

Computation of Rational Szegő-Lobatto Quadrature Formulas

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Joint work with:

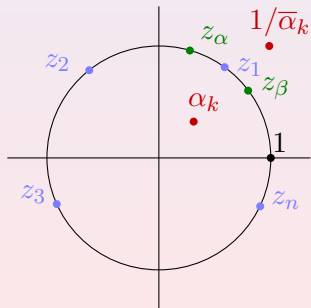
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Approximate calculation of integrals on the unit circle.

$$I_\omega(f) = \int_{-\pi}^{\pi} f(e^{i\theta})\omega(\theta)d\theta \approx Af(z_\alpha) + Bf(z_\beta) + \sum_{j=1}^n \lambda_j f(z_j) = I_{n+2}(f)$$



$$A, B, \lambda_j > 0$$

$$z_j \neq z_k, j \neq k$$

$$z_j \notin \{z_\alpha, z_\beta\}$$

$I_\omega(f) = I_{n+2}(f)$ with f in a subspace of rational functions with prescribed poles at α_k and $1/\bar{\alpha}_k$ of as high dimension as possible.

- I Introduction: Gauss, Gauss-Radau and Gauss-Lobatto formulas.
- II Periodic integrands: Szegő-type quadrature formulas.
- III Rational quadratures on the unit circle.
- IV Rational Szegő-Lobatto formulas.
- V Error and convergence.

$$I_\sigma(f) = \int_a^b f(x)\sigma(x), \quad -\infty \leq a < b \leq +\infty$$

There exist n distinct nodes $\{x_j\}_{j=1}^n \subset (a, b)$ and positive weights $\{A_j\}_{j=1}^n$ such that,

$$I_\sigma(P) = I_n(P) = \sum_{j=1}^n A_j P(x_j), \quad \forall P \in \mathbb{P}_{2n-1}$$

$$\mathbb{P}_{2n-1} = \text{span}\{x^k : 0 \leq k \leq 2n-1\}, \quad \mathbb{P} = \text{span}\{x^k, k = 0, 1, \dots\}$$



Gauss, Gauss-Christtophel or Gaussian quadrature formulas.

- Maximal domain of validity
- Positivity of the weights \Rightarrow Stability and convergence.
- Efficient computation in terms of an eigenvalue problem involving certain tridiagonal Jacobi matrices.

Gauss, Gauss-Radau and Gauss-Lobatto formulas

$[a, b]$ finite. $[a, b] = [-1, 1]$.

$$I_\sigma(f) = \int_{-1}^{+1} f(x)\sigma(x)dx$$

Theorem

Given $r, s \in \{0, 1\}$ there exist positive weights $A_+^{(r,s)}$, $A_-^{(r,s)}$ and $\{A_j^{(r,s)}\}_{j=1}^n$ along with n distinct nodes $\{x_j^{(r,s)}\} \subset (-1, 1)$ such that,

$$I_{n+r+s}(f) = rA_+^{(r,s)}f(1) + sA_-^{(r,s)}f(-1) + \sum_{j=1}^n A_j^{(r,s)}f(x_j^{(r,s)}) = I_\sigma(f)$$

$\forall f \in \mathbb{P}_{2n+r+s-1}$

- $r + s = 0 \Rightarrow$ Gauss Formulas.
- $r + s = 1 \Rightarrow$ Gauss-Radau Formulas.
- $r + s = 2 \Rightarrow$ Gauss-Lobatto Formulas.



Gauss type Formulas

Szegő-type quadratures

$$I_\omega(f) = \int_{-\pi}^{\pi} f(\theta)\omega(\theta)d\theta$$

f : 2π -periodic. $\omega(\theta)$: 2π -periodic weighted function.

$$I_n(f) = \sum_{j=1}^n \lambda_j f(\theta_j); \{\theta_j\} \subset [-\pi, \pi), \theta_j \neq \theta_k \text{ if } j \neq k$$

$$I_\omega(T) = I_n(T) : T(\theta) = \sum_{k=0}^N (a_k \cos k\theta + b_k \sin k\theta)$$

T : trigonometric polynomial with as high degree as possible.

$$N \leq n - 1$$

$N = n - 1 \Rightarrow$ Quadrature formulas with the highest trigonometric precision degree.

(Bi-orthogonal systems of trigonometric polynomials)

Szegő-type quadratures

Alternative approach

Passing to the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

Some notations

$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $\mathbb{E} = \{z \in \mathbb{C} : |z| > 1\}$.

$p, q \in \mathbb{Z}$, $p \leq q$, $\Lambda_{p,q} = \text{span}\{z^k : p \leq k \leq q\}$ (Laurent polynomials)

$\Lambda \equiv$ space of all Laurent polynomials.

$$T(\theta) = \sum_{k=0}^m (a_k \cos k\theta + b_k \sin k\theta) = L(e^{i\theta}), \quad L \in \Lambda_{-m,m}$$

$$I_\omega(f) = \int_{-\pi}^{\pi} f(e^{i\theta}) \omega(\theta) d\theta \approx I_n(f) = \sum_{j=1}^n \lambda_j f(z_j)$$

$z_j \in \mathbb{T}$, $z_j \neq z_k$ if $j \neq k$. $z_j = e^{i\theta_j}$, $\theta_j \in [-\pi, \pi)$, $\theta_j \neq \theta_k$.

$I_\omega(f) = I_n(f)$, $\forall f \in \Lambda_{-(n-1), (n-1)}$.

Szegő-type quadratures

Some notations

$\{\rho_k\}_0^\infty$: monic Szegő polynomials associated with $\omega(\theta)$, i.e.
 $\rho_0(z) = 1, \rho_n(z) = z^n + \dots + \delta_n, n = 1, 2, \dots, z = e^{i\theta}, n \geq 1,$
 $\langle \rho_n(z), z^k \rangle_\omega = \int_{-\pi}^{\pi} \rho_n(z) \overline{z^k} \omega(\theta) d\theta = 0, k = 0, 1, \dots, n-1.$
 $\rho_n^*(z) = z^n \overline{\rho_n(1/\bar{z})}$ (Reverse polynomial)

Theorem

Given $I_\omega(f) = \int_{-\pi}^{\pi} f(e^{i\theta}) \omega(\theta) d\theta$, then $I_n(f) = \sum_{j=1}^n \lambda_j f(z_j) = I_\omega(f), \forall f \in \Lambda_{-(n-1), (n-1)}$, if and only if,

- 1 The nodes $\{z_j\}_{j=1}^n$ are the zeros of:

$$B_n(z, \tau) = z \rho_{n-1}(z) + \tau \rho_{n-1}^*(z), |\tau| = 1.$$

- 2 $\lambda_j = \left[\sum_{k=0}^{n-1} \frac{|\rho_k(z_j)|^2}{\langle \rho_k, \rho_k \rangle_\omega} \right]^{-1}, j = 1, \dots, n.$

Szegő-type quadratures

- Szegő quadrature formulas ([W. B. Grags \(1982,1991\)](#), [Jones et als \(1989\)](#))
- Efficient computation in terms of an eigenvalue problem involving Hessenberg Matrices whose entries essentially depend on the numbers:

$$\delta_k = \rho_k(0), \rho_0 = 1, |\delta_k| < 1, k = 1, 2, \dots$$

(Verblunsky coefficients, Schur parameters, Reflection coefficients).

Szegő-Lobatto Quadrature rules

C. Jagels, L. Reichel (2007)

A. Bultheel, L. Daruis, P. G-V (2009)

$$I_{\omega}(f) = \int_{-\pi}^{\pi} f(e^{i\theta})\omega(\theta)d\theta$$

Given: $z_{\alpha} \neq z_{\beta}$ on \mathbb{T} , find n distinct nodes $\{z_j\}_{j=1}^n \subset \mathbb{T}$, $z_j \notin \{z_{\alpha}, z_{\beta}\}$ and $n + 2$ positive weights A, B and $\{\lambda_j\}_{j=1}^n$ such that:

$$I_{\omega}(f) = I_{n+2}(f) = Af(z_{\alpha}) + Bf(z_{\beta}) + \sum_{j=1}^n \lambda_j f(z_j), \forall f \in \Lambda_{-n,n}.$$

Szegő-type quadratures

Basic approach

$$\omega(\theta) \rightsquigarrow \{\rho_k\}_0^\infty$$

Available information: $\mu_k = \int_{-\pi}^{\pi} e^{-ik\theta} \omega(\theta) d\theta$, $k = 0, \pm 1, \pm 2, \dots$

Recursion: $\rho_k(z) = z\rho_{k-1}(z) + \delta_k \rho_{k-1}^*(z)$, $\rho_0 = 1$, $\delta_k = \rho_k(0)$,
 $k = 1, 2, \dots$ (Levinson algorithm).

From $\rho_n(z)$ and given $\tilde{\delta}_{n+1} \in \mathbb{D}$. ($|\tilde{\delta}_{n+1}| < 1$).

Define $\tilde{\rho}_{n+1}(z) = z\rho_n(z) + \tilde{\delta}_{n+1}\rho_n^*(z)$.

Now take $\tilde{\tau}_{n+2} \in \mathbb{T}$, ($|\tilde{\tau}_{n+2}| = 1$) and consider

$$\tilde{B}_{n+2}(z) = z\tilde{\rho}_{n+1}(z) + \tilde{\tau}_{n+2}\tilde{\rho}_{n+1}^*(z)$$

$\tilde{B}_{n+2}(z)$ has $n + 2$ distinct zeros on \mathbb{T} .

AIM

Determine $\tilde{\delta}_{n+1} \in \mathbb{D}$ and $\tilde{\tau}_{n+2} \in \mathbb{T}$ such that

$$\tilde{B}_{n+2}(z_\alpha) = \tilde{B}_{n+2}(z_\beta) = 0$$

Rational Quadrature Formulas on the Unit Circle

- Gauss-type Formulas: Exact integration of polynomials (all the poles at infinity)
- Szegő-type formulas: Exact integration of Laurent polynomials (all the poles at the origin and infinity)



Given $\int_{-\pi}^{\pi} f(e^{i\theta})\omega(\theta)d\theta \Rightarrow$ quadrature formulas exactly integrating more general rational functions with prescribed poles not on \mathbb{T} .

The poles: $\{\alpha_k\} \subset \mathbb{D}$; $\{1/\bar{\alpha}_k\} \subset \mathbb{E}$



Theory on Orthogonal Rational Functions
(A. Bultheel, E. Hendriksen, P. G-V, O. Njastad)

Rational Quadrature Formulas on the Unit Circle

The spaces of rational functions

$$\{\alpha_k\}_1^\infty \subset \mathbb{D}$$

Blaschke Factors

$$\xi_i(z) = \eta_i \frac{\alpha_i - z}{1 - \bar{\alpha}_i z}; \quad \eta_i = \frac{\alpha_i}{|\alpha_i|} \text{ if } \alpha_i \neq 0; \quad \eta_i = -1 \text{ if } \alpha_i = 0, \quad i=1,2,\dots$$

Blaschke Products

$$B_0 = 1, \quad B_n = B_{n-1} \xi_n, \quad n \geq 1.$$

$$\text{Set } \mathcal{L}_n = \text{span}\{B_k : k = 0, 1, \dots, n\}$$

$$\omega_0 = \pi_0 = 1; \quad \omega_k = \prod_{j=1}^k (z - \alpha_j); \quad \pi_k = \prod_{j=1}^k (1 - \bar{\alpha}_j z), \quad k = 1, 2, \dots$$

$$B_k(z) = \gamma_k \frac{\omega_k(z)}{\pi_k(z)}; \quad \gamma_k = (-1)^k \prod_{j=1}^k \eta_j;$$

$$\mathcal{L} = \left\{ R = \frac{P}{\pi_n} : P \in \mathbb{P}_n \right\} = \left\{ R(z) = \frac{P(z)}{(1 - \bar{\alpha}_1 z) \cdots (1 - \bar{\alpha}_n z)} \right\}$$

The spaces of rational functions

$$f_*(z) = \overline{f(1/\bar{z})} \Rightarrow \mathcal{L}_{n*} = \{f : f_* \in \mathcal{L}_n\} = \left\{f = \frac{Q}{\omega_n}, Q \in \mathbb{P}_n\right\}$$

$$= \left\{f = \frac{Q(z)}{(z - \alpha_1) \cdots (z - \alpha_n)}\right\}$$

$$\mathcal{R}_{p,q} = \mathcal{L}_{p*} + \mathcal{L}_q = \left\{f = \frac{P}{\omega_p \pi_q} : P \in \mathbb{P}_{p+q}\right\}$$

$$= \text{span}\{B_k, k = -p, \dots, -1, 0, 1, \dots, q\}$$

$$B_{-k} = B_{k*} = \frac{1}{B_k}$$

$$\alpha_k = 0, k = 1, 2, \dots, \mathcal{R}_{p,q} = \Lambda_{-p,q}$$

Rational Quadrature Formulas on the Unit Circle

$$I_\omega(f) = \int_{-\pi}^{\pi} f(e^{i\theta}) \omega(\theta) d\theta$$

$$I_n(f) = \sum_{j=1}^n \lambda_j f(z_j), \quad z_j \in \mathbb{T}, \quad z_j \neq z_k \text{ if } j \neq k$$

$I_\omega(R) = I_n(R), \forall R \in \mathcal{R}_{p(n),p(n)}, p(n)$ as large as possible.

$$\boxed{p(n) \leq n - 1} \quad p(n) = n - 1?$$

Orthogonal Rational Functions

$n \geq 1$, $\{\varphi_j\}_{j=0}^n$, $\varphi_j \in \mathcal{L}_j \setminus \mathcal{L}_{n-1}$ an orthogonal basis for \mathcal{L}_n
(Gram-Schmidt process on $\{B_k\}_0^n$)

$$\varphi_k(z) = \sum_{j=0}^k a_j B_j(z), \quad a_k \neq 0 \text{ leading coefficient}$$

$a_k = 1 \Rightarrow \varphi_k(z) \equiv$ monic.

When the process is repeated $\forall n \geq 1 \Rightarrow \{\varphi_k\}_0^\infty$ a sequence of ORF
w. r. t. $\omega(\theta)$ and the poles $\{\alpha_k\}_1^\infty$. (Rational Szegő Functions)

Drawback

The zeros of $\varphi_n(z)$ lie in \mathbb{D} .

Theorem (Rational Szegő Quadratures)

Let $\{\varphi_k\}_0^\infty$ be the sequence of monic Rational Szegő Functions.

For each $n \geq 1$, take $\tau \in \mathbb{T}$ and set

$$X_n(z) = X_n(z, \tau) = (z - \alpha_{n-1})\overline{\varphi_{n-1}(z)} + \tau(1 - \overline{\alpha_{n-1}}z)\varphi_{n-1}^*(z),$$

$(\varphi_k^*(z) = B_k(z)\varphi_{k*}(z) = B_k(z)\overline{\varphi_k(1/\bar{z})})$. Then,

- 1 $X_n(z)$ has exactly n distinct nodes z_1, \dots, z_n which lie on \mathbb{T} .
- 2 There exist positive numbers $\lambda_1, \dots, \lambda_n$ such that

$$I_n(f) = \sum_{j=1}^n \lambda_j f(z_j) = I_\omega(f) = \int_{-\pi}^{\pi} f(e^{i\theta})\omega(\theta)d\theta, \forall f \in \mathcal{R}_{n-1, n-1}$$

- 3
$$\lambda_j = \sum_{k=0}^{n-1} \frac{|\varphi_k(z_j)|^2}{\|\varphi_k\|_\omega^2}, j = 1, \dots, n,$$

$$\|\varphi_k\|_\omega^2 = \langle \varphi_k, \varphi_k \rangle_\omega = \int_{-\pi}^{\pi} \varphi_k(z)\overline{\varphi_k(z)}\omega(\theta)d\theta, (z = e^{i\theta})$$

- 4 There can not exist an n -point quadrature rule with nodes on \mathbb{T} to be exact either in $\mathcal{R}_{n, n-1}$ or $\mathcal{R}_{n-1, n}$.

An Illustrative Example

$$\omega(\theta) = \frac{1}{2\pi}, \theta \in [-\pi, \pi], \{\alpha_k\} \subset \mathbb{D}.$$

$\{\varphi\}_0^\infty$: the Orthogonal Sequence of monic Rational Szegő Function.

$$\langle \varphi_k, \varphi_k \rangle_\omega = 1, k = 0, 1, 2, \dots \varphi_n(z) = \kappa_n \frac{z B_n(z)}{z - \alpha_n}, \kappa_n \neq 0, n = 1, 2, \dots$$

$$\tau \in \mathbb{T} \Rightarrow I_n(f) = \sum_{j=1}^n \lambda_j f(z_j) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) d\theta, \forall f \in \mathcal{R}_{n-1, n-1}$$

$$z_j : z \omega_{n-1}(z) + \tau \Pi_{n-1}(z) = 0$$

$$\omega_{n-1}(z) = \prod_{k=1}^{n-1} (z - \alpha_k); \pi_{n-1}(z) = \prod_{k=1}^{n-1} (1 - \bar{\alpha}_k z)$$

$$\lambda_j = \frac{1}{1 + \sum_{k=1}^{n-1} \frac{1 - |\alpha_k|^2}{|z_j - \alpha_k|^2}}, j = 1, \dots, n$$

$$\alpha_k = 0, k = 1, 2, \dots \Rightarrow z_j : z^n + \tau = 0, \lambda_j = \frac{1}{n}, j = 1, \dots, n.$$

Rational Szegő-Lobatto Formulas

Given $x_1, x_2 \in \mathbb{T}$, $x_1 \neq x_2$, $n \geq 1$. Find n distinct nodes z_1, \dots, z_n on \mathbb{T} , $z_j \notin \{x_1, x_2\}$ and positive weights $A_1, A_2, \lambda_1, \dots, \lambda_n$ such that:

$$\tilde{I}_{n+2}(f) = A_1 f(x_1) + A_2 f(x_2) + \sum_{j=1}^n \lambda_j f(z_j) = I_\omega(f)$$

$\forall f \in \mathcal{R}_{p(n), p(n)}$, $p(n)$ as large as possible.

$n + 2$ nodes on $\mathbb{T} \Rightarrow p(n) \leq n + 1$.

Rational Szegő-Lobatto Formulas

- $p(n) = n + 1 \Leftrightarrow \tilde{I}_{n+2}(f)$ a Rational Szegő Quadrature Formula $\Leftrightarrow \gamma_n(x_1) = \gamma_n(x_2)$ with

$$\gamma_n(x_j) = \frac{(\alpha_{n-1} - x_j)\varphi_{n-1}(x_j)}{(1 - \bar{\alpha}_{n-1}x_j)\varphi_{n-1}^*(x_j)} \in \mathbb{T}, \quad j = 1, 2.$$

$\varphi_{n-1}(z)$: $(n - 1)$ -th monic Rational Szegő Function

- $\gamma_n(x_1) \neq \gamma_n(x_2) \Rightarrow p(n) \leq n$
If $p(n) = n \Rightarrow \tilde{I}_{n+2}(f)$ Rational Szegő-Lobatto Quadrature Formula.

Rational Szegő-Lobatto Formulas

The Approach

μ : positive Borel Measure on \mathbb{T} ;

$\{\alpha_k\}_1^\infty \subset \mathbb{D}$: The poles of the rational functions.

$\{\varphi_k\}_0^\infty$: Sequence of **Rational Szegő Functions**.

Recurrence Relation

$$\varphi_n(z) = \varepsilon_n \left[\left(\frac{z - \alpha_{n-1}}{1 - \bar{\alpha}_n z} \right) \varphi_{n-1}(z) + \left(\frac{\delta_n}{\varepsilon_n} \right) \left(\frac{1 - \bar{\alpha}_{n-1} z}{1 - \bar{\alpha}_n z} \right) \varphi_{n-1}^*(z) \right]$$

$$\varphi_n^*(z) = -\eta_n \bar{\varepsilon}_n \left[\overline{\left(\frac{\delta_n}{\varepsilon_n} \right)} \left(\frac{z - \alpha_{n-1}}{1 - \bar{\alpha}_n z} \right) \varphi_{n-1}(z) + \left(\frac{1 - \bar{\alpha}_{n-1} z}{1 - \bar{\alpha}_n z} \right) \varphi_{n-1}(z) \right]$$

$$\varphi_n^*(z) = B_n(z) \varphi_{n*}(z) = B_n(z) \overline{\varphi_n(1/\bar{z})}, \quad n \geq 1.$$

$$B_0(z) = 1, \quad B_k(z) = (-1)^k \prod_{j=1}^k \eta_j \frac{(z - \alpha_1) \cdots (z - \alpha_k)}{(1 - \bar{\alpha}_1 z) \cdots (1 - \bar{\alpha}_k z)}, \quad \eta_j = \begin{cases} \frac{\bar{\alpha}_j}{|\alpha_j|}, & \alpha_j \neq 0 \\ -1, & \alpha_j = 0 \end{cases}$$

$$\varepsilon_n \neq 0; \quad |\delta_n| < |\varepsilon_n|, \quad \beta_n = \frac{\delta_n}{\varepsilon_n}, \quad |\beta_n| < 1.$$

Theorem

Let μ be a positive Borel measure on \mathbb{T} , $\{\varphi_n\}_0^\infty$ being its sequence of monic Rational Szegő Functions with respect to the fixed sequence of poles $\{\alpha_k\}_1^\infty \subset \mathbb{D}$. Then, given $\beta_{n+1} \in \mathbb{D}$, there exists a positive Borel measure $\tilde{\mu}$ on \mathbb{T} such that if $\{\tilde{\varphi}_k\}_0^\infty$ represents its monic sequence, it holds that

- $\tilde{\varphi}_k(z) = \varphi_k(z), k = 0, 1, \dots, n.$
- $\tilde{\varphi}_{n+1}(z) = \tilde{\varepsilon}_{n+1} \left[\left(\frac{z - \alpha_n}{1 - \bar{\alpha}_{n+1}z} \right) \varphi_n(z) + \beta_{n+1} \left(\frac{1 - \bar{\alpha}_n z}{1 - \bar{\alpha}_{n+1}z} \right) \varphi_n^*(z) \right],$
 $\tilde{\varepsilon}_{n+1} \neq 0.$

Taking $d\mu(\theta) = \omega(\theta)d\theta$; $d\tilde{\mu}(\theta) = \tilde{\omega}(\theta)d\theta$;

$$\Rightarrow I_\omega(R) = I_{\tilde{\omega}}(R), \quad \forall R \in \mathcal{R}_{n,n}$$

Rational Szegő-Lobatto Formulas

Take $\tau_{n+2} \in \mathbb{T}$ and set

$$X_{n+2}(z) = (z - \alpha_{n+1})\tilde{\varphi}_{n+1}(z) + (1 - \bar{\alpha}_{n+1}z)\tilde{\varphi}_{n+1}^*(z)$$

$X_{n+2}(z)$ has $n+2$ distinct zeros $\tilde{z}_1, \dots, \tilde{z}_{n+2}$ on \mathbb{T} and there exist positive numbers $\tilde{\lambda}_1, \dots, \tilde{\lambda}_{n+2}$ such that

$$\tilde{I}_{n+2}(f) = \sum_{j=1}^{n+2} \tilde{\lambda}_j f(\tilde{z}_j) = I_{\tilde{\omega}}(f), \quad \forall f \in \mathcal{R}_{n+1, n+1}.$$

Since

$$I_{\omega}(R) = I_{\tilde{\omega}}(R), \quad \forall R \in \mathcal{R}_{n, n} \Rightarrow \boxed{\tilde{I}_{n+2}(R) = I_{\omega}(R), \quad \forall R \in \mathcal{R}_{n, n}}$$

$X_{n+2}(z) \in \mathcal{L}_{n+2}$ depends on two arbitrary parameters $\beta_{n+1} \in \mathbb{D}$ and $\tau_{n+2} \in \mathbb{T}$.

Theorem (Main Result)

Given a natural number n and two distinct complex numbers $x_1, x_2 \in \mathbb{T}$ along with a weight function $\omega(\theta)$ on \mathbb{T} there exist $\tilde{\beta}_{n+1} \in \mathbb{D}$ and $\tilde{\tau}_{n+2} \in \mathbb{T}$ such that:

$$B_{n+2}(x_1) = B_{n+2}(x_2) = 0$$

where

$$B_{n+2}(z) = (z - \alpha_{n+1})\tilde{\varphi}_{n+1}(z) + (1 - \bar{\alpha}_{n+1}z)\tilde{\tau}_{n+2}\tilde{\varphi}_{n+1}^*(z)$$

and

$$\tilde{\varphi}_{n+1}(z) = \left(\frac{z - \alpha_n}{1 - \bar{\alpha}_{n+1}z} \right) \varphi_n(z) + \tilde{\beta}_{n+1} \left(\frac{1 - \bar{\alpha}_n z}{1 - \bar{\alpha}_{n+1}z} \right) \varphi_n^*(z)$$

$\varphi_n(z)$ being the n -th monic Rational Szegő Function for $\omega(\theta)$ and the given sequence of poles $\{\alpha_k\}_1^\infty \subset \mathbb{D}$.

Corollary

Given x_1 and $x_2 \in \mathbb{T}$ ($x_1 \neq x_2$) there exist n distinct nodes $\tilde{z}_1, \dots, \tilde{z}_n$ on \mathbb{T} with $\tilde{z}_j \notin \{x_1, x_2\}$, $j = 1, \dots, n$ and $n + 2$ positive weights A_1, A_2 and λ_j , $j = 1, \dots, n$ such that

$$\tilde{I}_{n+2}(f) = A_1 f(x_1) + A_2 f(x_2) + \sum_{j=1}^n \tilde{\lambda}_j f(\tilde{z}_j) = I_\omega(f)$$

$$\forall f \in \mathcal{R}_{n,n} = \left\{ \frac{P(z)}{(z - \alpha_1) \cdots (z - \alpha_n)(1 - \bar{\alpha}_1 z) \cdots (1 - \bar{\alpha}_n z)}; P \in \mathbb{P}_{2n} \right\}$$

Error and Convergence

Error

Assume $x_1 \neq x_2$ both on \mathbb{T} . Consider

$$\tilde{I}_{n+2}(f) = A_1 f(x_1) + A_2 f(x_2) + \sum_{j=1}^n \tilde{\lambda}_j f(\tilde{z}_j)$$

and set $E_{n+2}(f) = I_\omega(f) - \tilde{I}_{n+2}(f)$. Suppose f analytic on a neighborhood of \mathbb{T} and let U be the domain of analyticity of $\omega_n(z)\pi_n(z)f(z)$. Then,

$$|E_{n+2}(f)| \leq 2 \|\omega_n \pi_n f\|_{\gamma_r \cup \gamma_R} \left(\frac{r^{n+1}}{1-r^2} g_n(r) + \frac{R^{1-n}}{R^2-1} g_n\left(\frac{1}{R}\right) \right)$$

where $r < 1 < R$ such that $\gamma_r \cup \gamma_R \subset U$ and

$$g_n(s) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \prod_{j=1}^n \frac{1}{|1 - se^{i\theta} \alpha_j|^2} d\theta, \quad |s| < 1$$

$a > 0$; $\gamma_a = \{z \in \mathbb{C} : |z| = a\}$, $\|f\|_K = \max_{z \in K} \{|f(z)|\}$

Error and Convergence

Convergence

For each $n \geq 1$ take $x_{1,n}, x_{2,n} \in \mathbb{T}$, $x_{1,n} \neq x_{2,n}$

$$\tilde{I}_{n+2}(f) = A_{1,n}f(x_{1,n}) + A_{2,n}f(x_{2,n}) + \sum_{j=1}^n \tilde{\lambda}_{j,n}f(\tilde{z}_{j,n})$$

(Rational Szegő-Lobatto formula)

Recall: $\tilde{I}_{n+2} = I_\omega(f)$, $\forall f \in \mathcal{R}_{n-n}$, $\lim_{n \rightarrow \infty} \tilde{I}_{n+2}(f) = I_\omega(f)$?

Lemma

Set $\mathcal{R} = \cup_{n=0}^{\infty} \mathcal{R}_{n,n}$, then \mathcal{R} is dense in the class $\mathcal{C}(\mathbb{T})$ of continuous functions on \mathbb{T} , iff $\sum_{k=1}^{\infty} (1 - |\alpha_k|) = +\infty$

Theorem

Set $R_\omega(\mathbb{T}) = \{f : \mathbb{T} \rightarrow \mathbb{C} : f\omega \text{ integrable on } \mathbb{T}\}$; then

$$\lim_{n \rightarrow \infty} \tilde{I}_{n+2}(f) = I_\omega(f) = \int_{-\pi}^{\pi} f(e^{i\theta})\omega(\theta)d\theta, \forall f \in R_\omega(\mathbb{T})$$

if $\sum_{k=1}^{\infty} (1 - |\alpha_k|) = +\infty$