On the convergence of rational Ritz values
Applications to rational interpolation of rational functions

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Joint work with
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Our problem: orthogonal polynomials (OP)

With $\Lambda = \{\lambda_1 < \lambda_2 < ... < \lambda_N\} \subset \mathbb{R}$, consider

$$\langle f, g\rangle_N = \sum_{\lambda_j \in \Lambda} w(\lambda_j)^2 f(\lambda_j)\overline{g(\lambda_j)}, \quad \text{weights } w(\lambda_j) > 0,\ \langle 1, 1\rangle_N = 1,$$

together with it's nth OP p_n with roots $\Theta = \{\theta_1 < ... < \theta_n\}$.

$$\deg p_n = n, \quad \forall j = 0, ..., n-1 : \langle p_n, x^j\rangle_N = 0.$$

Question

Does the set Θ approach Λ?
For which $\lambda \in \Lambda$ do we get small $\mathrm{dist}(\lambda, \Theta)$?

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Asymptotic answer: Kuijlaars '99, BB '00

For $n = n_N$, $N \to \infty$, $n/N \to t$, rate
$\limsup \mathrm{dist}(\lambda_{k_N,N}, \Theta_N)^{1/N} \leq \exp(U^\mu(\lambda) - F)$ if $\Lambda_N \ni \lambda_{k_N,N} \to \lambda$
and $\Lambda = \Lambda_N, \Theta = \Theta_N, w = w_N$ "nice".
Work on discrete OP: Rakhmanov '96, Dragnev & Saff '97

Our new problem: orthogonal rational functions (ORF)

Consider nth ORF with poles $\Xi = \{\xi_1, ..., \xi_n\}$ and roots $\Theta = \{\theta_1 < ... < \theta_n\}$.

$$\deg p_n = n, \quad \forall j = 0, ..., n-1 : \langle \frac{p_n}{q_n}, \frac{x^j}{q_n} \rangle_N = 0, \quad q_n(x) = \prod_{\xi_j \in \Xi} (1 - \frac{x}{\xi_j}).$$

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Similar asymptotic answer if poles far from support Λ

For $n = n_N, N \to \infty, n/N \to t$, rate
$\limsup \mathrm{dist}(\lambda_{k_N,N}, \Theta_N)^{1/N} \leq \exp(U^{\mu-\nu}(\lambda) - F)$ if
$\Lambda_N \ni \lambda_{k_N,N} \to \lambda$ and $\Lambda = \Lambda_N, \theta = \Theta_N, w = w_N, \Xi = \Xi_N$ "nice".

Answer with logarithmic potential theory

OP: Constrained energy problem with external field $Q = 0$

ORF: Constrained energy problem with external field $Q = -U^{\nu}$

(constrain σ asymptotics of supports, ν asymptotics of poles, more details later)

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... and what happens if poles are $\in \mathrm{conv}(\Lambda_N) \setminus \Lambda_N$?

Approach (poles of) RF by lower order RF

Consider the Markov function

$$\phi(z) = \sum_{\lambda \in \Lambda} \frac{w(\lambda)^2}{z-\lambda} = \langle \frac{1}{z - \cdot}, 1 \rangle_N .$$

OP p_n = denominator of nth Padé approximant at ∞.
ORF p_n/q_n: p_n denominator of $[n-1|n]$th rational interpolant r of ϕ at $\xi_1, \xi_1, \xi_2, \xi_2, ..., \xi_n, \xi_n \in \mathbb{R}$ (interpolation of value and first derivative).

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Error

$$\phi(z) - r(z) = \frac{q_n(z)^2}{p_n(z)^2} \sum_{\lambda\in\Lambda_N} \frac{w_N(\lambda)^2}{z-\lambda} \frac{p_n(\lambda)^2}{q_n(\lambda)^2}$$

Orthogonality: for $j = 0, 1, ..., n-1$

$$\sum_{\lambda\in\Lambda_N} w_N(\lambda)^2 \frac{p_n(\lambda)\lambda^j}{q_n(\lambda)^2} = \langle \frac{p_n}{q_n}, \frac{x^j}{q_n} \rangle_N = 0.$$

Approach eigenvalues by rational Ritz values

Given $A \in \mathbb{R}^{N \times N}$ symmetric, $b \in \mathbb{R}^N$, the rational Arnoldi method yields $V \in \mathbb{R}^{N \times n}$ with columns ONB of rational Krylov space

$$\text{span}\{q_n(A)^{-1}b, Aq_n(A)^{-1}b, \ldots, A^{n-1}q_n(A)^{-1}b\}.$$

Question

Does the set Θ of Ritz values := eigenvalues of $V_n^T AV_n$ approach spectrum of A?
Which eigenvalues are found by nth rational Ritz values?

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Link: Ruhe 84-94, Meerbergen 01, Decker & Bultheel '08, ...

Θ is just the set of roots of ORF p_n/q_n for the scalar product

$$\langle f, g \rangle_N = (g(A)b)^T (f(A)b) = \sum_{\lambda \in \Lambda} w(\lambda)^2 f(\lambda)\overline{g(\lambda)},$$

$\Lambda =$ spectrum of A, $w(\lambda)$ eigencomponents of b.

$\mathrm{dist}(\lambda, \Theta)$ and a polynomial extremal problem

Set as before

$$\begin{aligned} \Lambda &= \{\lambda_1 < \ldots < \lambda_N\} && \text{support of orthogonality, weights } w(\lambda)^2 > 0, \\ \Theta &= \{\theta_1 < \ldots < \theta_n\} && \text{zeros of } n\text{th ORF } p_n/q_n, \\ \Xi &= \{\xi_1, \ldots, \xi_n\} \subset \mathbb{R} && \text{poles of } n\text{th ORF } p_n/q_n. \end{aligned}$$

LEMMA (BB'00)

We have

$$\theta_1 - \lambda_1 = \min \left\{ \frac{\sum_{j=1, j\neq 1}^{N} \frac{w(\lambda_j)^2(\lambda_j - \theta_1) s(\lambda_j)^2}{q_n(\lambda_j)^2}}{\frac{w(\lambda_k)^2}{q_n(\lambda_1)^2} s(\lambda_1)^2} \;\middle|\; \deg(s) < n \right\},$$

and, if $\lambda_k \in [\theta_{\kappa-1}, \theta_\kappa]$, then $(\lambda_k - \theta_{\kappa-1})(\theta_\kappa - \lambda_k) =$

$$\min \left\{ \frac{\sum_{j=1, j\neq k}^{N} \frac{w(\lambda_j)^2(\lambda_j - \theta_{\kappa-1})(\lambda_j - \theta_\kappa) s(\lambda_j)^2}{q_n(\lambda_j)^2}}{\frac{w(\lambda_k)^2 s(\lambda_k)^2}{q_n(\lambda_k)^2}} \;\middle|\; \deg(s) < n-1 \right\}.$$

Notion from potential theory

Given a (signed) Borel measure μ, its logarithmic potential is defined as

$$U^{\mu}(z) = \int \log \frac{1}{|x-z|} d\mu(x).$$

The mutual logarithmic energy of measures μ_1 and μ_2 is defined as

$$I(\mu_1, \mu_2) = \int U^{\mu_1}(y)\, d\mu_2(y), \quad I(\mu) := I(\mu, \mu).$$

The normalized counting measure for $\Xi = \{\xi_1, ..., \xi_n\}$ is

$$\chi_N(\Xi) = \frac{1}{N} \sum_{\xi_j \in \Xi} \delta_{\xi_j}$$

Assumptions

Set with $n = n_N$ such that $n/N \to t \in (0,1)$

$$\begin{aligned}
\Lambda_N &= \{\lambda_{1,N}, \dots, \lambda_{N,N}\} \quad \text{support of orthogonality, weights } w_N(\lambda_{j,N})^2 > 0, \\
\Theta_N &= \{\theta_{1,N}, \dots, \theta_{n,N}\} \quad \text{zeros of } n\text{th ORF } p_n/q_n, \\
\Xi_N &= \{\xi_{1,N}, \dots, \xi_{n,N}\} \subset \mathbb{R} \quad \text{poles of } n\text{th ORF } p_n/q_n.
\end{aligned}$$

Assumptions:

(A1) $\exists\, \Lambda$ compact including all Λ_N, $\chi_N(\Lambda_N) \to \sigma$, $U^\sigma \in \mathcal{C}(\Sigma; \mathbb{R})$.

(A2) $\exists\, \Xi$ closed including all Ξ_N, $\chi_N(\Xi_N) \to \nu$, $U^\nu \in \mathcal{C}(\Sigma; \mathbb{R} \cup \{\infty\})$.

(A3) $\langle 1, 1 \rangle_N = 1$, $\lim_N \min_j w_N(\lambda_{j,N})^{1/N} = 1$.

(A4) Weak separation: for all $\Lambda_N \ni \lambda_{k_N,N} \to \lambda$

$$\limsup_{\delta \to 0+} \limsup_{N \to \infty} \sum_{0 < |\lambda_{j,N} - \lambda_{k_N,N}| \le \delta} \log \frac{1}{|\lambda_{j,N} - \lambda_{k_N,N}|} = 0.$$

Results

$\mathcal{M}_t^\sigma = \{\mu : \mu \geq 0, \mu \leq \sigma, \|\mu\| = t\}$.

THEOREM part 1 (BB, Güttel, Vandebril'09)

If (A1)–(A4) and $\Sigma \cap \Xi = \emptyset$, then $\chi_N(\Theta_N) \to \mu$, where μ unique minimizer of $\mu \mapsto I(\mu) - 2I(\mu, \nu)$ within $\mathcal{M}_t^\sigma$.

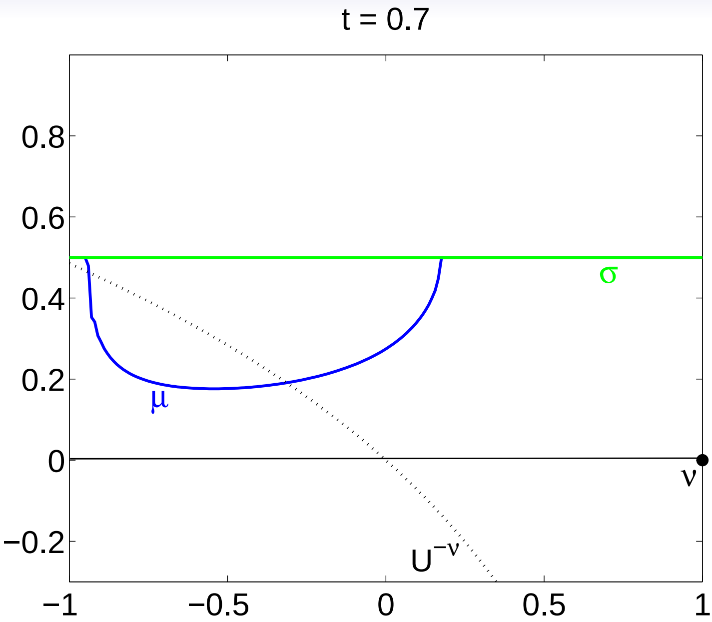
THEOREM part 2 (BB, Güttel, Vandebril'09)

Let (A1)–(A4) hold, and suppose that $\mathrm{supp}(\sigma), \mathrm{supp}(\nu_0)$ are finite unions of intervals in Jordan decomposition $\sigma - \nu = \sigma_0 - \nu_0$. Let μ as before and F the maximumum of $U^{\mu-\nu}$. Then for $\Lambda_N \ni \lambda_{k_N,N} \to \lambda$

$$\limsup_{N\to\infty} \mathrm{dist}(\lambda_{k_N,N}, \Theta_N)^{1/N} \leq \exp(U^{\mu-\nu}(\lambda) - F).$$

On each subinterval of $\{x : U^{\mu-\nu}(x) < F\}$, square rate for all but at most one exceptional index. Sharp.

Example Λ_N equidistant on $[-1,1]$, $d\sigma/dx = 1/2$, $\nu = t\delta_1$



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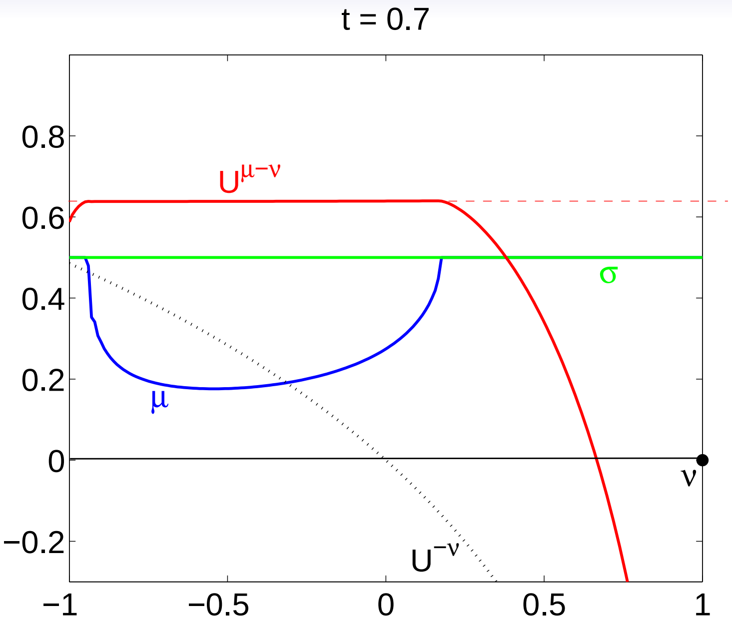

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Here $s_N(z) = \prod_{\beta \in B_N}(z-\beta)$ with $B_N \subset \Lambda_N$ s.t. $\chi_N(B_N) \to \mu$.

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blue term: principle of descent and $(A2)$.

red terms: $\Lambda_N \setminus B_N \ni \lambda_{j_N,N} \to \widetilde{\lambda}$: $|s_N(\lambda_{j_N,N})|^{1/N} \to U^\mu(\widetilde{\lambda})$ by $(A4)$.

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Bold term problems for $|q_N(\lambda_{j,N})|$ if $\lambda_{j,N}$ "close" to poles in Ξ_N.
... add such critical $\lambda_{j,N}$ to B_N...
Let $\delta > 0$. $\lambda_{j,N}$ is called critical if, for some $m \geq 0$, the following interval

$$\Big(\frac{\lambda_{j-m-1,N} + \lambda_{j-m,N}}{2}, \frac{\lambda_{j+m,N} + \lambda_{j+m+1,N}}{2} \Big) \cap \Big(\lambda_{j,N} - \delta, \lambda_{j,N} + \delta \Big).$$

contains $\geq 6m+1$ poles from Ξ_N.

Too many critical $\lambda_{j,N}$? For $I \subset \mathrm{supp}(\sigma)$:

$$\begin{cases} \sigma|_I \leq \nu|_I & \Longrightarrow \quad \mu|_I = \sigma|_I \\ \sigma|_I \geq \nu|_I & \Longrightarrow \quad \mu|_I \geq \nu|_I \end{cases}$$

Some conventions for examples

We do not draw $\operatorname{dist}(\lambda_{k,N}, \Theta_N)$ for fixed $N = 100$ and $n = 1, 2, \ldots, N$ but $\operatorname{dist}(\theta_{k,N}, \Lambda_N)$, each Ritz value $\theta_{k,N}$ being coded as follows

Symbol	Color	Distance of Ritz value θ to spectrum
$+$	Red	$\operatorname{dist}(\theta, \Lambda_N) < 10^{-7.5}$
$*$	Yellow	$10^{-7.5} \leq \operatorname{dist}(\theta, \Lambda_N) < 10^{-5}$
$\times$	Green	$10^{-5} \leq \operatorname{dist}(\theta, \Lambda_N) < 10^{-2.5}$
$\cdot$	Blue	$10^{-2.5} \leq \operatorname{dist}(\theta, \Lambda_N)$

$\mu_t \in \mathcal{M}_t^\sigma$ solution for external field $-U^\nu = -U^{\nu_t}$.

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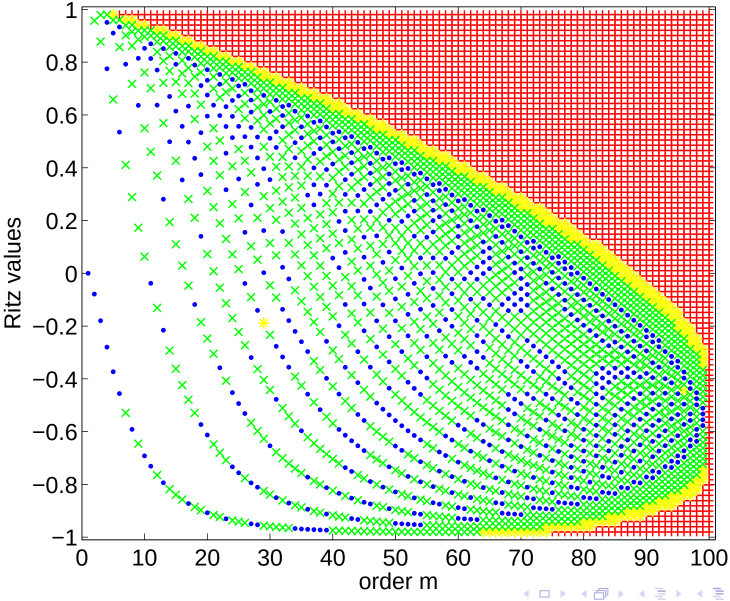

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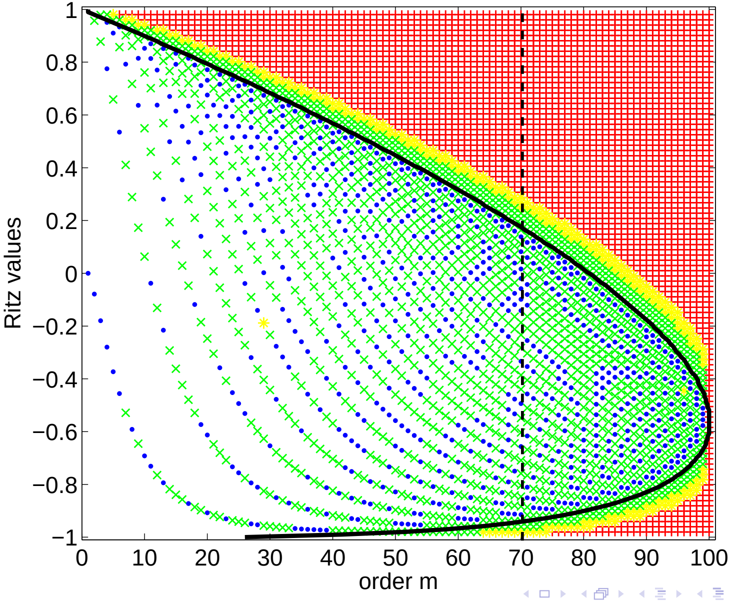
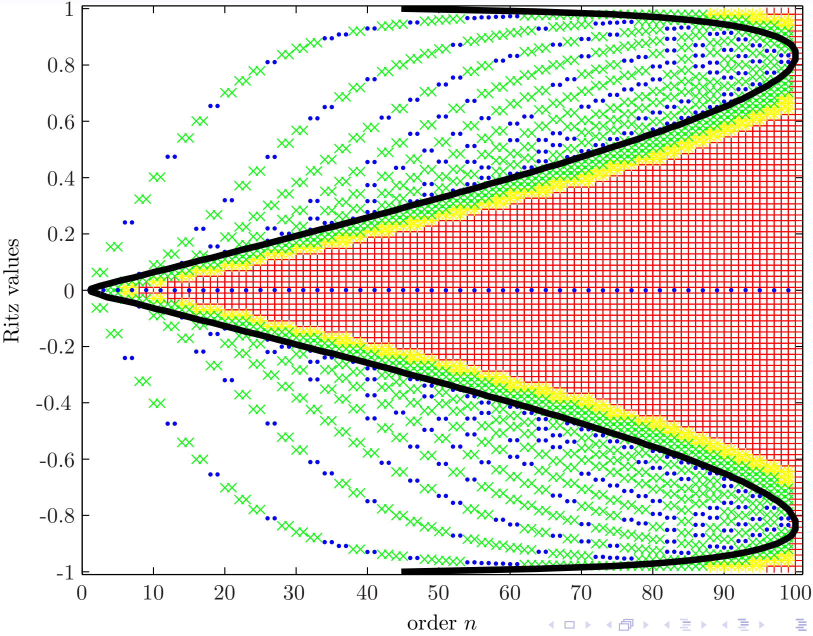
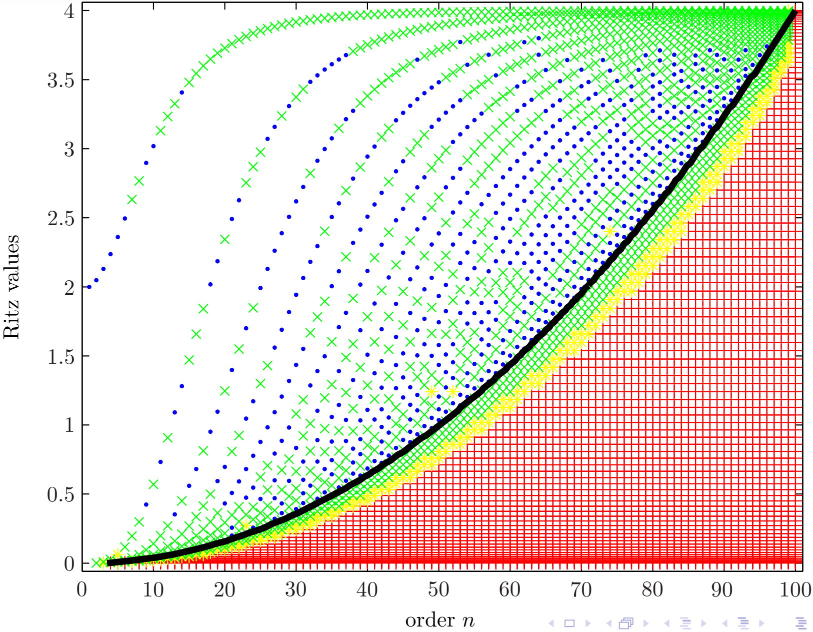


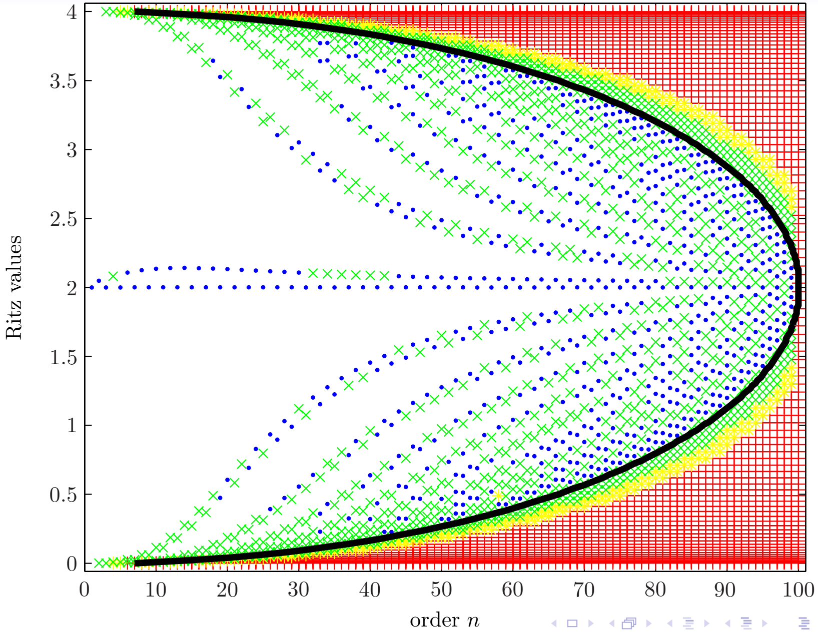
Example Λ_N equidistant on $[-1, 1]$, all poles at $x = 0$, $\nu_t = t\delta_0$



Example Λ_N Chebyshev points on $[0,4]$, poles $0,\infty$, $\nu_t = \frac{t}{2}\delta_0$



Example Λ_N Chebyshev points on $[0,4]$, poles $0,4$, $\nu_t = \frac{t}{2}(\delta_0 + \delta_4)$



An analytic example

Λ_N set of eigenvalues of

$$A_N = \begin{pmatrix} q^0 & q^1 & q^2 & \\ q^1 & q^0 & q^1 & \ddots \\ q^2 & q^1 & q^0 & \ddots \\ & \ddots & \ddots & \ddots \end{pmatrix} \in \mathbb{R}^{N \times N}, \quad q \in (0,1).$$

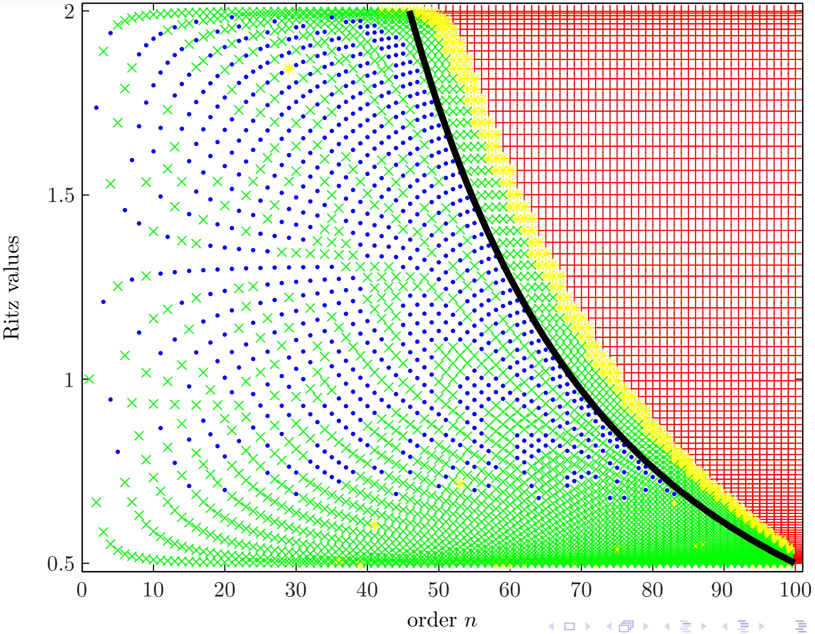
It is known [Kac, Murdock, Szegő '53] that

$$\frac{d\sigma}{dx}(x) = \frac{1}{\pi x \sqrt{(x-\alpha)(\beta - x)}}, \qquad \alpha = \frac{1}{\beta} = \frac{1-q}{1+q}.$$

For $\nu = t\delta_\xi$ for some $\xi > \beta$ we get $\{x : U^{\mu_t - \nu_t}(x) = F_t\} = [\alpha, b_t]$, where

$$b_t = \begin{cases} \beta, & \text{if } t < t_0, \\ \frac{\xi}{t^2\beta(\xi-\alpha)+1}, & \text{if } t \geq t_0, \end{cases} \qquad t_0 := \frac{1}{\beta}\sqrt{\frac{\xi-\beta}{\xi-\alpha}}.$$

Kac example, $q = 1/3$, $\nu_t = t\delta_{10}$



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