

Convergence of ray sequences of Padé approximants for ${}_2F_1(a, 1; c; z)$ when $c > a > 0$

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The Gauss hypergeometric function

Definition

$${}_2F_1(a, b; c; z) = 1 + \sum_{k=1}^{\infty} \frac{(a)_k (b)_k z^k}{(c)_k k!}, \quad |z| < 1,$$

where the parameters a, b, c and z may be real or complex and

$$(\alpha)_k = \alpha(\alpha + 1) \dots (\alpha + k - 1)$$

is Pochhammer's symbol.

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If $a = -n$, the series terminates and becomes a polynomial of degree n in x .

The Padé Approximant

The $[m/n]$ Padé approximant for a formal power series

$$f(z) = \sum_{k=0}^{\infty} t_k z^k$$

is a rational function

$$\frac{P_{mn}(z)}{Q_{mn}(z)}$$

of type (m, n) such that

$$f(z)Q_{mn}(z) - P_{mn}(z) = o(z^{m+n+1})$$

as $z \rightarrow 0$.

Padé table

The $[m/n]$ Padé approximants for f can be arranged to form the Padé table of f .

$[0/0]$	$[0/1]$	$[0/2]$	$[0/3]$	$[0/4]$...
$[1/0]$	$[1/1]$	$[1/2]$	$[1/3]$	$[1/4]$...
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The Padé table is normal if each entry in the table is normal.



Padé approximants for ${}_2F_1(a, 1; c; z)$

Theorem (Padé (1907))

Let $c \notin \mathbb{Z}^-$ and let $m \geq n - 1$. Then the denominator polynomial in the $[m/n]$ Padé approximant $P_{mn}(z)/Q_{mn}(z)$ for ${}_2F_1(a, 1; c; z)$ is given by

$$Q_{mn}(z) = {}_2F_1(-n, -a - m; -c - m - n + 1; z)$$

and

$$\begin{aligned} R_{mn}(z) &= Q_{mn}(z) {}_2F_1(a, 1; c; z) - P_{mn}(z) \\ &= S_{mn} z^{m+n+1} {}_2F_1(a + m + 1, n + 1; c + m + n + 1; z) \end{aligned}$$

where

$$S_{mn} = n! \frac{(a)_{m+1} (c - a)_n}{(c)_{m+n} (c + m)_{n+1}}$$



Padé approximants for ${}_2F_1(a, 1; c; z)$

Theorem (van Rossum, 1955)

If $a, c, c - a \notin \mathbb{Z}^-$, the Padé approximants for ${}_2F_1(a, 1; c; z)$ are normal for $m \geq n - 1$.

Padé approximants for ${}_2F_1(a, 1; c; z)$

Theorem (van Rossum, 1955)

If $a, c, c - a \notin \mathbb{Z}^-$, the Padé approximants for ${}_2F_1(a, 1; c; z)$ are normal for $m \geq n - 1$.

Theorem (de Bruin, 1976)

The Padé table for the hypergeometric series ${}_2F_1(a, 1; c; z)$ with $c > a > 0$ is normal.

The numerator polynomial

The numerator polynomial $P_{mn}(z)$ is determined by

$$f(z)Q_{mn}(z) - P_{mn}(z) = O(z^{m+n+1})$$

since $Q_{mn}(z)$ is known.

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$P_{mn}(z)$ is the polynomial we obtain from the first $(m+1)$ terms in the product

$${}_2F_1(a, 1; c; z) {}_2F_1(-n, -a-m; -c-m-n+1; z)$$



The numerator polynomial

Thus

$$P_{mn}(z) = \sum_{r=0}^m \sum_{l=0}^r \frac{(a)_{r-l}(-n)_l(-a-m)_l}{(-c-m-n+1)_l(c)_{r-l}l!} z^r$$

for $0 \leq r \leq m$.

The numerator polynomial

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for $0 \leq r \leq m$.

Example

For $a = 2$, $c = 6$, $m = 3$ and $n = 4$,

$$P_{34}(z) = 1 - \frac{4}{3}z + \frac{344}{693}z^2 - \frac{1}{22}z^3$$

which is not equal to ${}_2F_1(-3, \alpha; \beta; z)$ for any α, β .

Zero location of ${}_2F_1$ polynomials

Theorem

Let $F(z) = {}_2F_1(-n, b; d; z)$, where $n \in \mathbb{N}$, $b, d \in \mathbb{R}$ and $d \notin \mathbb{Z}^-$.

- (i) For $d > 0$ and $b > d + n - 1$, all n zeros of F are real and simple and lie in $(0, 1)$.
- (ii) For $b < 1 - n$ and $d < b + 1 - n$, all n zeros of F are real and simple and lie in $(1, \infty)$.
- (iii) For $b < 1 - n$ and $d > 0$, all zeros of F are real and simple and lie in $(-\infty, 0)$.

Proof:

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$$z^{c-1}(1-z)^{b-c-n} {}_2F_1 \left(\begin{matrix} -n, b \\ c \end{matrix} ; z \right) = \frac{1}{(c)_n} \frac{d^n}{dz^n} [z^{c-1+n}(1-z)^{b-c}]$$

Proof:

- Klein, 1890 - Geometric argument
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- Orthogonality of ${}_2F_1$'s and Rodrigues formula

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Application:

- Poles of Padé approximants for ${}_2F_1(a, 1; c; z)$
- Convergence of Padé approximants for ${}_2F_1(a, 1; c; z)$

Poles of the Padé approximant for ${}_2F_1(a, 1; c; z)$

Corollary

For $c \notin \mathbb{Z}^-$ and $m \geq n - 1$, the poles of the $[m/n]$ Padé approximant for ${}_2F_1(a, 1; c; z)$ lie in the intervals

- (i) $(0, 1)$ if $a < c < 1 - m - n$
- (ii) $(1, \infty)$ if $c > a > n - m - 1$
- (iii) $(-\infty, 0)$ if $a > n - m - 1$ and $c < 1 - m - n$.

Remark

(ii) Since $m \geq n - 1$, the condition $c > a > n - m - 1$ is stronger than the condition $c > a > 0$

- given by Kenney & Laub (1989)
- that guarantees normality of the Padé table.

The poles of the Padé approximant and convergence in the table

- The location and behavior of the zeros and poles of Padé approximants for various special functions, as well as the asymptotic zero and pole distribution, has been studied by many authors, most notably E. Saff and R. Varga [1978]
- The convergence of different types of sequences in the Padé table has been studied extensively.
 - Exponential function (Perron, 1957)
 - Convergence behavior in the Padé table for ${}_1F_1(1; c; z)$ with $c \notin \mathbb{Z}^-$ is similar to that of the exponential function (de Bruin, 1976)

Convergence in the Padé table for ${}_2F_1(a, 1; c; z)$, $c > a > 0$

Lemma

For $m \geq n - 1$ and $c > a > 0$ we have that

$$\begin{aligned} R_{mn}(z) &= Q_{mn}(z) {}_2F_1(a, 1; c; z) - P_{mn}(z) \\ &= S_{mn} z^{m+n+1} {}_2F_1(a + m + 1, n + 1; c + m + n + 1; z) \end{aligned}$$

tends to zero uniformly in z as $m \rightarrow \infty$, $n/m \rightarrow \rho$ with $0 < \rho \leq 1$ on compact subsets of $|z| < 1$.

Convergence in the Padé table for ${}_2F_1(a, 1; c; z)$, $c > a > 0$

Theorem

Let $a, c, c - a \notin \mathbb{Z}^-$ and $m \geq n - 1$. For $c > a > 0$, the sequence of $[m/n]$ Padé approximants

$$\frac{P_{mn}(z)}{Q_{mn}(z)}$$

converges to

$${}_2F_1(a, 1; c; z)$$

for $m \rightarrow \infty$, $n/m \rightarrow \rho$ with $0 < \rho \leq 1$, uniformly in z on compact subsets of $|z| < 1$.

The Padé table

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