Positive rational interpolatory quadrature formulas on the unit circle and the interval

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Approximation and extrapolation of convergent and divergent sequences and series. Luminy, France, 2009.

Motivation and aim of the talk

- Orthogonal Rational Functions are a generalization of Orthogonal Polynomials (poles at ∞) and Orthogonal Laurent Polynomials (poles at the origin and ∞).

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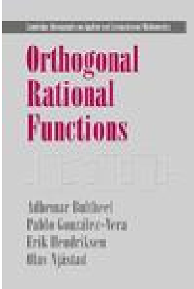

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- In the approximation of weighted integrals supported on the unit circle or the interval, if the integrand is a function with singularities (possible close to, but) outside the unit circle or the interval, then rational interpolatory quadrature formulas are often preferred than interpolatory rules.
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  rational Gauss-type quadrature formulas
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Notation

- $\mathbb{C}$: the complex plane. $\mathbb{R}$: the real line.
- $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$: Riemann sphere.
- $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$: Extended real line.
- $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$: unit circle.
- $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$: open disk.
- $I = [-1, 1]$: the interval.
- $X_Y = \{t \in X : t \notin Y\}$, for $X \subseteq \overline{\mathbb{C}}$ and $Y \subset \overline{\mathbb{C}}$.
- $\mathcal{P}_n$: the space of polynomials of degree less than or equal to n.
- Although z and x are both complex variables, we reserve the notation z for the unit circle and x for the interval.

Notation

- For any complex function $f(t)$, with $t = z$ or $t = x$ we define:

  substar conjugate: $f_*(t) = \overline{f(1/\overline{t})}$.

  super-c conjugate: $f^c(t) = \overline{f(\overline{t})}$.

  Consequently, $f_*^c(t) = f(1/t)$.

- Note that if $f(t)$ has a pole at $t = p$, then

  $f_*(t)$ has a pole at $t = 1/\overline{p}$,

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Spaces of rational functions

- Let a sequence of complex poles $\mathcal{A}_n = \{\alpha_1, \alpha_2, \dots, \alpha_n\} \subset \overline{\mathbb{C}}_I$ be fixed.
- In what follows we will assume that $\alpha_n \in \overline{\mathbb{R}}_I$.
- Define the factors

$$Z_k(x) = \frac{x}{1 - x/\alpha_k}, \quad k = 1, 2, \dots, n,$$

and the basis functions

$$b_0(x) \equiv 1, \quad b_k(x) = b_{k-1}(x) Z_k(x), \quad k = 1, 2, \dots, n.$$

- These basis functions generate the nested spaces of rational functions with poles in $\mathcal{A}_n$:

$$\mathcal{L}_j = \mathrm{span}\{b_0, \dots, b_j\}, \ 0 \leq j \leq n.$$

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  $$\pi_0(x) \equiv 1, \quad \pi_j(x) = \prod_{k=1}^{j} (1 - x/\alpha_k), \; 0 < j \leq n.$$

  Then we may write equivalently $\mathcal{L}_j = \{p_j/\pi_j : p_j \in \mathcal{P}_j\}.$

Spaces of rational functions

- $\mathcal{L}_j^c = \{f : f^c \in \mathcal{L}_j\}$.
- $\mathcal{L}_j$ and $\mathcal{L}_j^c$ are rational generalizations of $\mathcal{P}_j$: if all $\alpha_k = \infty$,

  $Z_k(x) = Z_k^c(x) = x$ and $b_k(x) = b_k^c(x) = x^k$.

- Consider the integral

$$J_\mu(F) = \int_{-1}^{1} F(x)d\mu(x),$$

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- An nth rational interpolatory rule is obtained by integrating an interpolating rational function of degree $n-1$, and is of the form

$$J_n(F) = \sum_{k=1}^{n} \lambda_k F(x_k), \quad \{x_k\}_{k=1}^{n} \subset I, \quad \{\lambda_k\}_{k=1}^{n} \subset \mathbb{R}, \quad x_j \neq x_k \text{ if } j \neq k,$$

so that $J_\mu(F) = J_n(F)$ for every $F \in \mathcal{R}_{p,q} = \mathcal{L}_p \cdot \mathcal{L}_q^c$,

$p + q \leq 2n - 1$ and $0 \leq q \leq p \leq n$.

Lemma

The weights λ_k in the quadrature formula $J_n(F)$ are real, if and only if,

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Orthogonal Rational Functions and Rational Gaussian quadrature formula

- $\varphi_n \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$: nth orthogonal rational function (ORF) w.r.t. the inner product

$$\langle F , G \rangle_\mu = \int_{-1}^{1} F(x) G^c(x) d\mu(x).$$

- Zeros x_k of $\varphi_n(x)$: all distinct and in $(-1, 1)$.
  Hence, they can be chosen as nodes for $J_n(F)$:
  n-point rational Gaussian quadrature formula, which has maximal domain of validity, i.e. the approximation is exact for every function $F \in \mathcal{R}_{n,n-1}$ (taking $p = n$ and $q = n - 1$).

- From the previous Lemma, the weights are real.
  Moreover they are all positive.

Rational Gauss-type quadrature formula

- For any other choice of the nodes: the weights may be non-positive and the quadrature will only be exact in a smaller set of rational functions.
- For each node that is fixed in advance: the domain of validity will generally decrease by one.

  Exactness at least for $p + q = n - 1$.

Special cases:

- If one node is fixed in advance: the weights are all positive and the quadrature is exact for every $F \in \mathcal{R}_{n-1,n-1}$.

  n-point rational Gauss-Radau quadrature formula.

- Whenever two nodes in an $(n+1)$-point quadrature formula are fixed in advance, so that the weights are all positive and the quadrature is exact for every $F \in \mathcal{R}_{n,n-1}$, we obtain the $(n+1)$-point rational Gauss-Lobatto quadrature formula.

- The existence depends on the choice of the prefixed nodes. Polynomial case characterized in

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- Given a sequence of complex numbers

$$\mathcal{B}_n = \{\beta_1, \beta_2, \ldots, \beta_n\} \subset \mathbb{D}, \text{ with } \beta_n \in (-1, 1),$$

we define the Blaschke factors for $\mathcal{B}_n$ as

$$\zeta_k(z) = \eta_k \frac{z - \beta_k}{1 - \overline{\beta}_k z}, \quad \eta_k = \begin{cases} -\frac{\overline{\beta}_k}{|\beta_k|}, & \beta_k \neq 0 \\ 1, & \beta_k = 0 \end{cases}, \; k = 1, 2, \ldots, n,$$

and the corresponding Blaschke products for $\mathcal{B}_n$ as

$$B_0(z) \equiv 1, \quad B_k(z) = B_{k-1}(z)\zeta_k(z), \; k = 1, 2, \ldots, n.$$

Another sequence of basic functions for the unit circle

- These Blaschke products generate the nested spaces of rational functions: $\mathring{\mathcal{L}}_j = \mathrm{span}\{B_0, \ldots, B_j\}$, $0 \leq j \leq n$.
- Define

$$\mathring{\pi}_0(z) \equiv 1, \quad \mathring{\pi}_j(z) = \prod_{k=1}^{j} (1 - \overline{\beta}_k z), \; 0 < j \leq n.$$

Then, we may write equivalently

$$B_j(z) = \upsilon_j \frac{\mathring{\pi}_j^*(z)}{\mathring{\pi}_j(z)}, \quad \upsilon_j = \prod_{k=1}^{j} \eta_k \in \mathbb{T},$$

where $\mathring{\pi}_j^*(z) = z^j \mathring{\pi}_{j*}(z)$, and

$$\mathring{\mathcal{L}}_j = \{p_j / \mathring{\pi}_j : p_j \in \mathcal{P}_j\}.$$

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- Now,
  $\mathring{\mathcal{L}}_{j*} = \{f : f_* \in \mathring{\mathcal{L}}_j\}$, $\mathring{\mathcal{L}}_j^c = \{f : f^c \in \mathring{\mathcal{L}}_j\}$ and
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  $\mathring{\mathcal{L}}_j$ and $\mathring{\mathcal{L}}_j^c$ are rational generalizations of $\mathcal{P}_j$ too:
  if all $\beta_k = 0$, then
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- Consider the integral

$$I_{\mathring{\mu}}(F) = \int_{-\pi}^{\pi} F(z)d\mathring{\mu}(\theta), \; z = e^{i\theta}.$$

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Rational interpolatory quadrature formulas

- Approximation of $I_{\mathring{\mu}}(F)$:

$$I_n(F) = \sum_{k=1}^{n} \mathring{\lambda}_k F(z_k), \quad \{z_k = e^{i\theta_k}\}_{k=1}^{n} \subset \mathbb{T},\ \{\mathring{\lambda}_k\}_{k=1}^{n} \subset \mathbb{R},$$
$$z_j \neq z_k \text{ if } j \neq k,$$

so that $I_{\mathring{\mu}}(F) = I_n(F)$ for every $F \in \mathring{\mathcal{R}}_{p,q} = \mathring{\mathcal{L}}_p \cdot \mathring{\mathcal{L}}_{q*}$, with $n-1 \leq p+q \leq 2n-2$ and $0 \leq q \leq p \leq n-1$.

Lemma

The weights are real now iff $\mathring{\mathcal{R}}_{p,q} = \mathring{\mathcal{R}}_{(p,q)*}$, implying that $p = q$.

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Orthogonal Rational Functions and Rational Szegő quadrature formula

- $\phi_n \in \mathring{\mathcal{L}}_n \setminus \mathring{\mathcal{L}}_{n-1}$: nth ORF with respect to the inner product
$$\langle F, G\rangle_{\mathring{\mu}} = \int_{-\pi}^{\pi} F(z)G_*(z)d\mathring{\mu}(\theta).$$

- Leading coefficient $\mathring{\kappa}_n$, i.e. the coefficient of $B_n(z)$ in the expansion of $\phi_n(z)$ in the basis $\{B_0, \ldots, B_n\}$, is then given by $\mathring{\kappa}_n = \overline{\phi_n^*(\beta_n)}$, where $\phi_n^*(z) = B_n(z)\phi_{n*}(z)$.
We will assume the ORF is monic, i.e. $\overline{\phi_n^*(\beta_n)} = 1$.

- Para-orthogonal rational function (pORF):
$$\mathring{Q}_{n,\tau}(z) = \phi_n(z) + \tau\phi_n^*(z), \quad \tau \in \mathbb{T}.$$

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  Hence, can be chosen as nodes for $I_n(F)$:
  n-point rational Szegő quadrature formula, which has maximal domain of validity $F \in \mathring{\mathcal{R}}_{p,q} = \mathring{\mathcal{L}}_{n-1} \cdot \mathring{\mathcal{L}}_{(n-1)*}$ $(p = q = n-1)$.

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Rational Szegő-type quadrature formulas

- Unlike in the case of the interval: nodes and weights in an n-point rational Szegő quadrature formula are not unique.

- Consequently, an n-point rational Szegő-Radau quadrature formula (one fixed node) always has positive weights and maximal domain of validity too.

- An n-point rational Szegő-Lobatto quadrature formula (two fixed nodes) is at least exact for every $F \in \mathring{\mathcal{R}}_{n-2,n-2}$ and again always has positive weights.

Propaganda for Pablo's talk in $\approx$ 45 minutes!

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Joukowski Transformation

- We denote $x = \frac{1}{2}(z + z^{-1})$ by $x = J(z)$, mapping the open unit disc $\mathbb{D}$ onto the cut Riemann sphere $\overline{\mathbb{C}}_I$ and the unit circle $\mathbb{T}$ onto the interval I.
- When $z = e^{\mathbf{i}\theta}$, then $x = J(z) = \cos\theta$.
- In what follows we will assume that x and z are related by this transformation.
- Inverse mapping: $z = J^{inv}(x)$, chosen so that $z \in \mathbb{D}$ if $x \in \overline{\mathbb{C}}_I$.
- With the sequence $\mathcal{A}_n = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subset \overline{\mathbb{C}}_I$ we associate a sequence $\mathcal{B}_n = \{\beta_1, \beta_2, \ldots, \beta_n\} \subset \mathbb{D}$, so that $\beta_k = J^{inv}(\alpha_k)$, and $\hat{\mathcal{B}}_{2n} = \{\hat{\beta}_1, \ldots, \hat{\beta}_{2n}\} \subset \mathbb{D}$ with

$$\hat{\beta}_{2k} = \overline{\hat{\beta}}_{2k-1} = \beta_k, \quad k = 1, \ldots, n.$$

Positive rational interpolatory quadrature formulas

- In what follows, the measures μ and $\mathring{\mu}$ are related by $\mathring{\mu}'(\theta) = \mu'(\cos\theta)\,|\sin\theta|$.
- By the Joukowski Transform, a function $F(x)$ transforms into a function $\mathring{F}(z) = (F \circ J)(z)$, so that $\mathring{F}(z) = \mathring{F}(z^{-1})$ and $J_{\mu}(F) = \frac{1}{2} I_{\mathring{\mu}}(\mathring{F})$.
- Moreover, every function $F \in \mathcal{L}_k \setminus \mathcal{L}_{k-1}$ transforms into a function $\mathring{F} \in (\mathring{\mathcal{L}}_k^c \cdot \mathring{\mathcal{L}}_{k*}) \setminus (\mathring{\mathcal{L}}_{k-1}^c \cdot \mathring{\mathcal{L}}_{(k-1)*})$.

Positive rational interpolatory quadrature formulas

Lemma

Suppose the numbers $\{\beta_1, \ldots, \beta_{n-1}\}$ are real or appear in complex conjugate pairs. Then,

1. *the zeros of an nth pORF $\mathring{Q}_{n,\tau}(z)$ w.r.t $\mathring{\mu}$ appear in complex conjugate pairs iff $\tau = \pm 1$.*
2. *the nth pORF $\mathring{Q}_{n,\tau_n}(z)$ w.r.t. $\mathring{\mu}$ has a zero in*
   - $z = 1$ *iff* $\tau_n = -\upsilon_n$,
   - $z = -1$ *iff* $\tau_n = (-1)^{n+1}\upsilon_n$,

   where $\upsilon_n \in \{\pm 1\}$ is defined as before.
3. *if $I_n(F) = \sum_{k=1}^{n} \mathring{\lambda}_k F(z_k)$ is an n-point rational Szegő quadrature formula for $I_{\mathring{\mu}}(F)$, based on the zeros of the pORF $\mathring{Q}_{n,\pm 1}(z)$, then for $k = 1, \ldots, n$, the weight $\mathring{\lambda}_k$ corresponding to the node z_k is equal to the weight $\mathring{\lambda}_j$ corresponding to the node $z_j = \overline{z}_k$.*

Generalization of Szegő's relation to the rational case

- $\varphi_n \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$: an nth ORF with respect to the measure μ on I,
  $\phi_n \in \mathring{\mathcal{L}}_n \setminus \mathring{\mathcal{L}}_{n-1}$: the nth monic ORF with respect to the measure $\mathring{\mu}$ on $\mathbb{T}$.

- There exists a nonzero constant C_n so that

$$\varphi_n(x) = C_n B_{n*}(z) \left\{ \hat{\phi}^c_{2n}(z) + \hat{\phi}^*_{2n}(z) \right\} = C_n B_{n*}(z) \hat{\mathring{Q}}_{2n,1}(z).$$

  In what follows, the hat refers to the sequence of numbers defined before.

- Generalization of the connection between Orthogonal Polynomials with respect to the measure $\mathring{\mu}$ on $\mathbb{T}$ and Orthogonal Polynomials with respect to the measure μ on the interval I.

Theorem (Connection between Gauss q.f. and Szegő q.f.)

Let $\mathring{\mu}$ and μ be positive Bounded Borel measures on $\mathbb{T}$ and I respectively and related as before. Let $\hat{I}_{2n}(F) = \sum_{k=1}^{2n} \mathring{\lambda}_k F(z_k)$ be a 2n-point rational Szegő quadrature formula for $I_{\mathring{\mu}}(F) = \int_{-\pi}^{\pi} F(z) d\mathring{\mu}(\theta)$, based on the zeros of the pORF $\mathring{\hat{Q}}_{2n,1}(z) = \hat{\phi}_{2n}(z) + \hat{\phi}^*_{2n}(z)$.

Suppose $z_k \neq \bar{z}_j$ for every $1 \leq k < j \leq n$ and set $z_k = e^{i\theta_k}$ for $k = 1, \ldots, n$. Then, when taking $x_k = \cos\theta_k$ and $\lambda_k = \mathring{\lambda}_k$ for $k = 1, \ldots, n$, the formula $J_n(F) = \sum_{k=1}^{n} \lambda_k F(x_k)$ coincides with the n-point rational Gaussian quadrature formula for $J_\mu(F) = \int_{-1}^{1} F(x) d\mu(x)$, based on the zeros of an nth ORF $\varphi_n \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$.

In the opposite direction,

Theorem (Connection between Gauss q.f. and Szegő q.f.)

Let $J_n(F) = \sum_{k=1}^{n} \lambda_k F(x_k)$ be the n-point rational Gaussian quadrature formula for $J_\mu(F) = \int_{-1}^{1} F(x) d\mu(x)$, based on the zeros of the nth ORF $\varphi_n \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$. Set $x_k = \cos\theta_k$ and define $\{z_k\}_{k=1}^{2n}$ and $\{\mathring{\lambda}_k\}_{k=1}^{2n}$ by means of

$$\left.\begin{array}{ll} z_k = e^{i\theta_k}, & \mathring{\lambda}_k = \lambda_k \\ z_{n+k} = e^{-i\theta_k}, & \mathring{\lambda}_{n+k} = \lambda_k \end{array}\right\} k = 1, \dots, n.$$

Then $\hat{I}_{2n}(F) = \sum_{k=1}^{2n} \mathring{\lambda}_k F(z_k) = \sum_{k=1}^{n} \mathring{\lambda}_k [F(z_k) + F(\overline{z}_k)]$ coincides with a 2n-point rational Szegő quadrature formula for $I_{\mathring{\mu}}(F) = \int_{-\pi}^{\pi} F(z) d\mathring{\mu}(\theta)$, when taking as nodes the zeros of the pORF $\mathring{\hat{Q}}_{2n,1}(z) = \hat{\phi}_{2n}(z) + \hat{\phi}^*_{2n}(z)$.

- Consider the complex varying measure μ_{n-1} defined by

$$d\mu_{n-1}(x) = \frac{1}{2}\left(\beta_{n-1} - \frac{1}{\overline{\beta}_{n-1}}\right)\sqrt{\alpha_n^2 - 1}\,\frac{(1-x^2)}{(\overline{\alpha}_{n-1} - x)\,|\alpha_n - x|}\,d\mu(x).$$

- Let $\tilde{\varphi}_{n-1}(x)$ denote an $(n-1)$th ORF on I w.r.t μ_{n-1}, associated to the sequence $\mathcal{A}_{n-1} \subset \overline{\mathbb{C}}_I$.

- Further, let $\phi_n \in \mathring{\mathcal{L}}_n \setminus \mathring{\mathcal{L}}_{n-1}$ denote the nth monic ORF with respect to the measure $\mathring{\mu}$ on $\mathbb{T}$.

Positive rational interpolatory quadrature formulas

- There exists a nonzero constant C_{n-1} so that

$$\tilde{\varphi}_{n-1}(x) = C_{n-1}\frac{B_{n*}(z)}{\zeta_n^c(z)-\zeta_{n*}(z)}\left\{\hat{\phi}_{2n}^c(z)-\hat{\phi}_{2n}^*(z)\right\}$$
$$= \tilde{C}_{n-1}(1-\beta_n z)^2 B_{(n-1)*}(z)\left\{\frac{\hat{\mathring{Q}}_{2n,-1}(z)}{z^2-1}\right\},$$
$$\tilde{C}_{n-1} = \frac{C_{n-1}}{1-\beta_n^2}$$

- Rational generalization of the connection between OPs with respect to the measure $\mathring{\mu}$ on the unit circle and OPs with respect to the measure $\tilde{\mu}$, with $d\tilde{\mu}(x)=(1-x^2)d\mu(x)$, on the interval, which has been established by Szegő too.

Positive rational interpolatory quadrature formulas

Theorem (Connection between Gauss-Lobatto q.f. and Szegő q.f.)

Let $\hat{I}_{2n}(F) = 2AF(-1) + 2BF(1) + \sum_{k=1}^{2n-2} \mathring{\lambda}_k F(z_k)$ *be a* $2n$*-point rational Szegő quadrature formula for* $I_{\mathring{\mu}}(F) = \int_{-\pi}^{\pi} F(z) d\mathring{\mu}(\theta)$*, based on the zeros of the pORF* $\hat{\mathring{Q}}_{2n,-1}(z) = \hat{\phi}_{2n}(z) - \hat{\phi}^*_{2n}(z)$.

Suppose $z_k \neq \overline{z}_j$ *for every* $1 \leq k < j \leq n-1$ *and set* $z_k = e^{i\theta_k}$ *for* $k = 1, \ldots, n-1$*. Then, when taking* $x_k = \cos\theta_k$ *and* $\lambda_k = \mathring{\lambda}_k$ *for* $k = 1, \ldots, n-1$*, the formula* $J_{n+1}(F) = AF(-1) + BF(1) + \sum_{k=1}^{n-1} \lambda_k F(x_k)$ *coincides with the* $(n+1)$*-point rational Gauss-Lobatto quadrature formula for* $J_{\mu}(F) = \int_{-1}^{1} F(x) d\mu(x)$ *with fixed nodes in* 1 *and* -1 *and based on the zeros of* $\tilde{\varphi}_{n-1}(x)$.

In the opposite direction:

Theorem (Connection between Gauss-Lobatto q.f. and Szegő q.f.)

Let $J_{n+1}(F) = AF(-1) + BF(1) + \sum_{k=1}^{n-1} \lambda_k F(x_k)$ *be the* $(n+1)$*-point rational Gauss-Lobatto quadrature formula for* $J_\mu(F) = \int_{-1}^{1} F(x)d\mu(x)$ *with fixed nodes in* 1 *and* -1*. Set* $x_k = \cos\theta_k$ *and define* $\{z_k\}_{k=1}^{2n-2}$ *and* $\{\mathring{\lambda}_k\}_{k=1}^{2n-2}$ *by means of*

$$\left.\begin{array}{ll} z_k = e^{\mathbf{i}\theta_k}, & \mathring{\lambda}_k = \lambda_k \\ z_{n-1+k} = e^{-\mathbf{i}\theta_k}, & \mathring{\lambda}_{n-1+k} = \lambda_k \end{array}\right\} k = 1, \ldots, n-1.$$

Then, $\hat{I}_{2n}(F) = 2AF(-1) + 2BF(1) + \sum_{k=1}^{2n-2} \mathring{\lambda}_k F(z_k) = 2AF(-1) + 2BF(1) + \sum_{k=1}^{n-1} \mathring{\lambda}_k [F(z_j) + F(\overline{z}_j)]$ *coincides with a* $2n$*-point rational Szegő quadrature formula for* $I_{\mathring{\mu}}(F) = \int_{-\pi}^{\pi} F(z)d\mathring{\mu}(\theta)$*, when taking as nodes the zeros of the pORF* $\hat{\mathring{Q}}_{2n,-1}(z) = \hat{\phi}_{2n}(z) - \hat{\phi}_{2n}^*(z)$.

Theorem

Let ϕ_n denote the monic ORF with respect to the measure $\mathring{\mu}$ on $\mathbb{T}$ and define $\breve{\varphi}_{n-1} \in \mathcal{L}_{n-1}$ by

$$\begin{aligned} \breve{\varphi}^{\pm}_{n-1}(x) &= \frac{B_{(n-1)*}}{1 \pm \eta_n \zeta_n(z)} \left\{ \hat{\phi}_{2n-1}(z) \pm \eta_n \hat{\phi}^{*}_{2n-1}(z) \right\} \\ &= \frac{B_{(n-1)*}}{1 \pm \eta_n \zeta_n(z)} \mathring{\hat{Q}}_{2n-1,\pm\eta_n}(z), \;\; \eta_n = \hat{\upsilon}_{2n-1} \in \{\pm 1\}. \end{aligned}$$

Then it holds that

$$Q^{\pm}_n(x) = \left(\frac{1 \pm x}{1 - x/\alpha_n} \right) \breve{\varphi}^{\pm}_{n-1}(x)$$

is orthogonal to $\mathcal{L}_{n-1}(\alpha_n) = \left\{ \frac{(1-x/\alpha_n)p_{n-2}(x)}{\pi_{n-1}(x)} : p_{n-2} \in \mathcal{P}_{n-2} \right\}$ with respect to the measure μ on I.

Positive rational interpolatory quadrature formulas

- $Q_n^{\pm}(x)$ orthogonal to $\mathcal{L}_{n-1}(\alpha_n)$ means that $Q_n^{\pm}(x)$ is a quasi-orthogonal rational function of the form

$$Q_n^{\pm}(x) = c_n \left\{ \varphi_n(x) + \rho^{\pm} \frac{Z_n(x)}{Z_{n-1}^c(x)} \varphi_{n-1}(x) \right\}, \quad c_n \in \mathbb{C}_0,$$

where $\varphi_k(x)$, $k = n-1, n$, denotes a kth ORF with respect to the measure μ on I, and

$$\rho^{\pm} = -\frac{\varphi_n(\pm 1)}{Z_n(\pm 1)} \cdot \frac{Z_{n-1}^c(\pm 1)}{\varphi_{n-1}(\pm 1)}.$$

- Previous Theorem is also a generalization of OPs with respect to the measure $\mathring{\mu}$ and OPs with respect to the measure $d\mu^{\pm}(x) = (1 \pm x) d\mu(x)$ on the interval.

Positive rational interpolatory quadrature formulas

Theorem (Connection between Gauss-Radau q.f. and Szegő q.f.)

Suppose $\xi \in \{\pm 1\}$. *Let* $\hat{I}_{2n}(F) = 2AF(\xi) + \sum_{k=1}^{2n-2} \mathring{\lambda}_k F(z_k)$ *be a* $(2n-1)$*-point rational Szegő quadrature formula for* $I_{\mathring{\mu}}(F) = \int_{-\pi}^{\pi} F(z) d\mathring{\mu}(\theta)$, *based on the zeros of the pORF* $\hat{\mathring{Q}}_{2n-1,-\xi\eta_n}(z) = \hat{\phi}_{2n-1}(z) - \xi\eta_n \hat{\phi}^*_{2n-1}(z)$.

Suppose $z_k \neq \overline{z}_j$ *for every* $1 \leq k < j \leq n-1$ *and set* $z_k = e^{\mathrm{i}\theta_k}$ *for* $k = 1, \ldots, n-1$. *Then, when taking* $x_k = \cos\theta_k$ *and* $\lambda_k = \mathring{\lambda}_k$ *for* $k = 1, \ldots, n$, *the formula* $J_n(F) = AF(\xi) + \sum_{k=1}^{n-1} \lambda_k F(x_k)$ *coincides with the n-point rational Gauss-Radau quadrature formula for* $J_\mu(F) = \int_{-1}^{1} F(x) d\mu(x)$ *with fixed node in* ξ.

In the opposite direction:

Theorem (Connection between Gauss-Radau q.f. and Szegő q.f.)

Suppose $\xi \in \{\pm 1\}$. *Let* $J_n(F) = AF(\xi) + \sum_{k=1}^{n-1} \lambda_k F(x_k)$ *be the n-point rational Gauss-Radau quadrature formula for* $J_\mu(F) = \int_{-1}^{1} F(x) d\mu(x)$ *with fixed node in* ξ. *Set* $x_k = \cos\theta_k$ *and define* $\{z_k\}_{k=1}^{2n-2}$ *and* $\{\mathring{\lambda}_k\}_{k=1}^{2n-2}$ *by means of*

$$\left.\begin{array}{ll} z_k = e^{\mathbf{i}\theta_k}, & \mathring{\lambda}_k = \lambda_k \\ z_{n-1+k} = e^{-\mathbf{i}\theta_k}, & \mathring{\lambda}_{n-1+k} = \lambda_k \end{array}\right\} k = 1, \ldots, n-1.$$

Then $\hat{I}_{2n-1}(F) = 2AF(\xi) + \sum_{k=1}^{2n-2} \mathring{\lambda}_k F(z_k) = 2AF(\xi) + \sum_{k=1}^{n-1} \mathring{\lambda}_k [F(z_k) + F(\overline{z}_k)]$ *coincides with a* $(2n-1)$*-point rational Szegő quadrature formula for* $I_{\mathring{\mu}}(F) = \int_{-\pi}^{\pi} F(z) d\mathring{\mu}(\theta)$, *when taking as nodes the zeros of the pORF* $\hat{\mathring{Q}}_{2n-1,-\xi\eta_n}(z) = \hat{\phi}_{2n-1}(z) - \xi\eta_n \hat{\phi}_{2n-1}^{*}(z)$.

A result for $\tau \neq \pm 1$

- For $\tau \neq \pm 1$ and under the condition that $\mathcal{L}_{n-1} = \mathcal{L}^c_{n-1}$, the *n*-point rational Szegő quadrature formulas transform into *n*-point rational interpolatory quadrature formulas on the interval with positive weights, which are only exact in $\mathcal{L}_{n-1}$ unless some particular choice of τ is made.

- Generalization to the rational case a result obtained in A. Bultheel, L. Daruis and P. González-Vera.- **Positive interpolatory quadrature formulas and para-orthogonal polynomials**, *Journal Computational and Applied Mathematics, 179(1–2):97–119, 2005.*

A result for $\tau \neq \pm 1$

Theorem

Suppose the poles $\{\alpha_1, \ldots, \alpha_{n-1}\}$ *(and hence, the numbers* $\{\beta_1, \ldots, \beta_{n-1}\}$*) are real or appear in complex conjugate pairs, i.e.* $\mathcal{L}_{n-1} = \mathcal{L}_{n-1}^c$ *and* $\mathring{\mathcal{L}}_{n-1} = \mathring{\mathcal{L}}_{n-1}^c$. *Let* $I_n(\mathring{F}) = \sum_{k=1}^{n} \mathring{\lambda}_k \mathring{F}(z_j)$ *be an nth rational Szegő quadrature formula with respect to* $\mathring{\mu}$*, where the nodes* $\{z_k = e^{i\theta_k}\}_{k=1}^{n}$ *are the zeros of the pORF* $\mathring{Q}_{n,\tau}(z) = \phi_n(z) + \tau\phi_n^*(z)$ *with* $\tau \neq \pm 1$. *Set* $x_k = \cos\theta_k$, $k = 1, \ldots, n$ *and* $\lambda_k = \mathring{\lambda}_k/2 > 0$, *and consider the n-point rational interpolatory quadrature formula based upon these nodes and weights* $J_n^{\tau}(F) = \sum_{k=1}^{n} \lambda_k F(x_k)$ *for* $J_\mu(F) = \int_{-1}^{1} F(x) d\mu(x)$. *Then this rational interpolatory quadrature formula is exact for every* $F \in \mathcal{L}_{n-1}$. *Furthermore, it is exact in* $\mathcal{L}_n$ *iff*

$$\tau = \tau_{optimal} = -\phi_n(\beta_n) \pm i\sqrt{1 - \phi_n^2(\beta_n)}.$$

Research in progress

- Numerical experiments.
- Error bounds for the quadrature rules considered in the talk.

Thanks for your attention!

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