The convergence of Fourier-Takenaka-Malmquist series

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Approximation and extrapolation of convergent and divergent sequences and series
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Survey

- Motivating applications
- Series tranformation = possible convergence acceleration
- The Takenake-Malmquist and Hambo bases
- Convergence theorems: Exponential decay
- Numerical experiments

The functions

- 2π-periodic functions $\sim$ functions on $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$
- Suppose f is real then Fourier series

$$f(z) = \sum_{k\in\mathbb{Z}} c_k z^k, \quad z = e^{i\omega}, \quad c_{-k} = \overline{c}_k$$
$$= \sum_{k=0}^{\infty} a'_k \cos k\omega + b'_k \sin k\omega$$

- If $f \in H(\mathbb{D})$, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$

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Motivating applications

- Impulse response (transfer function) of a LTI system
  - stable and causal = holomorphic in $\mathbb{D} \cup \mathbb{T}$
  - usually real and rational (meromorphic)

  Realization theory, model reduction,...
- Identification of a signal
  - want f stable, causal, minimal phase, real,...
  - given the spectrum $|f(z)|^2$, $z \in \mathbb{T}$
  - if $c_{-k} = \overline{c}_k$ then approximating $|f(z)|^2$ is like approximating its real part
- Lowest possible number of terms is an important issue
  - less delay, lower cost
  - rational is easily implemented with circuit loop

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Slow convergence possible

Convergence of power series may be slow if singularity close to $\mathbb{T}$

- Consider for example

$$\frac{1}{1-\overline{\beta}z} = 1 + \sum_{k=1}^{\infty} \overline{\beta}^k z^k, \quad z \in \mathbb{T}$$

- If $|\beta| = 1 - \varepsilon < 1$, ε small, then very slow convergence!
- The error

$$\left| f(z) - \sum_{k=0}^{n} \overline{\beta}^k z^k \right| \leq \frac{|\beta|^{n+1}}{1-|\beta|}$$

- More general, the closest-to-1 pole dictates convergence rate

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Change of basis

- Suppose f belongs to a separable space with bases $\{e_k\}$ and $\{e'_k\}$.

$$f(z) = \sum_k c_k e_k(z) = \sum_k c'_k e'_k(z)$$

- For example

$$f(z) = \mathrm{Re}(z) = \cos\omega = \sum_k \frac{\omega^{2k}}{(2k)!}$$

The basis used is $\{e_k(\omega) = \omega^k : k = 0, \ldots, \infty\}$, which gives infinitely many terms.

- But the basis $\{\cos k\omega : k = 0, \ldots, \infty\}$ requires only one term, hence extremely fast convergence.

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- A change of the basis from $e_k(\omega) = \omega^k$ to the Fourier basis $e'_k(\omega) = \cos k\omega$ speeds up the convergence.
- We may consider $f(\omega)$ also as a (continuous) series expansion with respect to the basis $\delta(k - \omega)$:

$$f(\omega) = \int_k f(k)\delta(k - \omega)dk$$

- The Fourier transform produces the coefficients for the Fourier basis, given the "coefficients" $f(\omega)$ with respect to the basis $\delta(k - \omega)$.
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The space

- We shall work in a Hilbert space $L^2([0,2\pi]) \sim L^2(\mathbb{T})$

$$\langle f,g\rangle = \frac{1}{2\pi i}\int_{\mathbb{T}} f(z)\overline{g(z)}\frac{dz}{z}$$

- Note that Fourier basis $\{z^k\}_{k\in\mathbb{Z}}$ is an orthonormal basis.
While $\{z^k\}_{k=0,1,\dots}$ complete in $H^2(\mathbb{D})$.
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Other bases for $H^2(\mathbb{D})$

- Takenake-Malmquist basis = any sequence of poles $\{1/\overline{\alpha}_k\}_{k=1,2,\ldots}$

$$e_0 = 1, \quad e_k(z) = \frac{\sqrt{1-|\alpha_k|^2}}{1-\overline{\alpha}_k z} z B_{k-1}(z)$$

$$B_{k-1}(z) = \prod_{t=1}^{k-1} \left(\frac{z-\alpha_t}{1-\overline{\alpha}_t z} \right)$$

with $\alpha_k \in \mathbb{D}$ if stable and limit points in $|z| \le r < 1$.

- Hambo basis = finite set of poles at $1/\overline{\alpha}_k, k = 1, \ldots, p$ repeated cyclically

$$B_{p\ell+j}(z) = B_j(z)[B_p(z)]^\ell, \quad 0 \le j \le p-1, \quad \ell = 1, 2, \ldots$$

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- Kautz basis = a complex conjugate pair $\{1/\overline{\alpha}, 1/\alpha\}$ repeated

$$e_0 = 1, \quad e_{2k-1}(z) = \frac{\sqrt{1-|\alpha|^2}}{1-\overline{\alpha}z} z \left|\frac{z-\alpha}{1-\overline{\alpha}z}\right|^{2(k-1)}$$

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Convergence

Theorem[1]

$f(\omega) := F(e^{i\omega}) \in C^q_{2\pi}$, $q > 2$, $c_k = \langle F, \varphi_k \rangle$, $\varphi_k(z) =$ TM basis

Then

$F_n(z) = \sum_{|k|<n} c_k \varphi_k(z)$ converges to $F(z)$ uniformly on $\mathbb{T}$ with rate at least $1/n^{q-2}$.

[1] A.B & P. Carrette: *Algebraic and spectral properties of general Toeplitz matrices* SIAM J. Control Optim, 2003.

Convergence

Theorem[1]

$f(\omega) := F(e^{i\omega}) \in C^q_{2\pi}$, $q > 4$, $c_k = \langle F, \varphi_k \rangle$,
$\varphi_k(z) = z^k$ Fourier basis,
$\Phi_n = [\varphi_0, \ldots, \varphi_n]^T$, $M_n = [\frac{1}{2\pi i} \int \Phi_n F \Phi_n^* \frac{dz}{z}]$ (Toepliz matrix),
$G_n = \Phi_n / \|\Phi_n\|$, ($\|\Phi_n(z)\|^2 = k_n(z,z)$ = reproducing kernel)
Then for $z \in \mathbb{T}$

$$F_n(z) = G_n^*(z) M_n G_n(z) = \sum_{|k|<n} (1 - \frac{|k|}{n}) c_k \varphi_k(z), \quad \text{(Cesàro sum)}$$

converges to $F(z)$ uniformly on $\mathbb{T}$ with rate at least $1/n$.

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$$F_n(z) = G_n^*(z) T(M_n) G_n(z)$$

converges to $T(F(z))$ uniformly on $\mathbb{T}$ with rate at least $1/n$.

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Preliminary remark

Remark

Since we are interested in either $F(z) \in H^2(\mathbb{D})$ or in $f(z) \in L^2(\mathbb{T})$ but real (hence $c_{-k} = \overline{c}_k$), studying the convergence to $F(z) = \sum_{k=0}^{\infty} c_k z^k$ or to $f(z) = \sum_{k=-\infty}^{\infty} c_k z^k$ is "equivalent".

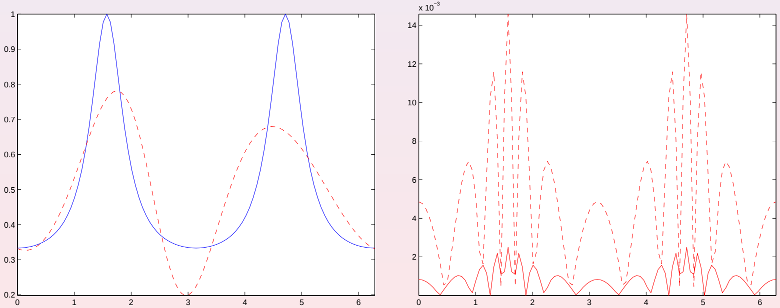
$$f(e^{i\omega}) = \mathrm{Re} F(e^{i\omega})$$

Examples

The examples will be stable rational functions with poles $\{1/\beta_j : j = 1, \ldots, r\}$ and we use the Hambo basis with poles $\{1/\alpha_j : j = 1, \ldots, p\}$

Example

$$f(\omega) = (\cos 2\omega + 2)/(4\cos 2\omega + 5)$$
$$f(\omega) = \mathrm{Re} F(e^{i\omega}) = (z^2+2)^{-1},\ z = e^{i\omega}.\ \text{Poles at } \pm i\sqrt{2}$$



left: f and f_2 with random poles; right: $|f - f_4|$ and $|f - f_6|$ when $\alpha_k = i^{2k+1}0.6$

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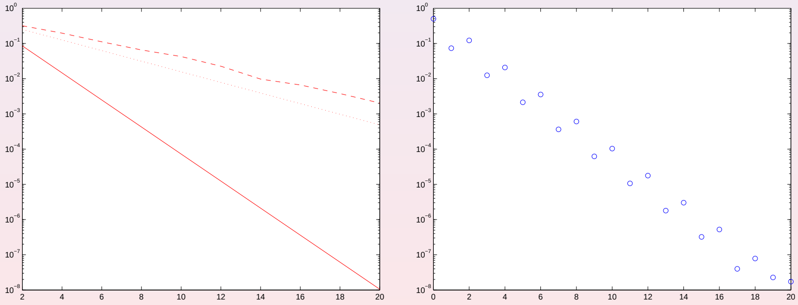

left: $\|f - f_n\|_\infty$ with (1) α_k random, (2) $\alpha_k = 0$, (3) $\alpha_k = i^{2k+2}0.6$;
right: Fourier coefs for (3)

Exponential decay

$$F(e^{i\omega}) = \sum_k \frac{a_k}{1-\overline{\beta}_k z}$$

$$\varphi_0 = 1,\ V_0 = [\varphi_1, \ldots, \varphi_p]^T,\ V_k = V_0(B_p)^k.$$

$$F(z) = F(0) + \sum_{k=1}^{\infty} L_k V_k(z), \quad L_k = \langle F, V_k \rangle$$

By orthogonality

$$\overline{L}_k = \sum_j \overline{a}_j \left[\frac{1}{2\pi i}\int_0^{2\pi} \frac{z}{z-\beta_j} V_0^T(z)\Big(B_p(z)\Big)^{k-1}\frac{dz}{z}\right]$$

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Exponential decay

$$F(e^{i\omega}) - F_n(e^{i\omega}) = \sum_{k=n+1}^{\infty} L_k V_k(z)$$

Thus elementwise:

$$\begin{aligned} |L_k| &\le \left|\sum_j \bar{a}_j V_k^T(\beta_j)\right| \le \left|\sum_j \bar{a}_j V_0^T(\beta_j)\right| \lambda^k, \quad \lambda = \max_j |B_p(\beta_j)| \\ &\le \mathbf{c}'\lambda^k, \quad \mathbf{c}' = \max_j \left|\sum_j a_j V_0^T(\beta_j)\right| \end{aligned}$$

Therefore

$$\begin{aligned} \|F - F_n\|_\infty &= \|\sum_{n+1}^{\infty} L_k V_k(z)\|_\infty \\ &\le \mathbf{c}' \sum_{n+1}^{\infty} \lambda^k \max_{z\in\mathbb{T}} |V_0(z)||B_p(z)|^k \\ &\le c\lambda^{n+1} \sum_{k=0}^{\infty} \lambda^k = c\frac{\lambda^{n+1}}{1-\lambda}, \quad c = \mathbf{c}' \max_{\mathbb{T}} |V_0(z)| \end{aligned}$$

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$$\begin{aligned} |L_k| &\leq \left|\sum_j \bar{a}_j V_k^T(\beta_j)\right| \leq \left|\sum_j \bar{a}_j V_0^T(\beta_j)\right| \lambda^k, \quad \lambda = \max_j |B_p(\beta_j)| \\ &\leq \mathbf{c}'\lambda^k, \quad \mathbf{c}' = \max_j \left|\sum_j a_j V_0^T(\beta_j)\right| \end{aligned}$$

Therefore

$$\begin{aligned} \|F - F_n\|_\infty &= \|\sum_{n+1}^{\infty} L_k V_k(z)\|_\infty \\ &\leq \mathbf{c}' \sum_{n+1}^{\infty} \lambda^k \max_{z\in\mathbb{T}} |V_0(z)||B_p(z)|^k \\ &\leq c\lambda^{n+1} \sum_{k=0}^{\infty} \lambda^k = c\frac{\lambda^{n+1}}{1-\lambda}, \quad c = \mathbf{c}' \max_{\mathbb{T}} |V_0(z)| \end{aligned}$$

Exponential decay

$$F(e^{i\omega}) - F_n(e^{i\omega}) = \sum_{k=n+1}^{\infty} L_k V_k(z)$$

Thus elementwise:

$$\begin{aligned} |L_k| &\leq \left|\sum_j \bar{a}_j V_k^T(\beta_j)\right| \leq \left|\sum_j \bar{a}_j V_0^T(\beta_j)\right| \lambda^k, \quad \lambda = \max_j |B_p(\beta_j)| \\ &\leq \mathbf{c}'\lambda^k, \quad \mathbf{c}' = \max_j \left|\sum_j a_j V_0^T(\beta_j)\right| \end{aligned}$$

Therefore

$$\begin{aligned} \|F - F_n\|_\infty &= \|\sum_{n+1}^{\infty} L_k V_k(z)\|_\infty \\ &\leq \mathbf{c}' \sum_{n+1}^{\infty} \lambda^k \max_{z\in\mathbb{T}} |V_0(z)||B_p(z)|^k \\ &\leq c\lambda^{n+1} \sum_{k=0}^{\infty} \lambda^k = c\frac{\lambda^{n+1}}{1-\lambda}, \quad c = \mathbf{c}' \max_{\mathbb{T}} |V_0(z)| \end{aligned}$$

Exponential decay

$$E_n(e^{i\omega}) = F(e^{i\omega}) - F_n(e^{i\omega}) = \sum_{k=n+1}^{\infty} c_k \varphi_k(z), \quad n = kp, \quad k \to \infty$$

Theorem

$$\lambda = \max_{1 \le j \le p} |B_p(\beta_j)| = \max_j \prod_{t=1}^{p} \left| \frac{\beta_j - \alpha_t}{1 - \overline{\alpha}_t \beta_j} \right| \quad \textit{(Hambo basis)}$$

The approximation error behaves like[1] $\|E_n\|_\infty \le c \frac{\lambda^{k+1}}{1-\lambda}$.

[1] P. Heuberger 1991

TM basis

With some more work, using the logarithmic potential of the α-distribution, analog results can be obtained for the TM basis[2].

Theorem

weak star convergence: $$\nu_n^\alpha = \frac{1}{n}\sum_{j=1}^{n} \delta_{\alpha_j} \xrightarrow{*} \nu^\alpha$$

set $$\lambda(z) = \int \log\left|\frac{x-z}{1-\bar{z}x}\right| d\nu^\alpha(x)$$

Then

$$\limsup_{n\to\infty} |E_n(z)|^{1/n} \leq \exp\{\lambda(z)\}$$

[2] A.B., Gonzalez-Vera, Hendriksen, Njåstad, 1997

What influences λ?

$$\lambda = \max_j \prod_{t=1}^{p} \left| \frac{\beta_j - \alpha_t}{1 - \overline{\alpha}_t \beta_j} \right| < 1$$

- λ small if all poles are approximated
- approximating 1 pole exactly does not speed up cvg
- if p poles are approximated, then a dip in the error every p steps
- the poles closest to the circle matter most

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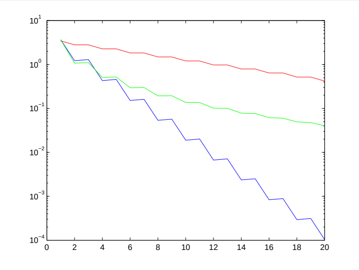
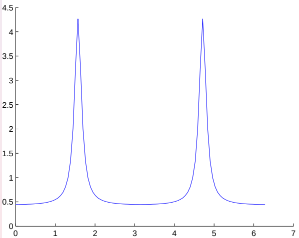
- if p poles are approximated, then a dip in the error every p steps
- the poles closest to the circle matter most

Example

$f = \mathrm{Re} F$, $F(e^{i\omega}) = (z^2 + \beta^2)^{-1}$. Poles at $\pm i\sqrt{9/10} \approx \pm i0.95$



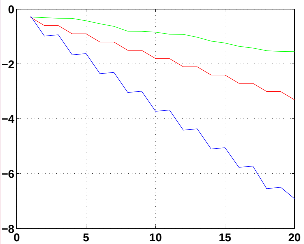


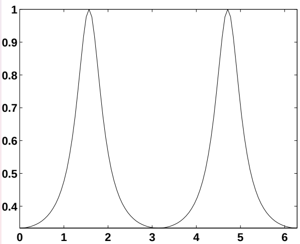
left: $\|f - f_n\|_\infty$ vs. n with
$\alpha_k = 0$,
$\alpha_k = 0.8i(-1)^k$,
$\alpha_k = i[0.9(1 - k/n) + 0.1k/n]$

right: $f(\omega)$

Example

$f(\omega) = \mathrm{Re} F(e^{i\omega})$, $F(z) = (z^2 + \beta^2)^{-1}$. Poles at $\pm i/\sqrt{2} \approx \pm i0.707$



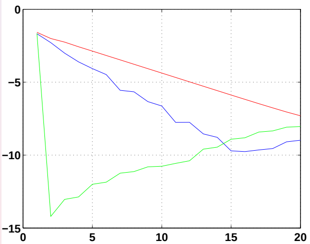


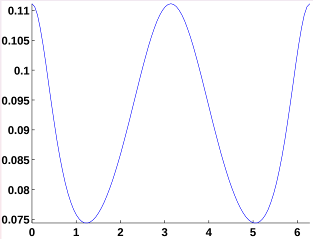
left: $\|f - f_n\|_\infty$ vs n with
$\alpha_k = 0$,
$\alpha_k = 0.8i(-1)^k$,
$\alpha_k =$ random in $\mathbb{D}$

right: $f(\omega)$

Example

$f(\omega) = \mathrm{Re}(|g(e^{i\omega})|^2)$, $g(z) = \frac{z-3}{(z-2)(z+5)}$. Poles in $\{0.5, -0.2\}$





left: $\|f - f_n\|_\infty$ vs n with right: $f(\omega)$

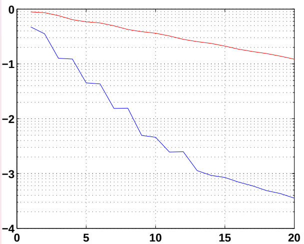
$\alpha_k = 0$,

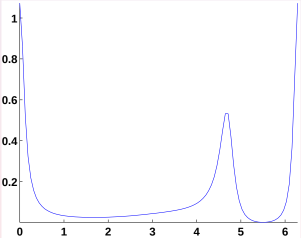
$\alpha_{2k} = 0.4$, $\alpha_{2k+1} = -0.3$,

$\alpha_{2k} = 0.5$, $\alpha_{2k+1} = -0.2$

Example

$f(\omega) = \mathrm{Re}(|g(e^{i\omega})|^2)$, $g(z) = \frac{z-1.1e^{-i\pi/4}}{(z-0.9)(z+1.2i)(z+4)}$. Poles in $\{0.9, -0.25, -0.8333i\}$



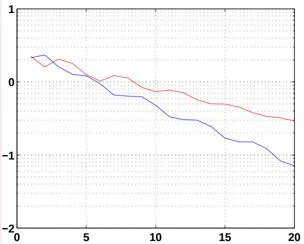


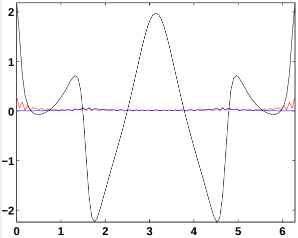
left: $\|f - f_n\|_\infty$ vs n with
$\alpha_k = 0$,
$\alpha_{2k} = 0.8$, $\alpha_{2k+1} = -0.8i$,

right: $f(\omega)$
error E_{20},
error E_{20},

Example

$f(\omega) = \mathrm{Re}(F(e^{i\omega}))$, $F(z) = \frac{(z-0.9e^{i\pi/4})(z-0.9e^{-i\pi/4})}{(z-0.9)(z-0.8i)(z+0.8i)(z+0.5)}$. Poles in $\{0.9, \pm 0.8i, -0.5\}$



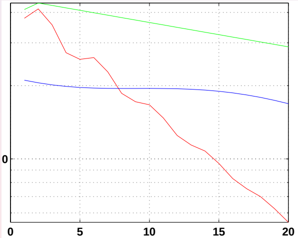


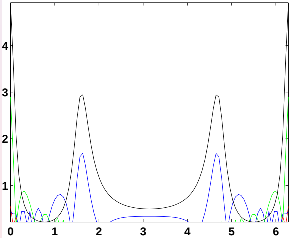
left: $\|f - f_n\|_\infty$ vs n with
$\alpha_k = 0$,
$\alpha_{1+4k} = 0.9$, $\alpha_{2+4k} = -0.4i$,
$\alpha_{3+4k} = 0.4i$, $\alpha_{4+4k} = -0.7$,

right: $f(\omega)$
error E_{20}
error E_{20}

Example

$f(\omega) = \mathrm{Re}(F(e^{i\omega}))$, $F(z) = \frac{(z-0.9e^{i\pi/4})(z-0.9e^{-i\pi/4})(z+0.5)}{(z-0.9)(z-0.8i)(z+0.8i)}$. Poles in $\{0.9, \pm 0.8i\}$



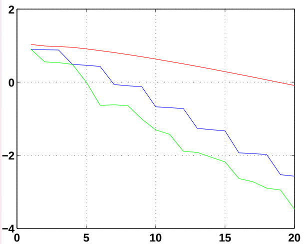


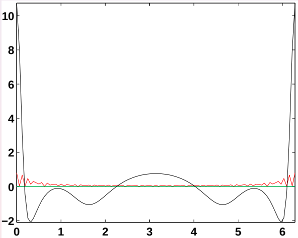
left: $\|f - f_n\|_\infty$ vs n with
$\alpha_k = 0$,
$\alpha_k = 0.9$,
$\alpha_{2k} = 0.8i$, $\alpha_{2k+1} = -0.8i$

right: $f(\omega)$
error E_{20}
error E_{20}
error E_{20}

Example

$f(\omega) = \mathrm{Re}(F(e^{i\omega}))$, $F(z) = \frac{(z-0.9e^{i\pi/4})(z-0.9e^{-i\pi/4})(z+0.5)}{(z-0.8)^2(z-0.5i)(z+0.5i)}$. Poles in $\{0.8, 0.8, \pm 0.5i\}$





left: $\|f - f_n\|_\infty$ vs n with

$\alpha_k = 0$,

$\alpha_{1+3k} = 0.7$,

$\alpha_{2+3k} = 0.7i$, $\alpha_{2+3k} = -0.7i$

$\alpha_{1+4k} = 0.7$, $\alpha_{2+4k} = 0.7$

$\alpha_{3+4k} = 0.7i$, $\alpha_{4+4k} = -0.7i$

right: $f(\omega)$

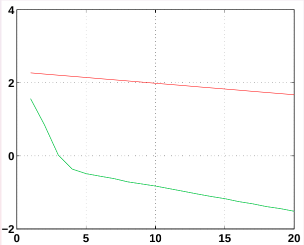
error E_{20}

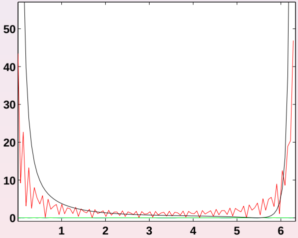
error E_{20}

error E_{20}

Example

$$f(\omega) = \mathrm{Re}(F(e^{i\omega})),\ F(z) = \frac{(z-1.1e^{-i\pi/4})}{(z-0.93)(z-0.2)}.$$ Poles in $\{0.93, 0.2\}$



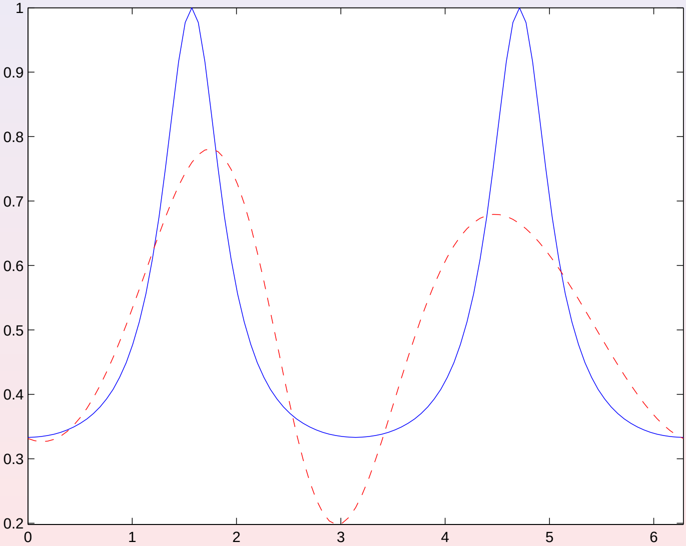


left: $\|f - f_n\|_\infty$ vs n with
$\alpha_k = 0$,
$\alpha_k = 0.97$,
$\alpha_{1+2k} = 0.9$, $\alpha_{2+2k} = 0.1$

right: $f(\omega)$
error E_{20}
error E_{20}= error E_{20}

The end

Thank you



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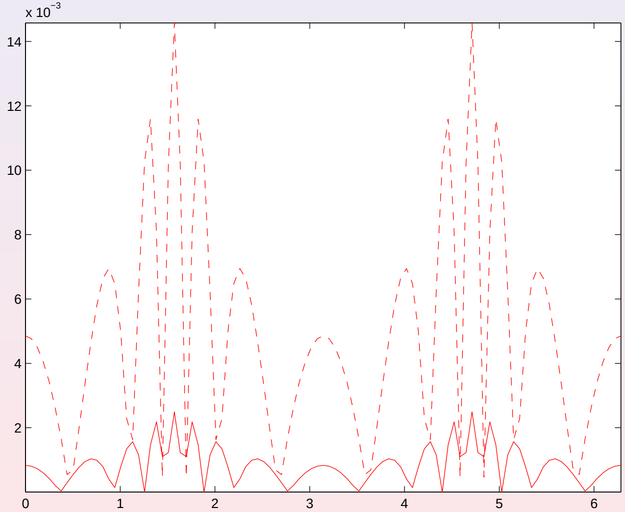

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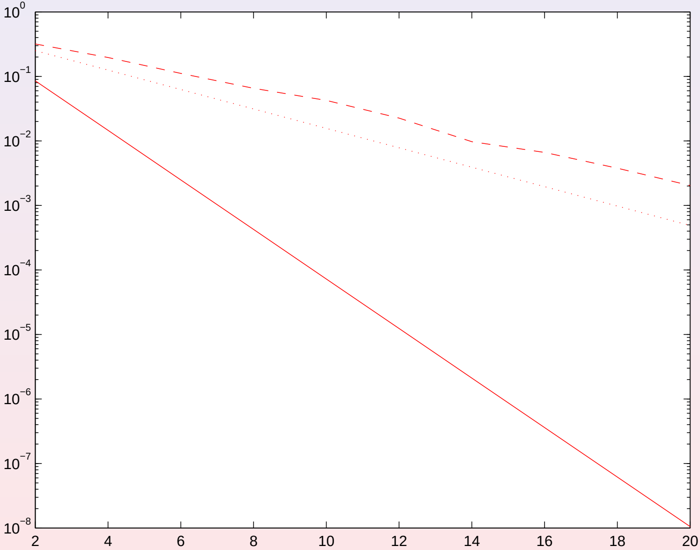

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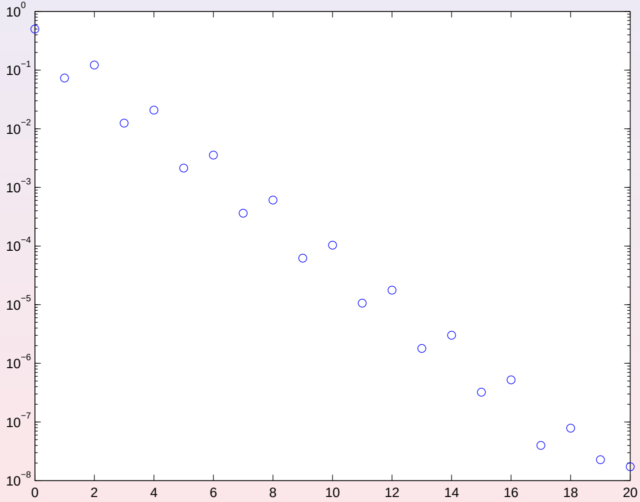

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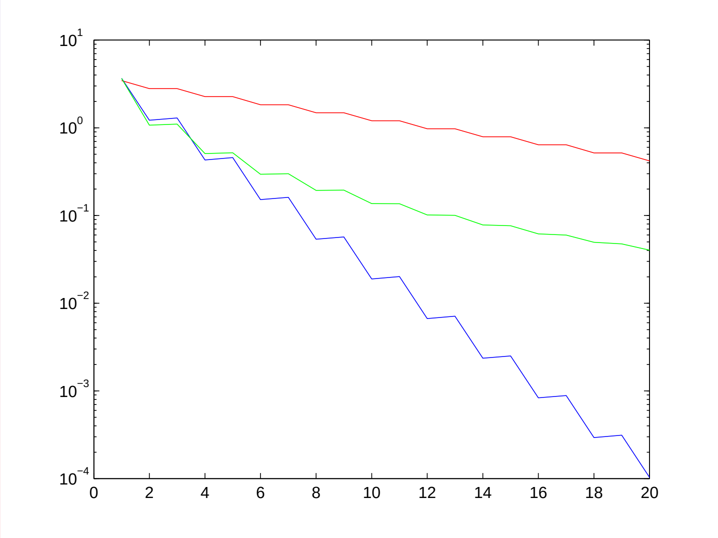

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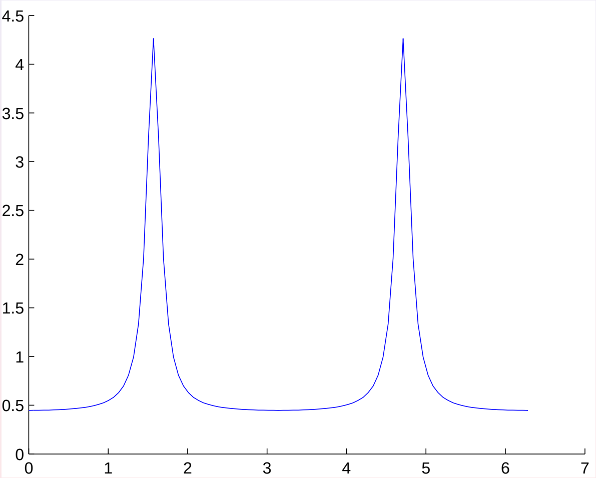

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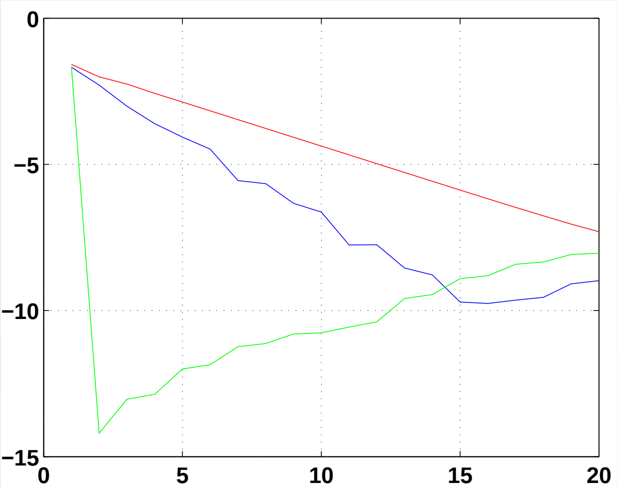

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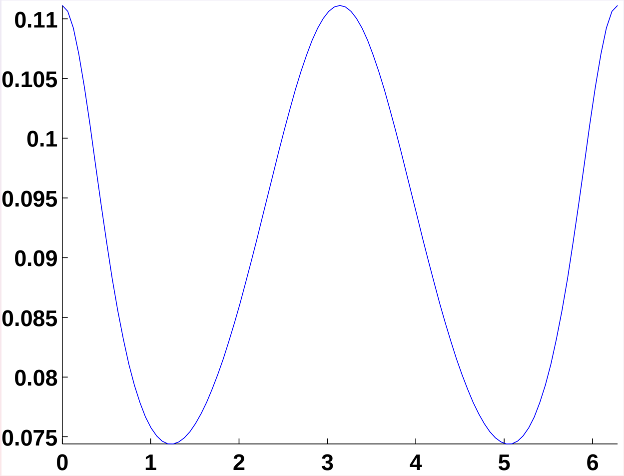

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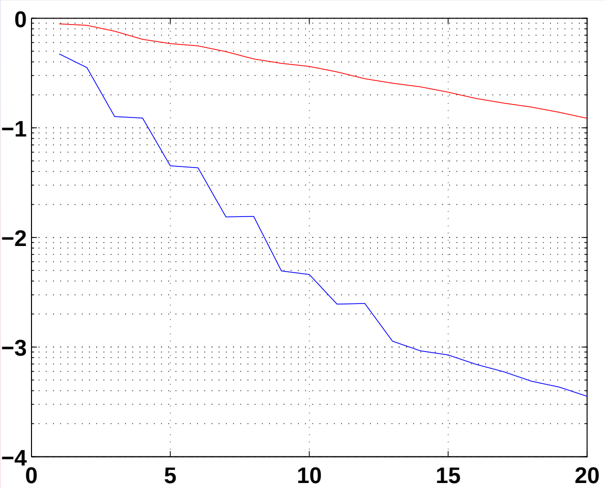

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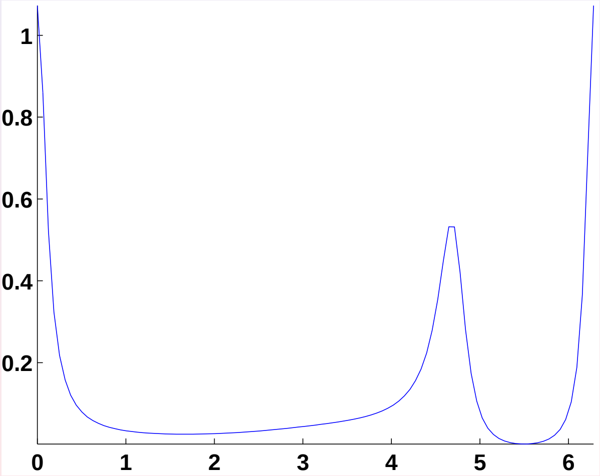

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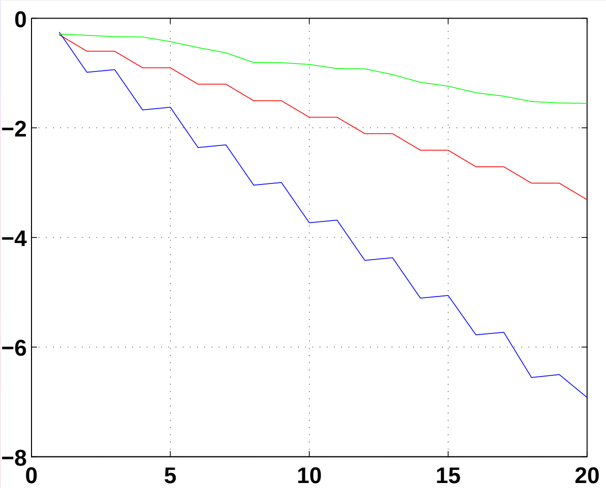

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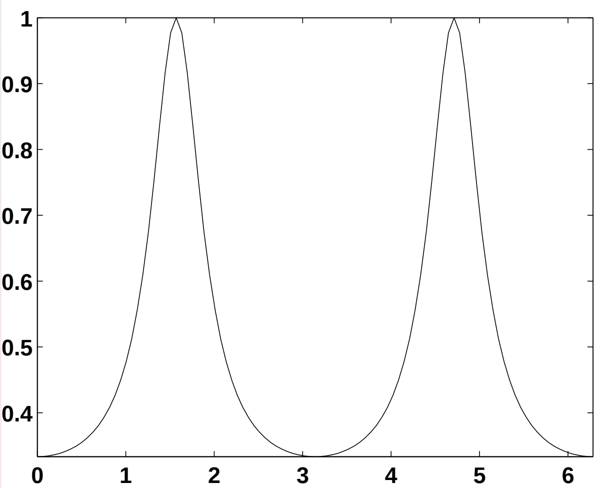

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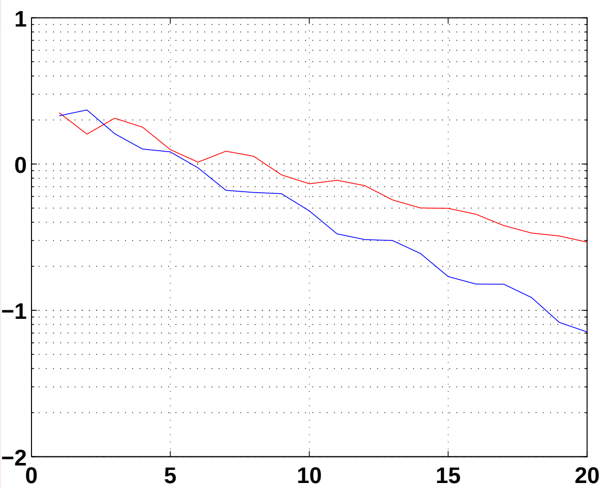

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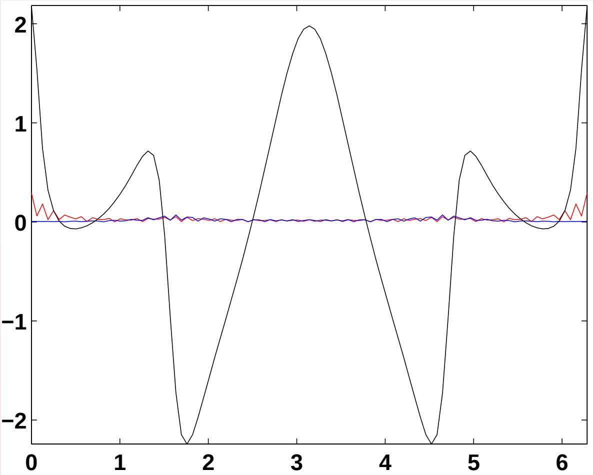

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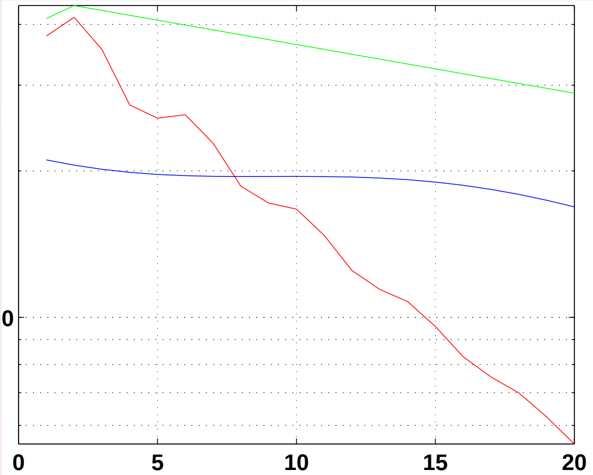

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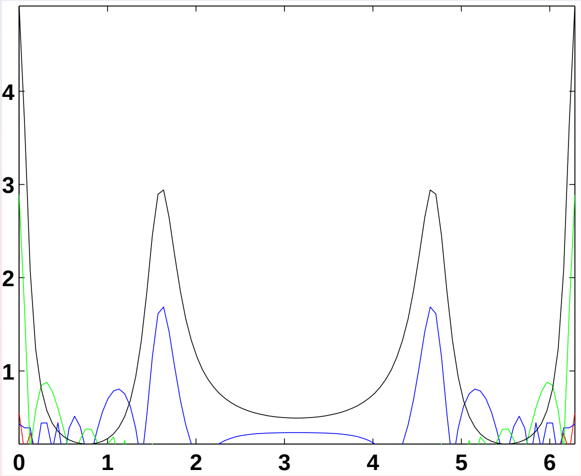

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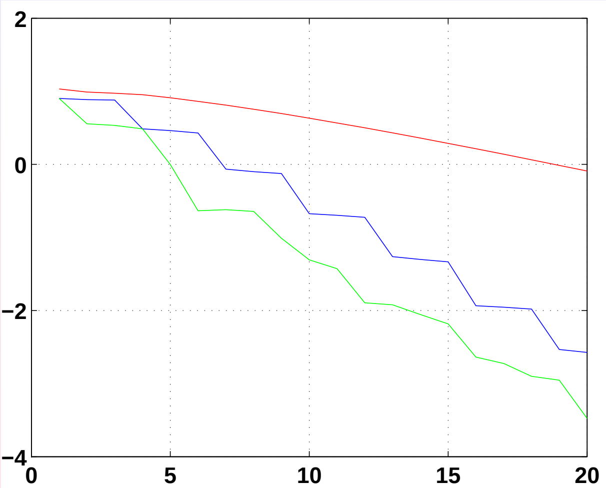

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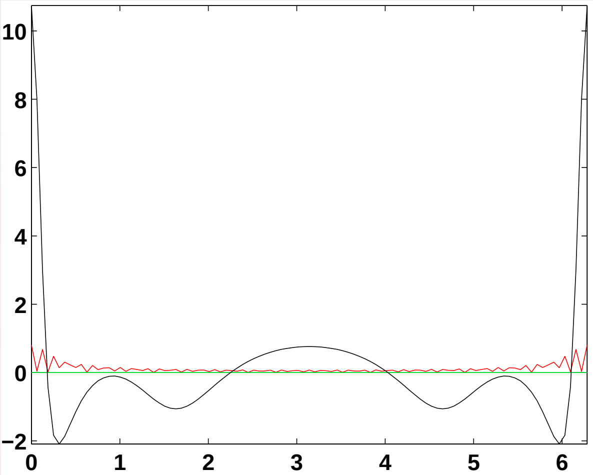

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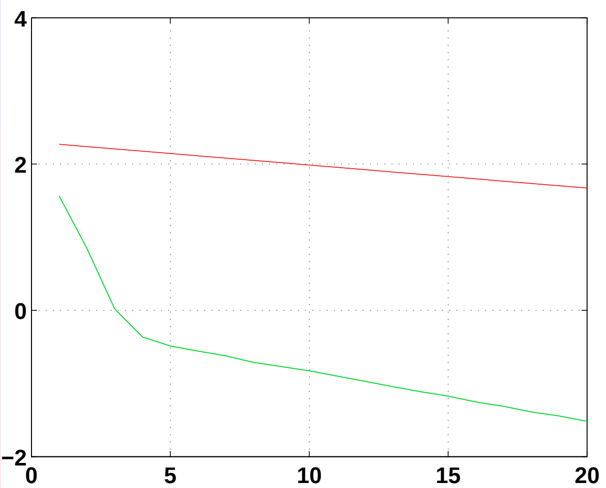

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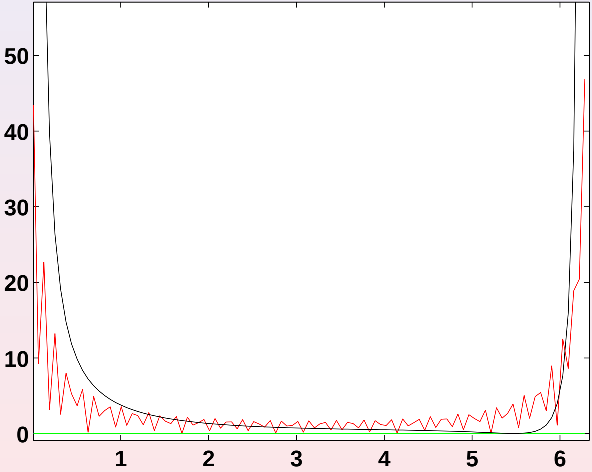

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