

# Asymptotic expansions and Mellin-Barnes representation

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# Context: Phenomenology in Quantum Field theory

Mellin-Barnes representation may be of use for at least two important questions in the phenomenology of QFT:

- 1 Perturbative calculations in QFT imply to evaluate numerous Feynman diagrams often with several masses and momenta. How can we evaluate them analytically ?
- 2 Perturbative expressions are series in powers of the coupling constants. How can we have non-perturbative (*i.e.* exponentially suppressed) informations ?

# Why using Mellin-Barnes representation?

- The first observation is that the Mellin transform has the following scale property

$$\mathcal{M}[f(ax)](s) = a^{-s} \mathcal{M}[f(x)](s)$$

- Renormalization Group solutions lead to consider asymptotic behaviours of the diagrams as  $a^\alpha \ln^\beta a$ . The Mellin transformation kernel is the most pertinent to obtain this type of asymptotic behaviour.
- The Mellin-Barnes representation allows expansion in several parameters and it also gives an explicit formula for the remainder that permits a control on perturbative and (in some cases) non-perturbative expansions.

# Mellin-Barnes representation

and

# Feynman diagrams calculation

S. Friot, D. Greynat and E. de Rafael, *Phys. Lett. B* **628**, 73 (2005)

J.-Ph. Aguilar, D. Greynat and E. de Rafael, *Phys. Rev. D* **77**, 093010 (2008)

# One dimensional Mellin Transform

The Mellin transform of a function  $f$  and its inverse transform are defined as

$$\mathcal{M}[f(x)](s) \doteq \int_0^{\infty} dx x^{s-1} f(x) \quad \longleftrightarrow \quad f(x) = \int_{c-i\infty}^{c+i\infty} \frac{ds}{2i\pi} x^{-s} \mathcal{M}[f(x)](s)$$

If and only if

$$c \doteq \operatorname{Re} s \in ]\alpha, \beta[ \quad \text{written} \quad \langle \alpha, \beta \rangle \quad \text{Fundamental strip}$$

It corresponds to the behaviours

$$f(x) \underset{x \rightarrow 0^+}{=} \mathcal{O}(x^{-\alpha}) \quad \& \quad f(x) \underset{x \rightarrow +\infty}{=} \mathcal{O}(x^{-\beta})$$

$$\begin{aligned} (1+x)^{-1} &\longleftrightarrow \frac{\pi}{\sin \pi s} && \langle 0, 1 \rangle \\ (1+x)^{-\nu} &\longleftrightarrow \frac{\Gamma(\nu-s)\Gamma(s)}{\Gamma(\nu)} && \langle 0, \operatorname{Re} \nu \rangle \\ \ln(1+x) &\longleftrightarrow \frac{\pi}{s \sin \pi s} && \langle -1, 0 \rangle \end{aligned}$$

## Idea

The singularities in the complex Mellin's plan govern completely the asymptotic behaviour of the associated function

We need to define the **singular expansion**

From the Laurent series of a function  $\varphi$  in  $p$

$$\varphi(s) = \frac{A_{-n}}{(s-p)^n} + \dots + \frac{A_{-1}}{s-p} + A_0 + A_1(s-p) + \dots$$

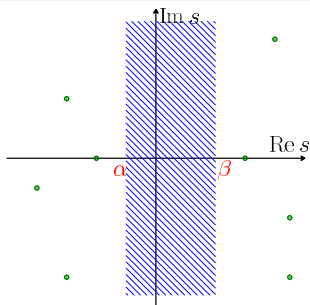
one can build the formal series by summing all over the poles of  $\varphi$  of the singular part:

$$\sum_p \left[ \frac{A_{-n}}{(s-p)^n} + \dots + \frac{A_{-1}}{s-p} \right]$$

this is the **singular expansion** of  $\varphi$  and it is written as

$$\varphi(s) \asymp \sum_p \left[ \frac{A_{-n}}{(s-p)^n} + \dots + \frac{A_{-1}}{s-p} \right]$$

# Converse Mapping Theorem



If  $f$  satisfies the condition to have a Mellin transform in the fundamental strip  $\langle \alpha, \beta \rangle$  and  $\mathcal{M}[f](s) = \mathcal{O}[|s|^{-\eta}]$  for  $\eta > 1$ .

## Converse Mapping Theorem

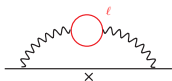
$$\mathcal{M}[f(x)]_{\text{right}}(s) \asymp \sum_{p > \beta, n} \frac{c_{p,n}}{(s-p)^n} \leftrightarrow f(x) \underset{x \rightarrow +\infty}{\sim} \sum_{p > \beta, n} \frac{(-1)^n c_{n,p}}{(n-1)!} x^{-p} \ln^{n-1} x$$

$$\mathcal{M}[f(x)]_{\text{left}}(s) \asymp \sum_{p < \alpha, n} \frac{d_{p,n}}{(s+p)^n} \leftrightarrow f(x) \underset{x \rightarrow 0}{\sim} \sum_{p < \alpha, n} \frac{(-1)^{n-1} d_{n,p}}{(n-1)!} x^p \ln^{n-1} x$$

A physical and practical example:  $g - 2$ 

Lautrup and de Rafael (1964)

Friot, Greynat and de Rafael (2005)



$$\begin{aligned}
 a_\mu &= \left(\frac{\alpha}{\pi}\right)^2 \int_0^1 \frac{dx}{x} (1-x) \Pi_R^{(\ell)} \left(-\frac{x^2}{1-x} m_\mu^2\right) \\
 &= \left(\frac{\alpha}{\pi}\right)^2 \int_0^1 \frac{dx}{x} (1-x)(2-x) \int_0^1 dy \frac{y(1-y)}{1 + \frac{m_\ell^2}{m_\mu^2} \frac{1-x}{x^2 y(1-y)}}
 \end{aligned}$$

Inverse Mellin Representation: 
$$\frac{1}{1+X} = \int_{c-i\infty}^{c+i\infty} \frac{ds}{2i\pi} X^{-s} \frac{\pi}{\sin \pi s} \quad \langle 0, 1 \rangle$$

$$\begin{aligned}
 a_\mu &= \left(\frac{\alpha}{\pi}\right)^2 \int_{c-i\infty}^{c+i\infty} \frac{ds}{2i\pi} \left(\frac{m_\ell^2}{m_\mu^2}\right)^{-s} \frac{\pi}{\sin \pi s} \int_0^1 dx x^{2s-1} (1-x)^{1-s} (2-x) \int_0^1 dy y^{1+s} (1-y)^{1+s} \\
 &= \left(\frac{\alpha}{\pi}\right)^2 \int_{c-i\infty}^{c+i\infty} \frac{ds}{2i\pi} \left(\frac{m_\ell^2}{m_\mu^2}\right)^{-s} \left(\frac{\pi}{\sin \pi s}\right)^2 \frac{1-s}{(2+s)(1+2s)(3+2s)}
 \end{aligned}$$

$$a_\mu = \left(\frac{\alpha}{\pi}\right)^2 \int_{c-i\infty}^{c+i\infty} \frac{ds}{2i\pi} \left(\frac{m_\ell^2}{m_\mu^2}\right)^{-s} \left(\frac{\pi}{\sin \pi s}\right)^2 \frac{1-s}{(2+s)(1+2s)(3+2s)}$$

- For  $m_\ell = m_\tau$  then  $\frac{m_\ell^2}{m_\mu^2} \gg 1$ : Right side of the fundamental strip

$$\left(\frac{\pi}{\sin \pi s}\right)^2 \frac{1-s}{(2+s)(1+2s)(3+2s)} \asymp \left(-\frac{1}{45}\right) \frac{1}{s-1} + \left(-\frac{1}{140}\right) \frac{1}{(s-2)^2} + \left(-\frac{9}{19600}\right) \frac{1}{s-2} + \dots$$

then

$$a_\mu = \left(\frac{\alpha}{\pi}\right)^2 \left[ \frac{1}{45} \frac{m_\mu^2}{m_\ell^2} + \frac{1}{140} \left(\frac{m_\mu^2}{m_\ell^2}\right)^2 \ln\left(\frac{m_\mu^2}{m_\ell^2}\right) + \frac{9}{19600} \left(\frac{m_\mu^2}{m_\ell^2}\right)^2 + \dots \right]$$

- For  $m_\ell = m_e$  then  $\frac{m_\ell^2}{m_\mu^2} \ll 1$ : Left side of the fundamental strip

$$\left(\frac{\pi}{\sin \pi s}\right)^2 \frac{1-s}{(2+s)(1+2s)(3+2s)} \asymp \frac{1}{6} \frac{1}{s^2} + \left(-\frac{25}{36}\right) \frac{1}{s} + \frac{\pi^2}{4} \frac{1}{s+\frac{1}{2}} + \dots$$

then

$$a_\mu = \left(\frac{\alpha}{\pi}\right)^2 \left[ \frac{1}{6} \ln\left(\frac{m_\mu^2}{m_\ell^2}\right) - \frac{25}{36} + \frac{9}{19600} \left(\frac{m_\mu^2}{m_\ell^2}\right)^2 + \frac{\pi^2}{4} \frac{m_\ell}{m_\mu} \dots \right]$$

# Exact solution

Flajolet et al. (1994)

Friot et Grunberg JHEP 0709:002 (2007)

Those two expansions of the anomaly are the exact representation for  $r = \frac{m_\ell^2}{m_\mu^2} \gg 1$  and  $r = \frac{m_\ell^2}{m_\mu^2} \ll 1$  because in the rest of the two asymptotic series

$$a_\mu = \sum_{n,p}^{[T]} c_{np} r^n \ln^p r + R(T)$$

there are no exponential corrections

$$|R(T)| = \left| \int_{\pm T - iT}^{\pm T + iT} \frac{ds}{2i\pi} r^{-s} \left( \frac{\pi}{\sin \pi s} \right)^2 \frac{1-s}{(2+s)(1+2s)(3+2s)} \right|$$

$$\leq r^{\pm T} 2T \pi^2 \left| \frac{1 \pm T}{(2 \pm T)(1 \pm 2T)(3 \pm 2T)} \right|$$

$$R(T) \underset{T \rightarrow \infty}{=} o\left(r^{\pm T}\right)$$

Therefore the resummation of all the contributions from each poles is convergent and give the exact function.

# Resummation

It is easy to perform resummations with the Mellin representation, in our example on right-side

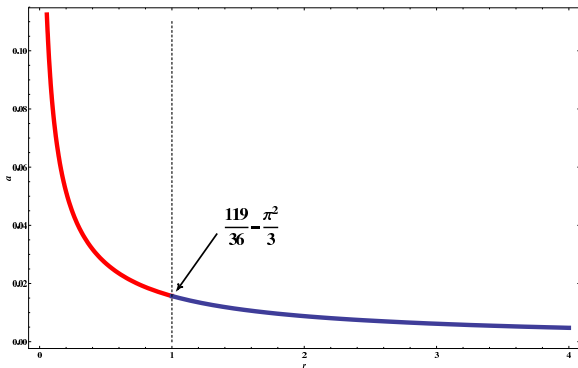
$$\left(\frac{\pi}{\sin \pi s}\right)^2 \frac{1-s}{(2+s)(1+2s)(3+2s)} \asymp \sum_{p=1}^{\infty} \left[ \frac{1}{2+n} + \frac{1}{4} \frac{1}{\frac{1}{2}+n} - \frac{5}{4} \frac{1}{\frac{3}{2}+n} \right] \frac{1}{(s-p)^2} \\ + \sum_{p=1}^{\infty} \left[ \frac{1}{(2+n)^2} + \frac{1}{4} \frac{1}{(\frac{1}{2}+n)^2} - \frac{5}{4} \frac{1}{(\frac{3}{2}+n)^2} \right] \frac{1}{s-p}$$

And using the *Converse Mapping Theorem* ( with  $r = \frac{m_r^2}{m_\mu^2} \gg 1$ ) we have the convergent expression by identification of the series as "usual" functions

$$a_\mu = \left(\frac{\alpha}{\pi}\right)^2 \left[ -\frac{1}{4} - r + \frac{\Phi\left(\frac{1}{r}, 2, \frac{3}{2}\right)}{4r} - \frac{5\Phi\left(\frac{1}{r}, 2, \frac{5}{2}\right)}{4r} + \frac{1}{2} \sqrt{r} \text{ArcCoth}(\sqrt{r}) \ln r - \frac{\ln r}{6} \right. \\ \left. + \frac{3}{2} r \ln(r) - \frac{5}{2} r^{3/2} \text{ArcCoth}(\sqrt{r}) \ln r - r^2 \ln\left(1 - \frac{1}{r}\right) \ln r + r^2 \text{Li}_2\left(\frac{1}{r}\right) \right]$$

Using the *Converse Mapping* Theorem ( with  $r = \frac{m_f^2}{m_\mu^2} \ll 1$ ) we have the expression

$$a_\mu = \left(\frac{\alpha}{\pi}\right)^2 \left[ -\frac{25}{36} - \frac{\pi^2}{4} r^{1/2} + 3r - \frac{5\pi^2}{4} r^{3/2} + \left(\frac{44}{9} + \frac{\pi^2}{3}\right) r^2 + \frac{5}{4} r^3 \Phi\left(r, 2, \frac{3}{2}\right) \right. \\ \left. - \frac{1}{4} r^3 \Phi\left(r, 2, \frac{5}{2}\right) - \frac{1}{6} \ln r + \frac{3}{2} r \ln r + \frac{1}{2} \sqrt{r} \text{ArcTanh}(\sqrt{r}) \right. \\ \left. - \frac{5}{2} r^{3/2} \text{ArcTanh}(\sqrt{r}) \ln r - r^2 \ln(1-r) \ln r + \frac{1}{2} r^2 \ln^2 r - r^2 \text{Li}_2(r) \right]$$



# Multi-dimensional Mellin Transform

We define the  $n$ -dimensional Mellin transform of a function  $f$  as

$$\mathcal{M}[f](s_1, \dots, s_n) \doteq \int_0^\infty dx_1 \cdots \int_0^\infty dx_n x_1^{s_1-1} \cdots x_n^{s_n-1} f(x_1, \dots, x_n)$$

and its inverse transformation

$$f(x_1, \dots, x_n) \doteq \int_{c_1+i\mathbb{R}} \frac{ds_1}{2i\pi} \cdots \int_{c_n+i\mathbb{R}} \frac{ds_n}{2i\pi} x_1^{-s_1} \cdots x_n^{-s_n} \mathcal{M}[f](s_1, \dots, s_n)$$

This inversion formula is of course valid in the **fundamental polyhedra** defined as all the constraints on  $\mathbf{c} \doteq {}^T(c_1, \dots, c_n)$  where the Mellin transform is completely analytic.

If we want to extend the *Converse Mapping* Theorem to the multi-dimensional case we need to introduce "briefly" the **Grothendieck Residue theory**.

# A few words on Grothendieck Residues theory

P. Griffiths, J.Harris, *Principles of Algebraic Geometry*, Wiley NYC 1978

A.K. Tsikh et al., hep-th 9609215

To simplify, we are considering the case of only **2 different scales** i.e. **2 complex variables:  $s$  and  $t$**

One way to see the residues in multi-dimensional complex analysis is to consider the quantity (for any  $h$  completely analytic)

$$\text{Res}_{(0,0)} \frac{h(s, t)}{\varphi_1(s, t) \varphi_2(s, t)} ds \wedge dt = \oint_0 \frac{h(s, t)}{\varphi_1(s, t) \varphi_2(s, t)} \frac{ds}{2i\pi} \wedge \frac{dt}{2i\pi} \doteq \oint_0 \omega$$

All the curves, the **divisors**, in the 4-dimension complex space given by  $j \in [1, 2]$

$$D_j \doteq \left\{ (s, t) \in \mathbb{C}^2, \varphi_j(s, t) = 0 \right\}$$

have intersections points in this space. They provide the calculation of the residue in a summation over

$$\bigcap_{j \in [1, 2]} D_j$$

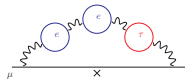
# Multi-dimensional Converse Mapping Theorem

J.-Ph. Aguilar, D. Greynat and E. de Rafael, hep-ph.0802.2618

## Idea

Idea: If you combine the calculation of the Grothendieck residues and the *multi-dimensional Jordan lemma* you can define sectors in complex space where the  $x_j$  are bigger or smaller than 1, these sectors then allowing to generate the complete asymptotic behaviour in each variables.

Let us consider the following example

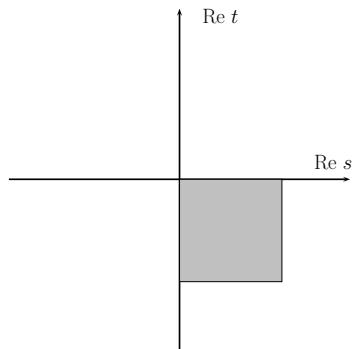
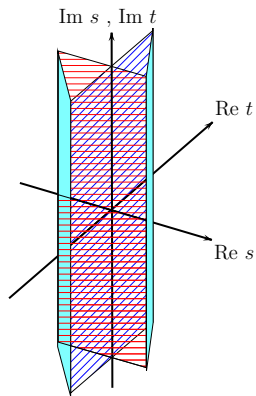


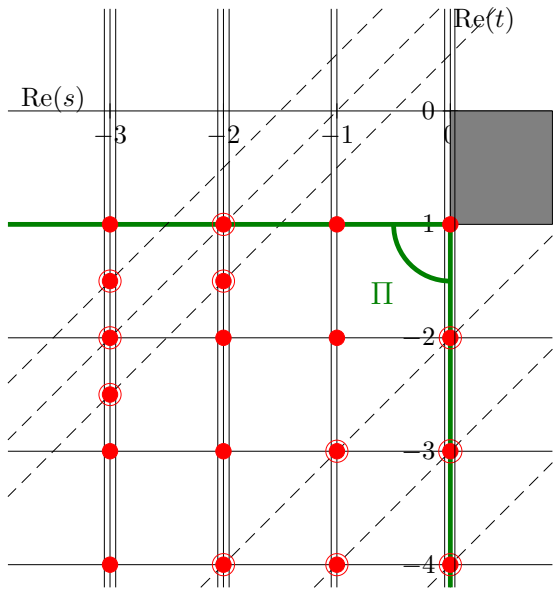
$$a_{\mu}^{(ee\tau)} = \left(\frac{\alpha}{\pi}\right)^4 \int_{c_s+i\mathbb{R}} \int_{c_t+i\mathbb{R}} \omega$$

$$\omega = \frac{2\sqrt{\pi}}{3} \left(\frac{4m_e^2}{m_{\mu}^2}\right)^{-s} \left(\frac{m_{\mu}^2}{m_{\tau}^2}\right)^{-t} \frac{(6 + 13s + 4s^2) \Gamma(t) \Gamma(1-t) \Gamma^2(2-t)}{s^3(2+s)(3+s) t \Gamma(4-2t)}$$

$$\times \frac{\Gamma^2(s+1) \Gamma(2-s) \Gamma(1+2s-2t) \Gamma(2-s+t)}{\Gamma(s+\frac{3}{2}) \Gamma(3+s-t)} ds \wedge dt$$

## Fundamental polyhedra and pertinent sector





# There are two kinds of singularities

The first kind: **only vertical and horizontal divisors (at least 2 or more)**

The 2-form  $\omega$  can be rewritten as ( $n$  and  $m$  are positive integers)

$$\omega = \frac{h(s, t)}{s^n t^m} ds \wedge dt$$

Therefore, using the Cauchy formula, we have obviously

$$\text{Res}_{(0,0)} \omega = \frac{1}{(n-1)!(m-1)!} \frac{\partial^{n+m-2} h(s, t)}{\partial s^{n-1} \partial t^{m-1}} \Big|_{(0,0)}$$

The second kind: **at least one oblique divisor**

The 2-form  $\omega$  can be rewritten as (for example)

$$\omega = \frac{h(s, t)}{s^n t^m (-s + t)} ds \wedge dt$$

For calculating the residue in this case we need more: **the Transformation Law.**

# The Transformation Law

## The Transformation Law

For  $U \subset \mathbb{C}^2$  an open set containing  $(0, 0)$

If  $\varphi = \begin{pmatrix} \varphi_1(s, t) \\ \varphi_2(s, t) \end{pmatrix}$  and  $\mathbf{g} = \begin{pmatrix} g_1(s, t) \\ g_2(s, t) \end{pmatrix}$  analytic mappings from  $U$  to  $\mathbb{C}^2$

and  $\varphi^{-1}(0, 0) = \mathbf{g}^{-1}(0, 0) = 0$

If it exists an analytic matrix  $A$  such that:  $\mathbf{g} = A\varphi$  then

$$\operatorname{Res}_{(0,0)} \frac{h(s, t)}{\varphi_1(s, t) \varphi_2(s, t)} ds \wedge dt = \operatorname{Res}_{(0,0)} \frac{h(s, t) \det A(s, t)}{g_1(s, t) g_2(s, t)} ds \wedge dt$$

Example:

$$\operatorname{Res}_{(0,0)} \frac{h(s, t)}{s^3 t(-s+t)} ds \wedge dt = \operatorname{Res}_{(0,0)} \frac{h(-s+t, t)}{st(-s+t)^3} ds \wedge dt$$

we take

$$\underbrace{\begin{pmatrix} -s^4 \\ t^4 \end{pmatrix}}_{\doteq \mathbf{g}} = \underbrace{\begin{pmatrix} -t^2 + 3st - 3s^2 & s \\ s^2 - 3st + 3t^2 & t \end{pmatrix}}_{\doteq A} \underbrace{\begin{pmatrix} st \\ (-s+t)^3 \end{pmatrix}}_{\doteq \varphi}$$

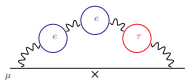
$$\det A = -s^3 - t^3$$

Therefore

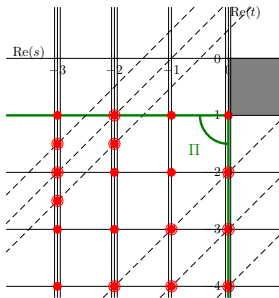
$$\begin{aligned}
 \operatorname{Res}_{(0,0)} \frac{h(-s+t, t)}{st(-s+t)^3} ds \wedge dt &= \operatorname{Res}_{(0,0)} h(-s+t, t) \frac{s^3 + t^3}{s^4 t^4} ds \wedge dt \\
 &= \operatorname{Res}_{(0,0)} h(-s+t, t) \left( \frac{1}{st^4} + \frac{1}{s^4 t} \right) ds \wedge dt \\
 &= \frac{1}{6} \left[ \frac{\partial^3 [h(-s+t, t)]}{\partial t^3} + \frac{\partial^3 [h(-s+t, t)]}{\partial s^3} \right]_{(0,0)}
 \end{aligned}$$

$$\operatorname{Res}_{(0,0)} \frac{h(s, t)}{s^3 t(-s+t)} ds \wedge dt = \frac{1}{2} \frac{\partial^3 h(s, t)}{\partial s^2 \partial t} \Big|_{(0,0)} + \frac{1}{2} \frac{\partial^3 h(s, t)}{\partial s \partial t^2} \Big|_{(0,0)} + \frac{1}{6} \frac{\partial^3 h(s, t)}{\partial t^3} \Big|_{(0,0)}.$$

# Coming back to our physical example



Along the line  $t = -1$ :



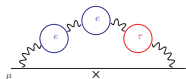
$$a_{\mu}^{(ee\tau)} = \left(\frac{\alpha}{\pi}\right)^4 \int_{c_s+i\mathbb{R}} \int_{c_t+i\mathbb{R}} \omega$$

$$\omega = \frac{h_{(0,-1)}(s, t)}{s^3 t} ds \wedge dt$$

$$h_{(0,-1)}(s, t) = \frac{2\sqrt{\pi}}{3} \left(\frac{4m_e^2}{m_{\mu}^2}\right)^{-s} \left(\frac{m_{\mu}^2}{m_{\tau}^2}\right)^{1-t} \frac{\Gamma(3+2s-2t)\Gamma(1-s+t)}{\Gamma(4+s-t)} \\ \times \frac{\Gamma^2(1+s)\Gamma(2-s)(6+13s+4s^2)}{(2+s)(s+3)\Gamma(\frac{3}{2}+s)} \frac{\Gamma^2(3-t)\Gamma(1+t)\Gamma(2-t)}{(-1+t)^2\Gamma(6-2t)}$$

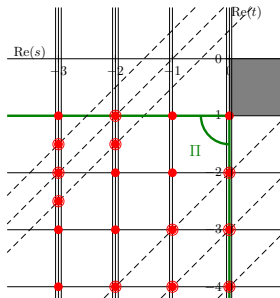
$$\text{Res}_{(0,-1)} \omega = \frac{1}{2} \frac{\partial^2 h_{(0,-1)}}{\partial s^2} \Big|_{(0,0)}$$

$$a_{\mu}^{(ee\tau)} = \left(\frac{\alpha}{\pi}\right)^4 \left(\frac{m_{\mu}^2}{m_{\tau}^2}\right) \left[ \frac{1}{135} \log^2 \frac{m_{\mu}^2}{m_e^2} - \frac{1}{135} \log \frac{m_{\mu}^2}{m_e^2} - \frac{61}{2430} + \frac{2}{405} \pi^2 \right]$$



$$a_{\mu}^{(ee\tau)} = \left(\frac{\alpha}{\pi}\right)^4 \int_{c_s+i\mathbb{R}} \int_{c_t+i\mathbb{R}} \omega$$

Along the line  $t = -2$ :



$$\omega = \frac{h_{(0,-2)}(s, t)}{s^3 t(-s+t)} ds \wedge dt$$

$$h_{(0,-2)}(s, t) = \left(\frac{4m_e^2}{m_{\mu}^2}\right)^{-s} \left(\frac{m_{\mu}^2}{m_{\tau}^2}\right)^{2-t} \frac{2\sqrt{\pi}}{3} \frac{\Gamma(5+2s-2t)\Gamma(1-s+t)}{\Gamma(5+s-t)}$$

$$\times \frac{(6+13s+4s^2)\Gamma^2(1+s)\Gamma(2-s)}{(2+s)(3+s)\Gamma\left(\frac{3}{2}+s\right)} \frac{\Gamma(1+t)\Gamma(3-t)}{(-1+t)(-2+t)^2} \frac{\Gamma^2(4-t)}{\Gamma(8-2t)}$$

$$\text{Res}_{(0,-2)}\omega = \frac{1}{2} \left. \frac{\partial^3 h_{(0,-2)}(s, t)}{\partial s^2 \partial t} \right|_{(0,0)} + \frac{1}{2} \left. \frac{\partial^3 h_{(0,-2)}(s, t)}{\partial s \partial t^2} \right|_{(0,0)} + \frac{1}{6} \left. \frac{\partial^3 h_{(0,-2)}(s, t)}{\partial t^3} \right|_{(0,0)}$$

Easily now the contribution to the anomaly is

$$\begin{aligned}
 a_{\mu}^{(ee\tau)} &= \left(\frac{\alpha}{\pi}\right)^4 \left(\frac{m_{\mu}^2}{m_{\tau}^2}\right)^2 \left[ \frac{1}{1260} \log^3 \frac{m_{\mu}^2}{m_{\tau}^2} - \left( \frac{1}{420} \log \frac{m_{\mu}^2}{m_e^2} + \frac{37}{44100} \right) \log^2 \frac{m_{\mu}^2}{m_{\tau}^2} \right. \\
 &\quad + \left( \frac{1}{420} \log^2 \frac{m_{\mu}^2}{m_e^2} + \frac{37}{22050} \log \frac{m_{\mu}^2}{m_e^2} + \frac{40783}{4630500} \right) \log \frac{m_{\mu}^2}{m_{\tau}^2} \\
 &\quad + \frac{3}{19600} \log^2 \frac{m_{\mu}^2}{m_e^2} + \left( \frac{\pi^2}{630} - \frac{229213}{12348000} \right) \log \frac{m_{\mu}^2}{m_e^2} \\
 &\quad \left. + \frac{\pi^2}{1512} - \frac{30026659}{5186160000} \right] \\
 &\quad + \dots + \mathcal{O} \left[ \left( \frac{m_{\mu}^2}{m_{\tau}^2} \right)^4 \log \frac{m_{\tau}^2}{m_e^2} \log \frac{m_{\mu}^2}{m_e^2} \log \frac{m_{\tau}^2}{m_{\mu}^2} \right] \\
 &= \left(\frac{\alpha}{\pi}\right)^4 0.0027486(9)
 \end{aligned}$$

# Mellin-Barnes representation

and

## asymptotic improvement of perturbative expansion

Friot, Greynat, hep-th 0907.5593

# Open Problems

- In QFT we have to deal with presumably divergent non-Borel summable series and expected asymptotic.
- But asymptotic to what ?
- One more difficulty: the general terms of the expansions are unknown.
- Consequence: theoretical errors are not under control then how can we find new physics effects in precision physics ?

# Introduction to the method

## Framework

The general term of a divergent (and supposed asymptotic) perturbative series is known.

## Aim

Find non perturbative effects directly from the perturbative series.

## Tools

- Terminant functions theory Dingle 1973
- Exponential improvement of asymptotic series : MB hyperasymptotic theory. Paris et al.'90, Berry 1989

## 0-dimensional euclidean example

Let us consider the 0-dimensional euclidean action  $S$  with a unit mass

$$S \doteq \frac{1}{2!} \phi^2 + \frac{\lambda}{4!} \phi^4$$

and the associated generating functional

$$Z(j) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\phi e^{-S+j\phi} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\phi e^{-\frac{1}{2!}\phi^2 - \frac{\lambda}{4!}\phi^4 + j\phi}$$

with  $\lambda \in \mathbb{C}$  and  $\text{Re } \lambda > 0$

We will focus on the vacuum-vacuum transitions

$$Z(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\phi e^{-\frac{1}{2!}\phi^2 - \frac{\lambda}{4!}\phi^4}$$

# Perturbative expansion

Taking  $\lambda$  small, one has the perturbative expansion of  $Z(0)$

$$Z(0) \underset{\lambda \rightarrow 0}{\sim} \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma\left(\frac{1}{2} + 2k\right)}{k!} \left(\frac{\lambda}{6}\right)^k = 1 - \frac{1}{8}\lambda + \frac{35}{384}\lambda^2 - \frac{385}{3072}\lambda^3 + \mathcal{O}(\lambda^4)$$

Yet

$$\forall \lambda, \left| \frac{(-1)^k \Gamma\left(\frac{1}{2} + 2k\right)}{k!} \left(\frac{\lambda}{6}\right)^k \right| \xrightarrow[k \rightarrow \infty]{} \infty$$

then this is a **divergent series**.

Let us define  $S_n^{\text{Pert.}}$  and  $R_n$

$$Z(0) \underset{\lambda \rightarrow 0}{\sim} \underbrace{\frac{1}{\sqrt{\pi}} \sum_{k=0}^{n-1} \frac{(-1)^k \Gamma\left(\frac{1}{2} + 2k\right)}{k!} \left(\frac{\lambda}{6}\right)^k}_{\doteq S_{n-1}^{\text{Pert.}}} + \underbrace{\frac{1}{\sqrt{\pi}} \sum_{k=n}^{\infty} \frac{(-1)^k \Gamma\left(\frac{1}{2} + 2k\right)}{k!} \left(\frac{\lambda}{6}\right)^k}_{\doteq R_n}$$

# Counting diagrams

One of the interests of the 0-dimensional theory is that  $S_N^{\text{Pert.}}$  counts the number of Feynman diagrams to a given order  $N$  in dimension 4.

$$Z(0) \underset{\lambda \rightarrow 0}{=} 1 - \frac{1}{8}\lambda + \frac{35}{384}\lambda^2 - \frac{385}{3072}\lambda^3 + \mathcal{O}(\lambda^4)$$

Order  $\lambda$

$$Z(0) = \frac{(-1)}{1!} \frac{\lambda}{4!} 3 \text{ (diagram)} + \mathcal{O}(\lambda^2) = -\frac{1}{8} \lambda \text{ (diagram)} + \mathcal{O}(\lambda^2)$$

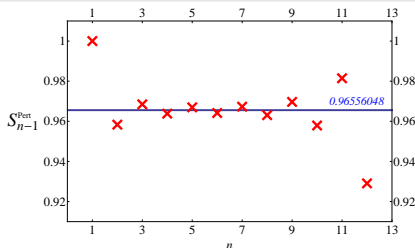
Order  $\lambda^2$

$$\frac{24}{384} \lambda^2 \left( \text{diagram} \right) + \frac{3}{384} \lambda^2 \text{ (diagram)} + \frac{8}{384} \lambda^2 \text{ (diagram)}$$

# Numerical analysis

From now, for all numerical calculations, we fix  $\lambda = \frac{1}{3}$ . Mathematica gives with 8 decimal precision

$$Z(0)|_{\lambda=\frac{1}{3}} \approx 0.96556048\dots$$



Choosing a resummation procedure for the perturbative series leads to the property that  $|R_n| < |u_n|$  and  $|R_n| < |u_{n-1}|$ . One then has

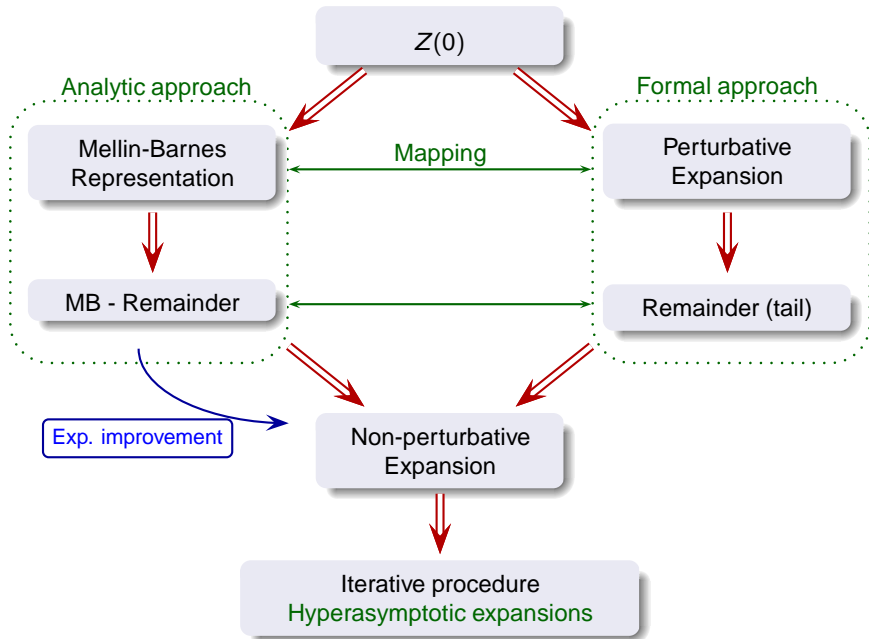
$$Z(0) = \sum_{k=0}^{\eta-1} u_k + \frac{1}{2}|u_\eta| \pm \frac{1}{2}|u_\eta| ,$$

here

$$Z(0)|_{\lambda=\frac{1}{3}} = 0.96555187 \pm 0.00140990$$

The central value corresponds to the standard [Stieltjes approximation](#) which is given, for the rank  $\eta$  defined  $\forall n, |u_\eta| \leq |u_n|$

$$Z(0) = \sum_{k=0}^{\eta-1} u_k + \frac{1}{2}|u_\eta| .$$



# Analytic approach

# Mellin-Barnes Hyperasymptotic Theory

## Main Idea

Construct an exponentially small remainder with its inverse Mellin-Barnes representation (non perturbative in  $\lambda$ ).

The first step is to obtain an inverse Mellin-Barnes representation of  $Z(0)$ :

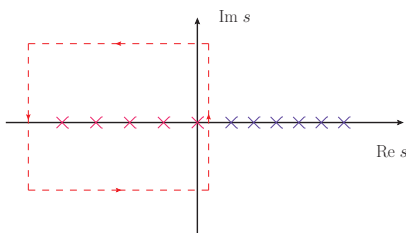
$$e^{-\frac{\lambda}{4!}\phi^4} = \int_{c+i\mathbb{R}} \frac{ds}{2i\pi} \left(\frac{\lambda}{4!}\phi^4\right)^{-s} \Gamma(s),$$

with  $\langle 0, +\infty \rangle$  and  $|\arg \lambda| < \frac{\pi}{2}$ . Then we obtain the following results

$$Z(0) = \frac{1}{\sqrt{\pi}} \int_{c+i\mathbb{R}} \frac{ds}{2i\pi} \left(\frac{\lambda}{6}\right)^{-s} \Gamma(s) \Gamma\left(\frac{1}{2} - 2s\right)$$

with  $\langle 0, \frac{1}{4} \rangle$  and  $|\arg \lambda| < \frac{3\pi}{2}$  (Analytic continuation). Paris et al. 90's

An interesting consequence is that one can calculate automatically  $Z(0)$  where its perturbative expansion is not Borel summable ( $\lambda < 0$ ).



Applying the Cauchy's theorem on the integral over  $s$  inside a rectangle horizontally sized  $n$ , one can prove that

$$Z(0) = \frac{1}{\sqrt{\pi}} \sum_{k=0}^{n-1} \frac{(-1)^k \Gamma\left(\frac{1}{2} + 2k\right)}{k!} \left(\frac{\lambda}{6}\right)^k + \frac{1}{\sqrt{\pi}} \int_{c-n+i\mathbb{R}} \frac{ds}{2i\pi} \left(\frac{\lambda}{6}\right)^{-s} \Gamma(s) \Gamma\left(\frac{1}{2} - 2s\right)$$

therefore

$$R_n = \frac{1}{\sqrt{\pi}} \int_{c-n+i\mathbb{R}} \frac{ds}{2i\pi} \left(\frac{\lambda}{6}\right)^{-s} \Gamma(s) \Gamma\left(\frac{1}{2} - 2s\right)$$

Using the Stirling's formula:  $|\Gamma(\sigma + i\tau)| \underset{\tau \rightarrow \pm\infty}{=} \mathcal{O}\left(|\tau|^{\sigma-\frac{1}{2}} e^{-\frac{\pi}{2}|\tau|}\right)$   
 and majoring the integrand one has Paris et al. 90's

$$|R_n| \underset{n \rightarrow \infty}{=} \left| \frac{2\lambda}{3} \right|^{n-c} \mathcal{O}\left(e^{-n} n^{n-c}\right)$$

### Superasymptotic Theorem

When  $|\lambda|$  is small, there exist  $a_0 > 0$  and  $|b_0| < \infty$  so that if

$$n = \frac{a_0}{|\lambda|} + b_0$$

then  $R_n$  is exponentially small in  $\lambda$  (non perturbative).

Here,

$$|R_n| \underset{|\lambda| \rightarrow 0}{=} \mathcal{O}\left(e^{-\frac{a_0}{|\lambda|}} \left(\frac{2a_0}{3}\right)^{\frac{a_0}{|\lambda|}}\right)$$

and for  $a_0 = \frac{3}{2}$ , we have the **Optimal Truncation Scheme** i.e. the smallest remainder

$$|R_n| \underset{n \rightarrow \infty}{=} \mathcal{O}\left(e^{-\frac{3}{2|\lambda|}}\right).$$

# The iterative procedure

## Main Idea

It is possible to construct more and more exponentially small remainders order by order in the expansion of  $R_n$ .

With the duplication formula

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right)$$

we can rewrite the remainder as

$$R_n = \frac{1}{\pi\sqrt{2}} \int_{-c+n+i\mathbb{R}} \frac{dt}{2i\pi} \left(\frac{3}{2\lambda}\right)^{-t} \frac{\pi}{\sin \pi t} \frac{\Gamma\left(t + \frac{1}{4}\right) \Gamma\left(t + \frac{3}{4}\right)}{\Gamma(t+1)}$$

We introduce now the main tool : the **inverse factorial expansion**.

# The inverse factorial expansion

## Barnes' lemma

$$\frac{\Gamma(s+a)\Gamma(s+b)}{\Gamma(s+c)} = \int_{i\mathbb{R}} \frac{dt}{2i\pi} \frac{\Gamma(t+c-a)\Gamma(t+c-b)\Gamma(s+\vartheta-t)\Gamma(-t)}{\Gamma(c-a)\Gamma(c-b)}$$

with  $\vartheta = a + b - c$ .

## Inverse factorial expansion

$$\begin{aligned} \frac{\Gamma(s+a)\Gamma(s+b)}{\Gamma(s+c)} &= \sum_{j=0}^{M-1} \frac{(-1)^j}{j!} (c-a)_j (c-b)_j \Gamma(s+\vartheta-j) \\ &+ \int_{M+i\mathbb{R}} \frac{dt}{2i\pi} \frac{\Gamma(t+c-a)\Gamma(t+c-b)\Gamma(s+\vartheta-t)\Gamma(-t)}{\Gamma(c-a)\Gamma(c-b)} \end{aligned}$$

$M > \operatorname{Re}(a-c) + \delta$ ,  $M > \operatorname{Re}(b-c) + \delta$ ,  $M > \operatorname{Re}(s+\vartheta)$

for  $0 < \delta < 1$  and  $|\arg s| < \frac{\pi}{2}$ .

Olver 1995

One has in our example,

$$\frac{\Gamma\left(t + \frac{1}{4}\right) \Gamma\left(t + \frac{3}{4}\right)}{\Gamma(t+1)} = \sum_{j=0}^{m-1} (-1)^j A_j \Gamma(t-j) + \int_{c+m+i\mathbb{R}} \frac{ds}{2i\pi} f(s) \Gamma(t-s)$$

where  $A_j = \frac{1}{j!} \left(\frac{1}{4}\right)_j \left(\frac{3}{4}\right)_j$  and  $f(s) = \frac{\Gamma(-s)\Gamma\left(s + \frac{1}{4}\right)\Gamma\left(s + \frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}$

The remainder is then

$$R_n = -\frac{1}{\pi\sqrt{2}} \sum_{j=0}^{m-1} (-1)^j A_j \int_{-c+n+i\mathbb{R}} \frac{dt}{2i\pi} \left(\frac{3}{2\lambda}\right)^{-t} \frac{\pi}{\sin \pi t} \Gamma(t-j) + R_{n,m}$$

with

$$R_{n,m} = -\frac{1}{\pi\sqrt{2}} \int_{\substack{(-c+n+i\mathbb{R}) \\ \times (-c+m+i\mathbb{R})}} \frac{dt}{2i\pi} \wedge \frac{ds}{2i\pi} \left(\frac{3}{2\lambda}\right)^{-t} \frac{\pi}{\sin \pi t} f(s) \Gamma(t-s)$$

Yet, we have  $(m \leq n)$ ,  $|R_{n,m}| \underset{n \rightarrow \infty}{=} \mathcal{O}\left(\left|\frac{3}{2\lambda}\right|^{-n-c} e^{-n} (n-m)^{n-m} m^{m+c-\frac{1}{2}}\right)$

# Optimal Truncation Scheme

For  $m = \frac{a_1}{|\lambda|} + b_1$ , we have

Paris et al. 90's

$$|R_{n,m}|_{|\lambda| \rightarrow 0} = \mathcal{O} \left( \sqrt{|\lambda|} e^{-\frac{a_0 + \ln 3 - \ln 2}{|\lambda|} a_1 \frac{a_1}{|\lambda|}} (a_0 - a_1)^{\frac{a_0 - a_1}{|\lambda|}} \right)$$

$$|R_n|_{|\lambda| \rightarrow 0} = \mathcal{O} \left( e^{-\frac{a_0}{|\lambda|}} \left( \frac{2a_0}{3} \right)^{\frac{a_0}{|\lambda|}} \right)$$

We can optimize the remainders in two ways, first satisfying the condition on  $R_n$  and then optimizing  $R_{n,m}$  or optimizing directly  $R_{n,m}$ :

			$R_n = \mathcal{O}(\cdot)$	$R_{n,m} = \mathcal{O}(\cdot)$
OTS 1	$a_0 = \frac{3}{2}$	$a_1 = \frac{3}{4}$	$e^{-\frac{3}{2 \lambda }}$	$\sqrt{ \lambda } e^{-\frac{3}{2 \lambda }(1+\ln 2)}$
OTS 2	$a_0 = 3$	$a_1 = \frac{3}{2}$	—	$\sqrt{ \lambda } e^{-\frac{3}{ \lambda }}$

Hyperasymptotic expansion of  $Z(0)$  at first hyperasymptotic level

$$\begin{aligned}
 Z(0) &= \sum_{k=0}^{n-1} (-1)^k A_k \left( \frac{3}{2\lambda} \right)^k \\
 &\quad - \frac{1}{\pi\sqrt{2}} \sum_{j=0}^{m-1} (-1)^j A_j \int_{-c+n+i\mathbb{R}} \frac{dt}{2i\pi} \left( \frac{3}{2\lambda} \right)^{-t} \frac{\pi}{\sin \pi t} \Gamma(t-j) \\
 &\quad - \frac{1}{\pi\sqrt{2}} \int_{\substack{(-c+n+i\mathbb{R}) \\ \times (-c+m+i\mathbb{R})}} \frac{dt}{2i\pi} \wedge \frac{ds}{2i\pi} \left( \frac{3}{2\lambda} \right)^{-t} \frac{\pi}{\sin \pi t} f(s) \Gamma(t-s)
 \end{aligned}$$

## OTS1 at first hyperasymptotic level

$$\begin{aligned}
 Z(0) &= \sum_{k=0}^{\left[\frac{3}{2|\lambda|}\right]-1} (-1)^k A_k \left(\frac{3}{2\lambda}\right)^k \\
 &\quad - \frac{1}{\pi\sqrt{2}} \sum_{j=0}^{m-1} (-1)^j A_j \int_{-c+\left[\frac{3}{2|\lambda|}\right]+i\mathbb{R}} \frac{dt}{2i\pi} \left(\frac{3}{2\lambda}\right)^{-t} \frac{\pi}{\sin \pi t} \Gamma(t-j) \\
 &\quad + \mathcal{O}\left(\sqrt{|\lambda|} e^{-\frac{3(1+\ln 2)}{2|\lambda|}}\right)
 \end{aligned}$$

## OTS2 at first hyperasymptotic level

$$\begin{aligned}
 Z(0) &= \sum_{k=0}^{\left[\frac{3}{|\lambda|}\right]-1} (-1)^k A_k \left(\frac{3}{2\lambda}\right)^k \\
 &\quad - \frac{1}{\pi\sqrt{2}} \sum_{j=0}^{\left[\frac{3}{2|\lambda|}\right]-1} (-1)^j A_j \int_{-c+\left[\frac{3}{|\lambda|}\right]+i\mathbb{R}} \frac{dt}{2i\pi} \left(\frac{3}{2\lambda}\right)^{-t} \frac{\pi}{\sin \pi t} \Gamma(t-j) \\
 &\quad + \mathcal{O}\left(\sqrt{|\lambda|} e^{-\frac{3}{|\lambda|}}\right)
 \end{aligned}$$

## Iterations...

We have shown that it is possible to write the remainder as

$$R_n = -\frac{1}{\pi\sqrt{2}} \sum_{j=0}^{m-1} (-1)^j A_j \int_{-c+n+i\mathbb{R}} \frac{dt}{2i\pi} \left(\frac{3}{2\lambda}\right)^{-t} \frac{\pi}{\sin \pi t} \Gamma(t-j)$$

$$- \frac{1}{\pi\sqrt{2}} \int_{\substack{(-c+n+i\mathbb{R}) \\ \times (-c+m+i\mathbb{R})}} \frac{dt}{2i\pi} \wedge \frac{ds}{2i\pi} \left(\frac{3}{2\lambda}\right)^{-t} \frac{\pi}{\sin \pi t} f(s)\Gamma(t-s)$$

where  $f(s) = \frac{\Gamma(-s)\Gamma(s+\frac{1}{4})\Gamma(s+\frac{3}{4})}{\Gamma(\frac{1}{4})\Gamma(\frac{3}{4})} = -\frac{1}{\pi\sqrt{2}} \frac{\Gamma(s+\frac{1}{4})\Gamma(s+\frac{3}{4})}{\Gamma(s+1)} \frac{\pi}{\sin \pi s}$

Yet applying the **Inverse factorial expansion** one has

## Functional equation

$$f(s) = -\frac{1}{\pi\sqrt{2}} \frac{\pi}{\sin \pi s} \left[ \sum_{l=0}^{m'-1} (-1)^l A_l \Gamma(s-l) + \int_{c+m'-i\mathbb{R}} \frac{dt}{2i\pi} f(t) \Gamma(s-t) \right]$$

Then using this expression of  $f$  in  $R_n$  one may get straightforwardly a hyperasymptotic expansions at an arbitrary hyperasymptotic level.

# Formal approach

The starting point is now the expression of  $Z(0)$  as a divergent series

$$Z(0) \underset{\lambda \rightarrow 0}{\sim} \underbrace{\frac{1}{\sqrt{\pi}} \sum_{k=0}^{n-1} \frac{(-1)^k \Gamma\left(\frac{1}{2} + 2k\right)}{k!} \left(\frac{\lambda}{6}\right)^k}_{\doteq S_{n-1}^{\text{Pert.}}} + \underbrace{\frac{1}{\sqrt{\pi}} \sum_{k=n}^{\infty} \frac{(-1)^k \Gamma\left(\frac{1}{2} + 2k\right)}{k!} \left(\frac{\lambda}{6}\right)^k}_{\doteq R_n}$$

Using the duplication formula,

$$R_n = \frac{1}{\pi\sqrt{2}} \sum_{k=n}^{\infty} \frac{\Gamma\left(k + \frac{1}{4}\right) \Gamma\left(k + \frac{3}{4}\right)}{\Gamma(k+1)} \left(-\frac{2\lambda}{3}\right)^k$$

We can apply the **Inverse factorial expansion** for

$$\frac{\Gamma\left(k + \frac{1}{4}\right) \Gamma\left(k + \frac{3}{4}\right)}{\Gamma(k+1)} = \sum_{j=0}^{m-1} (-1)^j A_j \Gamma(k-j) + \int_{c+m+i\mathbb{R}} \frac{ds}{2i\pi} f(s) \Gamma(k-s)$$

$$\text{where } A_j = \frac{1}{j!} \left(\frac{1}{4}\right)_j \left(\frac{3}{4}\right)_j \quad \text{and} \quad f(s) = \frac{\Gamma(-s) \Gamma\left(s + \frac{1}{4}\right) \Gamma\left(s + \frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right) \Gamma\left(\frac{3}{4}\right)}$$

## Resummation and terminant function

The remainder is then

$$R_n = \frac{1}{\pi\sqrt{2}} \sum_{j=0}^{m-1} (-1)^j A_j \sum_{k=n}^{\infty} \Gamma(k-j) \left(\frac{-2\lambda}{3}\right)^k + \frac{1}{\pi\sqrt{2}} \int_{c+m+i\mathbb{R}} \frac{ds}{2i\pi} f(s) \sum_{k=n}^{\infty} \Gamma(k-s) \left(\frac{-2\lambda}{3}\right)^k$$

One can now perform a Borel resummation on the two infinite sums in  $R_n$ ,

$$\mathcal{B} \left[ \sum_{k=n}^{\infty} \Gamma(k-j) \left(\frac{-2\lambda}{3}\right)^k \right] = \Gamma(n-j) \left(\frac{-2\lambda}{3}\right)^n \Lambda_{n-j-1} \left(\frac{3}{2\lambda}\right)$$

The function  $\Lambda_\ell$  is known as a **terminant function**, here [Dingle 1973](#)

$$\Lambda_\ell(x) = x^{\ell+1} e^x \Gamma(-\ell, x)$$

where

$$\Gamma(a, x) = \int_x^\infty dy y^{a-1} e^{-y}$$

is the **incomplete Gamma function**.

Therefore performing the summation, one has the expansion ( $m \leq n$ )

$$\begin{aligned}
 Z(0) &= \frac{1}{\sqrt{\pi}} \sum_{k=0}^{n-1} \frac{(-1)^k \Gamma\left(\frac{1}{2} + 2k\right)}{k!} \left(\frac{\lambda}{6}\right)^k \\
 &+ \frac{(-1)^n}{\pi\sqrt{2}} e^{\frac{3}{2\lambda}} \sum_{j=0}^{m-1} (-1)^j A_j \Gamma(n-j) \left(\frac{2\lambda}{3}\right)^j \Gamma\left(-n+j+1, \frac{3}{2\lambda}\right) \\
 &+ \frac{(-1)^n}{\pi\sqrt{2}} e^{\frac{3}{2\lambda}} \int_{c+m+i\mathbb{R}} \frac{ds}{2i\pi} \left(\frac{2}{3\lambda}\right)^{-s} f(s) \Gamma(n-s) \Gamma\left(-n+s+1, \frac{3}{2\lambda}\right)
 \end{aligned}$$

where the sub-dominant terms appear explicitly through the expression of the tail of the series.

Using the MB representation of the incomplete Gamma function we have exactly the hyperasymptotic expansion of  $Z(0)$  at first level obtained in the analytic approach.

# Resurgence phenomenon

If we consider a few terms in the perturbative expansion of  $Z(0)$

$$Z(0) = 1 - \frac{1}{8}\lambda + \frac{35}{384}\lambda^2 - \frac{385}{3072}\lambda^3 + \dots$$

we have the same coefficient in the non-perturbative expansion of  $R_n$  as

$$R_n = -\frac{(-1)^n}{\pi\sqrt{2}} e^{\frac{3}{2\lambda}} \times \left\{ 1 \Gamma(5)\Gamma\left(-4, \frac{3}{2\lambda}\right) - \frac{1}{8}\lambda \Gamma(4)\Gamma\left(-3, \frac{3}{2\lambda}\right) + \frac{35}{384}\lambda^2 \Gamma(3)\Gamma\left(-2, \frac{3}{2\lambda}\right) - \frac{385}{3072}\lambda^3 \Gamma(2)\Gamma\left(-1, \frac{3}{2\lambda}\right) + \dots \right\}$$

This resurgence phenomenon also appears in higher order hyperasymptotic levels.

## Numerical analysis

		$n$	$m$	$m'$	$m''$
OTS 1	1	$\left[\frac{3}{2 \lambda }\right] = 4$	$\left[\frac{3}{4 \lambda }\right] = 2$		
	2	$\left[\frac{3}{2 \lambda }\right] = 4$	$\left[\frac{3}{4 \lambda }\right] = 2$	$\left[\frac{3}{8 \lambda }\right] = 1$	
	3	$\left[\frac{3}{2 \lambda }\right] = 4$	$\left[\frac{3}{4 \lambda }\right] = 2$	$\left[\frac{3}{8 \lambda }\right] = 1$	$\left[\frac{3}{8 \lambda }\right] = 0$
OTS 2	1	$\left[\frac{3}{ \lambda }\right] = 9$	$\left[\frac{3}{2 \lambda }\right] = 4$		
	2	$\left[\frac{9}{2 \lambda }\right] = 13$	$\left[\frac{3}{ \lambda }\right] = 9$	$\left[\frac{3}{2 \lambda }\right] = 4$	
	3	$\left[\frac{6}{ \lambda }\right] = 18$	$\left[\frac{9}{2 \lambda }\right] = 13$	$\left[\frac{3}{ \lambda }\right] = 9$	$\left[\frac{3}{2 \lambda }\right] = 4$

		$Z(0)$	$S_n$	$S_m$	$S_{m'}$	$S_{m''}$
Mathematica		0.965560481..				
Pertur. expa.		0.96555187 $\pm 0.001410990$				
OTS 1	1	0.96552297	0.9638	0.0017		
	2	0.96556492	0.9638	0.0017	0.000041	
	3	0.96556492	0.9638	0.0017	0.000041	0
OTS 2	1	0.965562911	0.9696	-0.0040		
	2	0.965560477	1.0573	-0.0917	0.0000061	
	3	0.965560486	-27.696	28.662	-0.0001292	$9 \times 10^{-9}$

- Numerical stability

Olver et al. '95

It has been proven that it is possible to obtain numerical stability by choosing another OTS, taking

$$a_j = \frac{n+1-j}{n+1}$$

For the third hyperasymptotic level we have

$$n = 8 \qquad m = 6 \qquad m' = 4 \qquad m'' = 2$$

and

$$Z(0) = 0.965560480$$

- The non-Borel summable sector

For  $\lambda = -\frac{1}{3}$ , we have

Mathematica	$1.05995021 - 0.00758472 i$
Stieljes	$1.06098837 + 0 i \pm 0.00140990$
$S_5 + S_2^{\text{NP}}$	$1.05990083 - 0.00752794 i$

## CONCLUSIONS

- The Mellin Barnes representation is a perfect tool to obtain asymptotic expansions of Feynman diagrams containing one or several scales. It assures the calculation at any order and under certain conditions the exact representation.
- The Mellin Barnes representation is also a powerful instrument to deal with perturbative divergent series and their resummation. It allows also through the hyperasymptotic method to get non perturbative contributions directly from the perturbative expansion itself.