase with β (cf. [12, Theorem 7.1.2]), we can also easily decide between cases (3) and (4).

Theorem 2.1. *Let $\beta > 0$, $0 < c < 1$ and let*

$$\begin{aligned} 0 < x_1 < x_2 < \dots < x_n &\text{ be the zeros of } M_n(x; \beta + 2, c) \\ 0 < y_1 < y_2 < \dots < y_{n-1} &\text{ be the zeros of } M_{n-1}(x; \beta + 2, c) \text{ and} \\ 0 < t_1 < t_2 < \dots < t_n &\text{ be the zeros of } M_n(x; \beta, c). \end{aligned}$$

Then (4) holds.

Remark: Note that it is also possible to derive (10) using (8) and the contiguous function relations for hypergeometric polynomials. Various algorithms have been developed to compute these types of relations. The most efficient algorithm for large shifts of the parameters is available as a Maple program (cf. [14]).

Theorem 2.1 is stated in terms of an integer shift in the parameter. In fact the result holds more generally for a continuous shift in the parameter.

Corollary 2.2. *Let $\beta > 0$, $0 < c < 1$ and let*

$$\begin{aligned} 0 < p_1 < p_2 < \dots < p_n &\text{ be the zeros of } M_n(x; \beta + t, c) \\ 0 < q_1 < q_2 < \dots < q_{n-1} &\text{ be the zeros of } M_{n-1}(x; \beta + t, c) \text{ and} \\ 0 < t_1 < t_2 < \dots < t_n &\text{ be the zeros of } M_n(x; \beta, c) \end{aligned}$$

where $0 < t \leq 2$. Then

$$t_i < p_i < q_i < t_{i+1} < p_{i+1} \text{ for } i = 1, 2, \dots, n-1.$$

Proof. Since the zeros of a Meixner polynomial of fixed degree increase as the parameter increases (cf. [12, Theorem 7.1.2]), we know that $t_i < p_i$ and $q_i < y_i$, where y_i is as in Theorem 2.1. Moreover, since the zeros of two consecutive terms in a sequence of orthogonal polynomials separate each other, $p_i < q_i$. By Theorem 2.1 $y_i < t_{i+1}$ and the result follows. ■

Remark: The triple interlacing is not retained in general for shifts of $t > 2$ or $t < 0$. For example, when $n = 5$, $\beta = 0.24$ and $c = 0.9987$, the zeros of $M_n(x; \beta, c)$ are

$$40.0935, 692.827, 2167.01, 4659.17 \text{ and } 8737.39,$$

those of $M_n(x; \beta + 2.5, c)$ are

$$704.789, 2033.33, 4117.61, 7187.41 \text{ and } 11856.2$$

those of $M_{n-1}(x, \beta + 2.2, c)$ are

$$728.73, 2258.8, 4802.57 \text{ and } 8932.59$$

while those of $M_{n-1}(x, \beta - 0.1, c)$ are

$$28.1867, 805.979, 2675.79 \text{ and } 6145.02.$$

Note however that the zeros of $M_n(x, \beta, c)$ and $M_n(x, \beta + t, c)$ will interlace for $-2 < t < 2$ by symmetry.

Clearly, when $c \in (0, 1)$ and $\beta > 0$ for the Meixner polynomials, it is not possible to study the interlacing of the zeros when the parameter c is shifted.

The Meixner polynomials also are orthogonal with respect to a positive measure for $x < -\beta$ when $c > 1$ and $\beta > 0$ (cf. [15, p.346]). The orthogonality relation may be obtained from (9) by an application of Pfaff's transformation (cf. [15, Thm. 2.2.5]). Hence when $c > 1$ and $\beta > 0$ the zeros are real, simple and lie in the interval $(-\infty, \beta)$. Numerical evidence suggests that, when $c > 1$, the zeros of ${}_2F_1(-n, -x; \beta; 1 - \frac{1}{c})$ and ${}_2F_1(-n, -x; \beta; 1 - \frac{1}{c+t})$, $t \in (0, 2)$ interlace on $(-\infty, 0)$. However, even in the simplest case, when $t = 1$, no suitable relation between ${}_2F_1(-n, -x; \beta; d)$, ${}_2F_1(-n + 1, -x; \beta; d)$ and ${}_2F_1(-n, -x; \beta; \frac{1}{2-d})$ or ${}_2F_1(-n + 1, -x; \beta; 2 - \frac{1}{d})$ could be found to prove this.

3. Interlacing of the zeros of different Meixner-Pollaczek polynomials

The Meixner-Pollaczek polynomials are defined by (cf. [12])

$$P_n^\lambda(x; \phi) = \frac{(2\lambda)_n}{n!} e^{in\phi} {}_2F_1 \left(\begin{matrix} -n, & \lambda + ix \\ & 2\lambda \end{matrix} \middle| 1 - e^{-2i\phi} \right).$$

They are orthogonal on the interval $(-\infty, \infty)$ for $\lambda > 0$ and $0 < \phi < \pi$ with respect to the weight function $e^{(2\phi-\pi)x} |\Gamma(\lambda + ix)|^2$.

Using the contiguous function relation for hypergeometric polynomials

$$\begin{aligned} & \frac{b(c-a)(b-c)z^2}{c(c+1)} {}_2F_1 \left(\begin{matrix} a+1, & b+1 \\ & c+2 \end{matrix} \middle| z \right) \\ &= (cz - c - bz) {}_2F_1 \left(\begin{matrix} a, & b \\ & c \end{matrix} \middle| z \right) - c(z-1) {}_2F_1 \left(\begin{matrix} a+1, & b \\ & c \end{matrix} \middle| z \right), \end{aligned}$$

obtained using the Maple program available on [5], letting $a = -n$, $b = \lambda + ix$, $c = 2\lambda$, $z = 1 - e^{-2i\phi}$ and then taking the real part of the equation, we obtain

$$\begin{aligned} & P_{n-1}^{\lambda+1}(x; \phi) \\ &= \frac{nx \sin 2\phi - \lambda - \lambda \cos 2\phi}{(\lambda^2 + x^2)(\cos \phi - \cos 3\phi)} P_n^\lambda(x; \phi) + \frac{2\lambda(2\lambda + n - 1) \cos \phi}{(\lambda^2 + x^2)(\cos \phi - \cos 3\phi)} P_{n-1}^\lambda(x; \phi). \end{aligned}$$

We can now apply Corollary 1.2.

Theorem 3.1. *Let $\lambda > 0$, $\phi \in (0, \pi)$. Then the zeros of $P_n^\lambda(x; \phi)$ and $P_{n-1}^{\lambda+1}(x; \phi)$ interlace.*

We remark that although numerical evidence suggests that the zeros of $P_n^\lambda(x; \phi)$ and $P_{n-1}^{\lambda+t}(x; \phi)$, $t \in (0, 1)$ interlace, it is not possible to prove this using Markov's monotonicity theorem since the monotonicity of the zeros changes at the origin.

To show that the zeros of $P_n^\lambda(x; \phi)$ do not interlace in general with those of $P_{n-1}^{\lambda+1}(x; \phi)$, consider the case where $n = 5$, $\lambda = 0.13$ and $\phi = \frac{\pi}{2} + 0.2$. Then the zeros of $P_n^\lambda(x; \phi)$ are given by

$$-1.69632, -0.34109, 0.114137, 1.1007, 2.98143$$

and those of $P_n^{\lambda+1}(x; \phi)$ are

$$-2.29027, -0.70018, 0.43275, 1.80228, 3.92784.$$

We also note that since the zeros of $P_5^{1.13}(x; 0.1)$ are

$$-73.2588, -43.574, -24.2651, -11.3913, -3.4887,$$

those of $P_4^{2.33}(x; 0.1)$ are

$$-74.2285, -43.9502, -24.0442, -10.466$$

and those of $P_4^{0.13}(x; 0.1)$ are

$$-40.8994, -18.084, -5.66064, -0.338508$$

it is clear that the zeros of $P_n^\lambda(x; \phi)$ and $P_{n-1}^{\lambda+t}(x; \phi)$ do not interlace in general when $t > 1$ or $t < 0$.

4. Interlacing of the zeros of Krawtchouk polynomials from different sequences

Krawtchouk polynomials are defined by (cf. [12])

$$K_n(x; p, N) = {}_2F_1 \left(\begin{matrix} -n, & -x \\ & -N \end{matrix} \middle| \frac{1}{p} \right), \quad n = 0, 1, \dots, N$$

and are orthogonal when $p \in (0, 1)$ with the discrete orthogonality relation given by

$$\sum_{x=0}^N \binom{N}{x} p^x (1-p)^{N-x} K_m(x; p, N) K_n(x; p, N) = \frac{(-1)^n n!}{(-N)_n} \left(\frac{1-p}{p} \right)^n \delta_{nm}$$

when $m < n$. Using the contiguous relations satisfied by the hypergeometric polynomials (cf. [16], p 71) it can be shown that

$$\begin{aligned} {}_2F_1 \left(\begin{matrix} a, & b \\ & c \end{matrix} \middle| z \right) &= \frac{(z - cz + bz + a)(c - 1)}{z(a - c + 1)(c - b - 1)} {}_2F_1 \left(\begin{matrix} a, & b \\ & c - 1 \end{matrix} \middle| z \right) \\ &+ \frac{a(c - 1)(z - 1)}{z(a - c + 1)(c - b - 1)} {}_2F_1 \left(\begin{matrix} a + 1, & b \\ & c - 1 \end{matrix} \middle| z \right). \end{aligned}$$

Letting $z = \frac{1}{p}$, $a = -n$, $b = -x$ and $c = -N$, we obtain

$$\begin{aligned} K_n(x; p, N) &= \frac{(N - 1)(1 - p)n}{(N - n + 1)(x - N - 1)} K_{n-1}(x; p, N + 1) \\ &- \frac{(N + 1 - x - np)(N + 1)}{(N - n + 1)(x - N - 1)} K_n(x; p, N + 1). \end{aligned}$$

Theorem 4.1. *Let $p \in (0, 1)$ and $n = 0, 1, \dots, N$. If*

$$\begin{aligned} 0 < x_1 < \dots < x_n < N & \text{ are the zeros of } K_n(x; p, N), \\ 0 < y_1 < \dots < y_{n-1} < N & \text{ are the zeros of } K_{n-1}(x; p, N + 1) \text{ and} \\ 0 < t_1 < t_2 < \dots < t_n < N & \text{ are the zeros of } K_n(x; p, N + 1) \end{aligned}$$

then (5) holds. Furthermore, when $n \leq \frac{1}{p}$, then (3) holds.

Proof. Since the coefficient of $K_{n-1}(x; p, N+1)$ is continuous and does not change sign for $x \in (0, N)$, we can apply Corollary 1.2. To see which polynomial has the smallest zero we note that if $w_0 = \binom{N}{x} p^x (1-p)^{N-x}$ denotes the weight function of $K_n(x; p, N)$ and $w_1 = \binom{N+1}{x} p^x (1-p)^{N-x+1}$ is the weight function of $K_n(x; p, N+1)$ then the ratio

$$\begin{aligned} \frac{w_1}{w_0} &= \frac{(N+1)(1-p)\Gamma(N-x)}{\Gamma(N-x+1)} \\ &= \frac{(N+1)(1-p)}{(N-x)} \end{aligned}$$

is an increasing function of x . It then follows from the monotonicity result of Markov (cf. [1], p. 116, Theorem 6.12.2) that $x_k < t_k$ for $k = 1, 2, \dots, n$.

To prove the triple interlacing (3), we note that the coefficient of $K_n(x; p, N+1)$ does not change sign on $(0, N)$ when $n \leq \frac{1}{p}$ and the result follows from Lemma 1.1. ■

Next we show that the zeros of $K_n(x; p, N)$ and those of $K_{n-1}(x; p, N-1)$ interlace. For this we make use of the relation

$$\begin{aligned} &{}_2F_1 \left(\begin{matrix} a, & b \\ & c \end{matrix} \middle| z \right) \\ &= \frac{a}{c} {}_2F_1 \left(\begin{matrix} a+1, & b \\ & c+1 \end{matrix} \middle| z \right) - \frac{a-c+2}{c} {}_2F_1 \left(\begin{matrix} a, & b \\ & c+1 \end{matrix} \middle| z \right) \end{aligned}$$

obtained from replacing c by $c+1$ in the contiguous relation (2) given in [15, p. 71]. Letting $z = \frac{1}{p}$, $a = -n$, $b = -x$ and $c = -N$, we obtain

$$K_n(x; p, N) = \frac{n}{N} K_{n-1}(x; p, N-1) + \frac{N-n+2}{N} K_n(x; p, N-1).$$

Since the coefficients are constant we can apply Lemma 1.1 and, together with the monotonicity of the zeros of $K_n(x; p, N)$ with increasing N as discussed above, we obtain the following.

Theorem 4.2. *Let $p \in (0, 1)$ and $n = 0, 1, \dots, N$. If*

$$\begin{aligned} 0 < x_1 < \dots < x_n < N & \text{ are the zeros of } K_n(x; p, N), \\ 0 < y_1 < \dots < y_{n-1} < N & \text{ are the zeros of } K_{n-1}(x; p, N-1) \text{ and} \\ 0 < t_1 < t_2 < \dots < t_n < N & \text{ are the zeros of } K_n(x; p, N-1) \end{aligned}$$

then (4) holds.

Note that the zeros of $K_n(x; p; N)$ and $K_n(x; p, N+t)$ do not interlace in general when t is an integer greater than 1 as can be seen from the following example. Let $n = 5$, $p = 0.9$ and $N = 7$, the zeros of $K_n(x; p; N)$ are

$$2.00941, 3.61178, 4.89421, 5.9853, 6.9993 \tag{11}$$

and those of $K_n(x; p, N+2)$ are

$$3.47107, 5.3137, 6.76437, 7.95403, 8.99683.$$

The zeros of $K_n(x; p; N)$ do not generally interlace with the zeros of $K_{n-1}(x; p, N+1)$ or those of $K_{n-1}(x; p, N-2)$ since, for the same parameter values as above, the zeros of $K_4(x, 0.9, 8)$ are

$$3.66902, 5.47901, 6.86554, 7.98644,$$

those of $K_4(x, 0.9, 5)$ are

$$1.40309, 2.826, 3.97241, 4.99851$$

and clearly neither of these sets of zeros interlace with the zeros of $K_5(x; 0.9, 7)$ given in (11).

5. Interlacing of the zeros of different Hahn polynomials

The Hahn polynomials may be defined by (cf. [12])

$$Q_n(x; \alpha, \beta, N) = {}_3F_2 \left(\begin{matrix} -n, -x, n + \alpha + \beta + 1 \\ 1 + \alpha, -N \end{matrix} \middle| 1 \right).$$

They constitute a finite analogue of the Jacobi polynomials when $\alpha, \beta > -1$. In this case they have the discrete orthogonality property

$$\sum_{x=0}^N w(x) Q_m(x; \alpha, \beta, N) Q_n(x; \alpha, \beta, N) = 0, \text{ for } m < n.$$

The weight function $w(x) = \binom{\alpha+x}{x} \binom{\beta+N-x}{N-x}$ is positive for $x = 0, 1, \dots, n-1$, so it follows that when $\alpha, \beta > -1$ the zeros are simple and lie in $(0, N)$.

From the contiguous relation for generalised hypergeometric polynomials (cf. [16, eqn.14, p.82]), letting $\alpha_1 = -n$ and $\alpha_k = n + \alpha + \beta + 1$, we obtain

$$\begin{aligned} & {}_3F_2 \left(\begin{matrix} -n, -x, n + \alpha + \beta + 1 \\ 1 + \alpha, -N \end{matrix} \middle| 1 \right) \\ &= \frac{n}{2n + \alpha + \beta + 1} {}_3F_2 \left(\begin{matrix} -n + 1, -x, n + \alpha + \beta + 1 \\ 1 + \alpha, -N \end{matrix} \middle| 1 \right) \\ & \quad + \frac{n + \alpha + \beta + 1}{2n + \alpha + \beta + 1} {}_3F_2 \left(\begin{matrix} -n, -x, n + \alpha + \beta + 1 \\ 1 + \alpha, -N \end{matrix} \middle| 1 \right). \end{aligned}$$

This yields the relation

$$\begin{aligned} & Q_n(x; \alpha, \beta, N) \\ &= \frac{n}{2n + \alpha + \beta + 1} Q_{n-1}(x; \alpha, \beta + 1, N) + \frac{n + \alpha + \beta + 1}{2n + \alpha + \beta + 1} Q_n(x; \alpha, \beta + 1, N) \end{aligned}$$

and, since the coefficients are constant, we obtain triple interlacing from Lemma 1.1. The zeros of $Q_n(x; \alpha, \beta, N)$ increase with β (cf. [12, Theorem 7.1.2]) and this allows us to decide between cases (3) and (4). In addition, we obtain the interlacing of the zeros for the continuous shift of the parameter β from the monotonicity of the zeros as in Corollary 2.2.

Theorem 5.1. *Let $\beta, \alpha > -1$ and let*

$$\begin{aligned} & 0 < p_1 < p_2 < \dots < p_n \quad \text{be the zeros of } Q_n(x; \alpha, \beta + t, N) \\ & 0 < q_1 < q_2 < \dots < q_{n-1} \quad \text{be the zeros of } Q_{n-1}(x; \alpha, \beta + t, N) \text{ and} \\ & 0 < t_1 < t_2 < \dots < t_n \quad \text{be the zeros of } Q_n(x; \alpha, \beta, N) \end{aligned}$$

where $0 < t \leq 1$. Then

$$t_i < p_i < q_i < t_{i+1} < p_{i+1} \text{ for } i = 1, 2, \dots, n-1.$$

Letting $\alpha_1 = -n$ and $\beta_k = \alpha + 1$ in [16, eqn.15, p.82] we obtain

$$\begin{aligned} & {}_3F_2 \left(\begin{matrix} -n, -x, n + \alpha + \beta + 1 \\ 1 + \alpha, -N \end{matrix} \middle| 1 \right) \\ &= \frac{n}{n + \alpha} {}_3F_2 \left(\begin{matrix} -n + 1, -x, n + \alpha + \beta + 1 \\ 1 + \alpha, -N \end{matrix} \middle| 1 \right) \\ & \quad + \frac{\alpha}{n + \alpha} {}_3F_2 \left(\begin{matrix} -n, -x, n + \alpha + \beta + 1 \\ \alpha, -N \end{matrix} \middle| 1 \right) \end{aligned}$$

or

$$\begin{aligned} Q_n(x; \alpha, \beta, N) & \\ &= \frac{n}{n + \alpha} Q_{n-1}(x; \alpha, \beta + 1, N) + \frac{\alpha}{n + \alpha} Q_n(x; \alpha - 1, \beta + 1, N). \end{aligned} \tag{12}$$

Theorem 5.2. *The zeros of $Q_n(x; \alpha, \beta, N)$ and $Q_n(x; \alpha - t, \beta + s, N)$ interlace when $0 < t, s \leq 1$.*

Proof. Since $Q_n(x; \alpha, \beta, N)$ and $Q_{n-1}(x; \alpha, \beta + 1, N)$ have interlacing zeros, evaluating (12) at consecutive zeros of $Q_n(x; \alpha, \beta, N)$ shows that the zeros of $Q_n(x; \alpha, \beta, N)$ and $Q_n(x; \alpha - 1, \beta + 1, N)$ interlace. The zeros of $Q_n(x; \alpha, \beta, N)$ increase when α decreases and β increases (cf. [12, Theorem 7.1.2]) and this yields the result. ■

Remark: We note that similar results can be proven for the zeros of the Dual Hahn, Continuous Hahn and Continuous Dual Hahn polynomials using the contiguous relations for generalised hypergeometric polynomials (cf [16, eq. 14, 15, p. 82]) as above.

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