# *Padé approximants and the prediction of non-perturbative parameters in particle physics*

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# *Outline*

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*(2.3) Padé approximants and OPE/ChPT relations*

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(3) Summary

# *Statement of the problem(s) I*

- The Lagrangian of the strong interactions can be written down in a very easy fashion

$$\mathcal{L}_{QCD} = -\frac{1}{2}\mathrm{Tr}\, G^{\mu\nu} G_{\mu\nu} + \sum_{q=1}^{6} \bar{\psi}_q [i\gamma_\mu D^\mu - m_q] \psi_q \qquad (1)$$

  where $D^\mu = \partial^\mu - i g_s G^\mu$ and $[D_\mu, D_\nu] = -i g_s G_{\mu\nu}$.

- The theory is requested to be invariant under local $SU(3)_c$ transformations, which is at the origin of the gluon field $G_\mu$. The 6 quark species (flavours) transform in the fundamental representation and the gauge potential $G_\mu$ in the adjoint.

- The simplicity of formulating the strong interactions as a $SU(3)_c$ gauge theory does however no justice to the complexities it hides.

# *Statement of the problem(s) II*

- In fact, Eq. (1) was received with a lot of skepticism, because quarks and gluons had never (and have never) been observed. Experimentalists only detect hadrons, and a large diversity of them.
- Eq. (1) assumes that hadrons are composite objects, and the quarks and gluons the building blocks. Even though this complied very well with the observed phenomenology (some static properties of hadrons), there was no dynamical explanation of this confinement. The equations of motion cannot be solved analytically, and the option left, namely perturbation theory, seemed crazy.
- The Lagrangian was made computationally friendly with the discovery of asymptotic freedom (and infrared slavery)

$$\alpha_s(\mu) \to 0, \qquad (\mu >> 0) \tag{2}$$

- At high Euclidean momentum the QCD Lagrangian works as a more complicated (non-Abelian) version of the well-known QED.

# *Statement of the problem(s) III*

- *Some* of the non-perturbative effects can be captured by the OPE, but not all. In practice one has a double expansion

$$\Pi(q^2) = -\frac{1}{4\pi^2}\log\frac{-q^2}{\mu^2}[1+\alpha_s+\alpha_s^2+\cdots]+\frac{\langle\alpha_s G_{\mu\nu}G^{\mu\nu}\rangle}{12q^4}[1+\alpha_s+\alpha_s^2+\cdots]+\cdots \quad (3)$$

  where each of the series ($\alpha_s$,$1/q^2$) is believed to be asymptotic.

- At very low energies, there is also information that can be extracted. The so-called chiral symmetry is broken and Goldstone theorem requires the presence of massless particles, whose interactions are highly constrained by symmetry properties. The strong interactions can then be cast as an effective field theory of pions (ChPT), where most of the parameters can be determined experimentally or estimated theoretically.

- Bottom line: For every correlator in QCD there is accessible information at very high and very low Euclidean momenta. It seems like an ideal setting to use interpolators to fill in the unknown region...

# *Padé approximants and non-perturbative physics I*

- Some definitions to set my notations: Padé approximants are rational approximants. Given a function $f(z)$, its $[N/M]$ Padé approximant satisfies

$$f(z) = \frac{\mathcal{P}_M(z)}{\mathcal{Q}_N(z)} + \mathcal{R}_{[N,M]}(z) \tag{4}$$

  As $N, M$ are increased, so the residue $\mathcal{R}_{[N,M]}(z)$ decreases.

- The use of Padé approximants in QCD is hindered by the obiquitous presence of physical cuts (typically through logarithms).

- One of the most fruitful approaches to QCD in the non-perturbative regime comes from the so-called large$-N_c$ limit of QCD, where one works with an enlarged gauge group $SU(N_c)$. The number of quarks and gluons grows as $N_c$ and $N_c^2$ respectively but the resulting picture is greatly simplified.

# *Padé approximants and non-perturbative physics II*

- Assuming confinement, the theory can be shown to lead to an almost idylic hadronic world of stable and non-interacting resonances.
- Besides its numerous phenomenological successes, the most interesting thing is that all correlators are meromorphic at leading order in $1/N_c$.
- If one adopts the framework of large-$N_c$ QCD, one can use the theory of Padé approximants to meromorphic functions.

# *Padé approximants and the hadronic spectrum I*

- One of the first attempts to derive the hadronic spectrum of vector mesons was actually inspired by Padé approximants. The easiest way is to start with the two-point correlator

$$\begin{aligned} \Pi_V^{\mu\nu}(q) &= i\int \mathrm{d}^4x\, e^{iq\cdot x}\langle\, 0\,|\, T\{\, V_{\bar{u}d}^{\mu}(x)\, V_{\bar{d}u}^{\nu}(0)\,\}|\, 0\,\rangle \\ &= (q^\mu q^\nu - q^2 g^{\mu\nu})\Pi_V(q^2) \qquad (5) \end{aligned}$$

  whose singularities are poles (for stable particles) and cuts (due to decaying resonances). In the large-$N_c$ limit only the former survive.

- Next, one can try to build the $[N/M]$ Padé approximant for $\Pi_V(q^2)$ as follows

$$\Pi_V(q^2) = \Pi_V^{[N,M]}(q^2) + \mathcal{R}_{[N,M]}(q^2)\,, \qquad \Pi_V^{[N,M]}(q^2) \equiv \frac{\mathcal{P}_M(q^2)}{\mathcal{Q}_N(q^2)}, \quad (6)$$

  The correlator is of Stieltjes type, and therefore the Padé poles are going to lie on the physical axis.

# *Padé approximants and the hadronic spectrum II*

- By definition, the following equation is satisfied:

$$\frac{d^n}{d(q^2)^n}\left[\Pi_V(q^2)\mathcal{Q}_N(q^2)-\mathcal{P}_N(q^2)\right]\Big|_{q^2=-\mu^2}=0, \qquad n=0,\ldots,2N\,. \tag{7}$$

- Note that after the first $N$ equations there is no contribution from $\mathcal{P}_N(q^2)$. The analytical structure of $\Pi_V(q^2)\mathcal{Q}_N(q^2)$ contains only single poles (the ones from $\Pi_V(q^2)$). Cauchy's theorem therefore leads to

$$\int_0^\infty \frac{dt}{(t+\mu^2)^{n+1}}\,\mathrm{Im}\Pi_V(t)\mathcal{Q}_N(t)=0, \qquad n=N+1,\ldots,2N\,, \tag{8}$$

- The problem is to sort out the input for $\mathrm{Im}\Pi_V(t)$ to be used in the previous equation. In QCD at large enough energies (in the Euclidean) the leading term is a constant. This was originally taken as the seed, with the hope that more information (both perturbative and OPE) could be added later on to improve the result.

# Padé approximants and the hadronic spectrum III

- The solution for $\mathcal{Q}_N(t)$ can then be shown to lead to Legendre polynomials $P_N^{(0,0)}$,

$$\mathcal{Q}_N(q^2) = {}_2F_1\left(-N, -N; 1; -\frac{q^2}{\mu^2}\right) = (q^2+\mu^2)^N P_N^{(0,0)}\left(\frac{\mu^2-q^2}{\mu^2+q^2}\right) \tag{9}$$

which is actually a result due originally to Gau$\beta$. Eq.(9) can now be plugged back in Eq.(7) to yield

[Weideman'05]

$$\Pi_V^N(q^2) \simeq \frac{2}{(q^2+\mu^2)^N} \sum_{k=0}^{N} \left(\begin{array}{c} k \\ j \end{array}\right)^2 \left[\frac{H_{N-k}-H_k}{P_N^{(0,0)}(\chi)}\right] \left(-\frac{q^2}{\mu^2}\right)^k , \ \chi = \frac{\mu^2-q^2}{\mu^2+q^2} , \tag{10}$$

which is the Padé approximant to the logarithm, *i.e.*, in the continuum limit $N \to \infty$ we recover the logarithm we started from.

# *Padé or not Padé? I*

- The main motivation behind [Migdal'77] was to infer information from the hadronic spectrum (discrete) starting from the conformal limit of the correlator. The following correlated limit was taken instead,

$$q^2 << \mu^2, \quad N \to \infty, \quad \frac{N}{\mu} = ct. \tag{11}$$

- Then

$$\lim_{q^2 \to 0} \chi = \left(1 - 2\frac{q^2}{\mu^2}\right) \tag{12}$$

and

$$\lim_{N\to\infty} P_N^{(0,b)}\left(1 - \frac{\xi^2}{2N^2}\right) = J_0(\xi) \tag{13}$$

# Padé or not Padé? II

- The vector correlator takes the form

$$\Pi_V^N(q^2) = -\frac{4}{3}\frac{N_c}{(4\pi)^2}\left[\log\frac{q^2}{\mu^2} - \pi\frac{Y_0\left(q\Lambda\ \right)}{J_0\left(q\Lambda\ \right)}\right], \qquad \Lambda = \frac{2N}{\mu}. \quad (14)$$

  and the poles of the propagator (the masses of the physical particles in the large-$N_c$ limit) are located at the zeroes of the $J_0$ Bessel function.

- Obviously, the correlated limit does not retrieve the original function and *should not* be called a Padé approximant. The true Padé approximant takes the form

$$\Pi_V^N(q^2) = -\frac{4}{3}\frac{N_c}{(4\pi)^2}\log\frac{-q^2}{\mu^2} - \frac{\mathcal{R}_N(q^2)}{\mathcal{Q}_N(q^2)}, \quad (15)$$

  where the residue is a Meijer G function that vanishes in the continuum limit, thus ensuring convergence. The correlated limits taken by Migdal freeze the residue, thereby spoiling the convergence of the rational approximant. The non-vanishing residue also explains why the resulting spectrum has distinct resonances *after* $N \to \infty$.

# *Padé or not Padé? III*

- In the far Euclidean it can be shown that the correlator of Eq.(14) reduces to ($Q^2 = -q^2$)

$$\Pi_V(Q^2) = -\frac{4}{3}\frac{N_c}{(4\pi)^2}\log\frac{Q^2}{\mu^2} + \mathcal{O}(e^{-2Q\Lambda})\,, \qquad (Q^2 \gg 0). \tag{16}$$

The last term is not part of the OPE. It is generically referred to as quark-hadron duality violations, defined as

$$\Pi_V(q^2) \simeq \Pi_V^{OPE}(q^2) + \Delta(q^2)\,, \qquad (q^2 > 0). \tag{17}$$

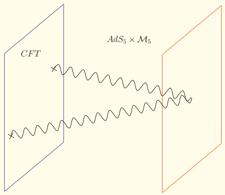
In Migdal's model, it can be shown that

$$\Delta(q^2) = \frac{N_c}{12\pi}\frac{Y_0\,(q\Lambda\;)}{J_0\,(q\Lambda\;)}\,. \tag{18}$$

In other words, duality is broken and the duality violating pieces collect the singularities of the spectrum. This is exactly the role that duality violations have to play, since the OPE, being a regular expansion, is unable to capture the resonance poles.

# *Connection with 5-dimensional models I*

- The AdS/CFT philosophy: given a string theory in $AdS_5 \times \mathcal{M}_5$ with a 4-dimensional boundary, then the values of on-shell fields (in the supergravity limit) on the boundary are the external sources of a CFT living on the boundary.




- Applications for non-conformal field theories (like QCD) rather *ad hoc*. For instance, the simplest action in the gravity side:

$$S = -\int d^4x\, dz\, e^{-\Phi(z)} \sqrt{g} \frac{1}{4g_5^2} \mathrm{Tr}\left[F_{\hat{\mu}\hat{\nu}} F^{\hat{\mu}\hat{\nu}}\right], \qquad (19)$$

where $g_{\hat{\mu}\hat{\nu}}$ is the AdS metric

$$g_{\hat{\mu}\hat{\nu}}\, dx^{\hat{\mu}} dx^{\hat{\nu}} = \frac{1}{z^2}\left(\eta_{\mu\nu} dx^\mu dx^\nu + dz^2\right), \qquad \eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$$

# *Connection with 5-dimensional models II*

- $\Phi(z)$ is the dilaton field. In the *hard wall model* (HWM),

$$\Phi(z) = \phi, \quad \epsilon \leq z \leq \Lambda\,,$$

  where $z = \epsilon$ and $z = \Lambda$ are the four-dimensional boundary branes.

- The problem therefore reduces to QFT in a box. It turns out that this action with these particular boundary conditions is dual to the (non)-Padé construction before.

- I regard this as a mere curiosity: after all, Migdal's approach is not a Padé approximant; and the spectrum inferred by the HWM does not comply with expected properties of string theory (and phenomenology) such as Regge trajectories.

# *Padé approximants and electroweak observables I*

- Computing the spectrum of large-$N_c$ QCD is extremely complicated, and even so it is not clear its connection with the one in QCD.
- Unlike in the Minkowski axis, large-$N_c$ in the Euclidean approximates better QCD. We can use the drawbacks of last section to our advantage: since one cannot infer the QCD spectrum from the OPE, a reasonable job can be made with an arbitrary spectrum as long as it satisfies the OPE. For many applications (those involving Euclidean regimes) one does not need the spectrum.
- Electroweak observables fall into this cathegory. In full generality, they can all be expressed as integrals over Euclidean regimes of QCD correlators. For instance,

$$\Delta m_\pi^2 = m_{\pi^+}^2 - m_{\pi^0}^2 = -\frac{3\alpha_{EM}}{4\pi f_\pi^2}\int_0^\infty dQ^2 Q^2 \Pi_{LR}(Q^2) \qquad (20)$$

# *Padé approximants and electroweak observables II*

- The MHA is a prescription to compute such observables using truncated large-$N_c$ ansätze for the correlators, while paying attention to fulfill what is known from QCD. For instance, we can choose as ansatz:

$$\Pi_{LR}(q^2) = \frac{f_\pi^2}{q^2} + \frac{f_V^2}{-q^2 + m_V^2} - \frac{f_A^2}{-q^2 + m_A^2} \tag{21}$$

and require the two following superconvergence relations (Weinberg sum rules):

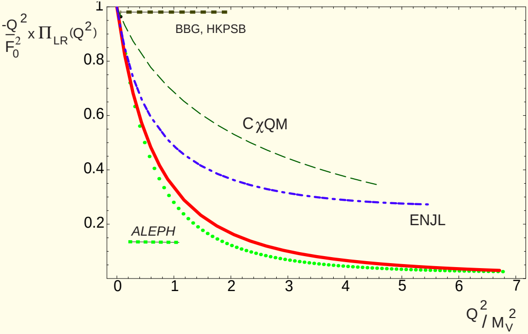
$$\begin{aligned} \int_0^\infty dt \, \mathrm{Im}\, \Pi_{LR}(t) &= 0 \\ \int_0^\infty dt \, t \, \mathrm{Im}\, \Pi_{LR}(t) &= 0 \end{aligned} \tag{22}$$

The correlator then looks like

$$\Pi_{LR}(q^2) = \frac{m_V^2 \, m_A^2 \, f_\pi^2}{q^2(-q^2 + m_V^2)(-q^2 + m_A^2)} \tag{23}$$

# *Padé approximants and electroweak observables III*

- The free parameters $m_V$, $m_A$, $f_\pi$ are fixed from phenomenology. One can now compare the ansatz with experimental data:




- The MHA is both an approximation to large-$N_c$ QCD and a Padé-type approximant. Ideally, when $N$ is infinitely large, the Padé-type to the correlator is actually the spectrum of large-$N_c$ QCD.

# *Padé approximants and electroweak observables IV*

Shopping list:

- identify a suitable underlying correlator (both simple and *order parameter*) to the observable under study.
- Compute its high energies with pQCD (in the same scheme as the Wilson coefficients) and low energies with ChPT.
- Build a Padé-type interpolator, where the number of states (poles) is dictated by

$$N - P = \frac{1}{2\pi i} \oint \frac{\Sigma'(z)}{\Sigma(z)} dz \simeq -p_{OPE} \tag{24}$$

  where $\Sigma(z)$ is the propagator with no poles at the origin.
- The correlator so constructed can be matched to high and/or low energies. In practice, a balance between the two regimes tends to be optimal.

# *Padé approximants and OPE/ChPT relations I*

- One of the main advantages of the simplicity of the large-$N_c$ limit is that, once the poles and residues are known, the full correlator is known. In particular, there is a relation between OPE and ChPT coefficients which can be used to test phenomenological analyses.

- Consider in particular the $\Pi_{LR}(q^2)$ correlator, whose spectrum has been reconstructed by very precise data on the decays of the tau lepton. At high energies, it can be parameterised by OPE condensates, starting with $\xi_6$ and $\xi_8$ ($\xi_2 = 0 = \xi_4$):

$$\lim_{q^2 \to (-\infty)} \Pi_{LR}(q^2) = \sum_{n=3}^{\infty} \frac{\xi_{2n}}{q^{2n}} \tag{25}$$

whereas at low energies the ChPT Lagrangian yields

$$\lim_{q^2 \to 0} \Pi_{LR}(q^2) = \frac{f_\pi^2}{q^2} - 8\,L_{10} + 16\,C_{87}\,q^2 + \mathcal{O}(q^4) \tag{26}$$

# *Padé approximants and OPE/ChPT relations II*

- At present there is a puzzle: many groups have performed analyses on the tau data, and while they find agreement on $L_{10}$, the values for the OPE condensates are controversial. If we succeed in connecting the two regimes, we can make, within our approximation, a consistency check.
- The ansatz for the correlator will be

$$\Pi_{LR}(q^2) = \frac{f_\pi^2}{q^2} + \frac{f_V^2}{-q^2 + m_V^2} - \frac{f_A^2}{-q^2 + m_A^2} \tag{27}$$

If we want to leave poles and residues free we have to solve the following system of equations:

$$\begin{aligned} f_A^2 - f_V^2 &= -f_\pi^2 \\ f_A^2 m_A^2 - f_V^2 m_V^2 &= 0 \\ f_A^2 m_A^4 - f_V^2 m_V^4 &= \xi_6 \\ f_A^2 m_A^6 - f_V^2 m_V^6 &= \xi_8 \end{aligned} \tag{28}$$

# *Padé approximants and OPE/ChPT relations III*

- Its solution for the hadronic parameters in terms of $f_\pi$, $\xi_6$ and $\xi_8$ is highly non-linear. There are 4 sets of solutions and (not surprisingly) in all of which the majority of parameters take on complex values. Indeed, $\Pi_{LR}$ is not of Stieltjes type.

- Surprisingly (at least to me) the connection between low and high energy parameters is extremely simple:

$$L_{10} = \frac{1}{8}\left[\frac{f_A^2}{m_A^2} - \frac{f_V^2}{m_V^2}\right] = -\frac{1}{8}\frac{\xi_8}{\xi_6^2} f_\pi^4 \tag{29}$$

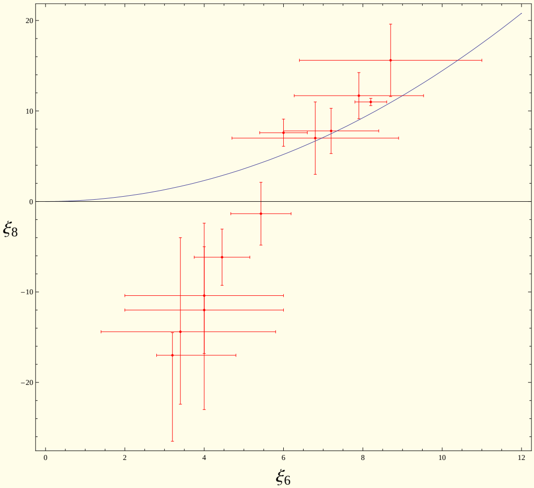
- The first thing to realize is that, since $L_{10} = (-5.22 \pm 0.06) \cdot 10^{-3}$, $\xi_8$ is predicted to be positive. There is a rigorous theorem by Witten which already fixes the sign of $\xi_6$ to be positive.

- Even more importantly, Eq. (29) is satisfied at the quantitative level by the phenomenological analyses that already predict $\xi_8 > 0$.

# *Padé approximants and OPE/ChPT relations IV*

| | $\xi_6$ | $\xi_8$ | $\xi_8 = -8\,L_{10}\,f_\pi^{-4}\ \xi_6^2$ |
|---|---|---|---|
| Friot *et al.* | $+7.90 \pm 1.63$ | $+11.69 \pm 2.55$ | $+9.0 \pm 3.7$ |
| Ioffe *et al.* | $+6.8 \pm 2.1$ | $+7 \pm 4$ | $+6.7 \pm 4.1$ |
| Zyablyuk | $+7.2 \pm 1.2$ | $+7.8 \pm 2.5$ | $+7.5 \pm 2.5$ |
| Narison | $+8.7 \pm 2.3$ | $+15.6 \pm 4.0$ | $+10.9 \pm 5.8$ |
| ALEPH | $+8.2 \pm 0.4$ | $+11.0 \pm 0.4$ | $+9.71 \pm 0.96$ |
| OPAL | $+6.0 \pm 0.6$ | $+7.6 \pm 1.5$ | $+5.2 \pm 1.0$ |
| Cirigliano *et al.* on ALEPH | $+4.45 \pm 0.70$ | $-6.16 \pm 3.11$ | $+2.86 \pm 0.90$ |
| Cirigliano *et al.* on OPAL | $+5.43 \pm 0.76$ | $-1.35 \pm 3.47$ | $+4.3 \pm 1.2$ |
| Bijnens *et al.* on ALEPH | $+3.4^{+2.4}_{-2.0}$ | $-14.4^{+10.4}_{-8.0}$ | $+1.7 \pm 2.4$ |
| Bijnens *et al.* on OPAL | $+4.0 \pm 2.0$ | $-10.4^{+8.0}_{-6.4}$ | $+2.3 \pm 2.3$ |
| Latorre *et al.* | $+4.0 \pm 2.0$ | $-12^{+7}_{-11}$ | $+2.3 \pm 2.3$ |
| Almasy *et al.* | $3.2^{+1.6}_{-0.4}$ | $-17.0^{+2.5}_{-9.5}$ | $+1.5 \pm 1.5$ |

# Padé approximants and OPE/ChPT relations V



# Summary I

- *Padé approximants are not the right tool to determine the spectrum of QCD*. Such a Padé can only be the approximant to something that is already known from the beginning. Inserting as input the leading order in the Euclidean regime is flawed for many reasons: first and foremost, different spectrums may lead to the same (asymptotic) OPE. In other words, the analytical continuation from the Euclidean to the physical axis depends crucially on the singularities of the correlator, which is actually the result one would like to find. The naive analytical continuation of Migdal is clearly inconsistent with the (assumed) large-$N_c$ limit. Actually, even inserting an input into the correlator is an exercise in circular logic.

# *Summary II*

- In order to study electroweak observables, Padé-type approximants to meromorphic functions, realized in the form of the MHA, have provided valuable aid as interpolators between the high and low energy of the underlying correlators. The results seem to be numerically sound but since the correlators are order parameters, only Pommerenke's theorem can be invoked.
- There is a common misconception: large-$N_c$ and Padé poles and residues can be identified. This is only true when there is infinite information (and then we do not need to use Padé's at all). For a finite amount of information, one needs to leave at least the residues to be fixed.

# *Summary III*

- For some applications (when interpolation is not the main goal) it is preferable to use conventional Padés. For instance, when looking for analytical relations between low and high energy parameters. Given a fixed number of poles, a conventional Padé requires more constraints than a Padé-type. Rule of thumb: if interpolator is needed and there is scarcity of constraints, Padé-types are the best choice. If, on the other hand, one is interested in high energy prediction, we might not want the masses to be fixed. Conventional Padé's are then the right choice.