Convergent and divergent series, solutions of the Prolate Spheroidal differential equation

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Context: the DESIR *package (1)*

A linear homogeneous ODE

$$L(y) = a_d y^{(d)} + a_{d-1} y^{(d-1)} + \cdots + a_0 y = 0$$

in the neighborhood of the singularities: $a_d(t) = 0$

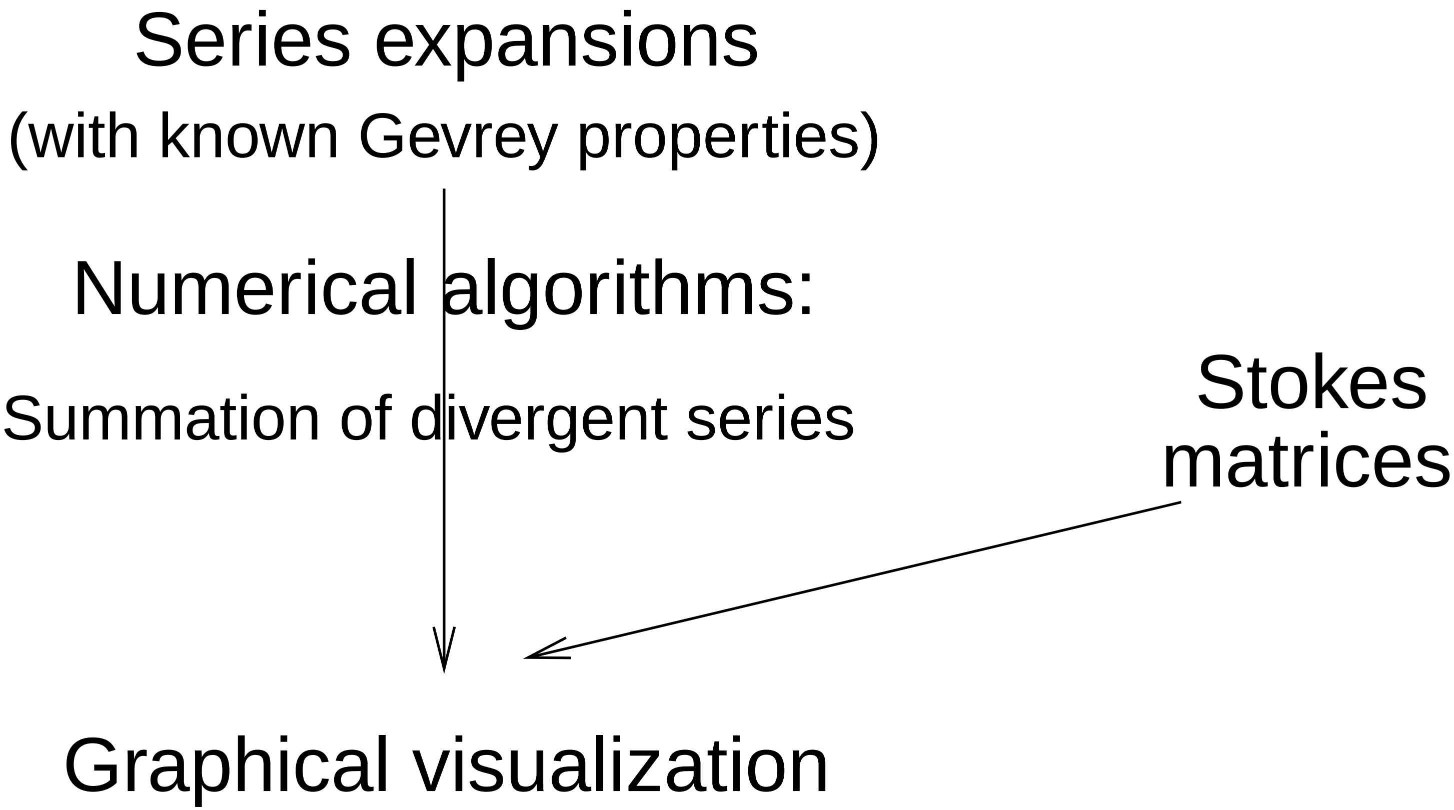
Formal algorithms

Series expansions

$$y(x) = exp(Q(1/x)) x^{\lambda} \hat{\varphi}(x), \hat{\varphi}(x) \in \mathbb{C}[[x]][\ln x],$$

$$t = x^n, n \in \mathbb{N}^*$$

Context: the DESIR *package (2)*



Series expansions

(with known Gevrey properties)

Numerical algorithms:

Summation of divergent series

Stokes
matrices

Graphical visualization

Outline

- The prolate spheroidal wave functions (definitions and basic properties)
- Experimental results on Stokes phenomenon and monodromy
- Proof of the conjecture

The prolate spheroidal wave functions

The prolate spheroidal wave functions, (PSWFs) $\{\varphi_{n,\sigma,\tau}\}$, constitute an orthonormal basis of the space of σ-bandlimited functions on the real line,
i. e. functions whose Fourier transforms have support on the interval $[-\sigma,\sigma]$.

The PSWFs are maximally concentrated on the interval $[-\tau,\tau]$ (as precised below) and depend on parameters σ and τ.

Properties

They can be characterized by one of the following properties:

- as the maximum energy concentration of a σ-bandlimited function on the interval $[-\tau, \tau]$; that is, $\varphi_{0,\sigma,\tau}$ is the function of total energy 1 ($= ||\varphi_{0,\sigma,\tau}||^2$) such that

$$\int_{-\tau}^{\tau} |f(t)|^2$$

is maximized, $\varphi_{1,\sigma,\tau}$ is the function with the maximum energy concentration among those functions orthogonal to $\varphi_{0,\sigma,\tau}$, etc...;

Properties

- as the eigenfunctions of an integral operator with kernel arising from the sinc function $S(t) = \sin(\pi t)/\pi t$:

$$\frac{\sigma}{\pi}\int_{-\tau}^{\tau}\varphi_{n,\sigma,\tau}S\Big(\frac{\sigma}{\pi}(t-x)\Big)dx = \lambda_{n,\sigma,\tau}\varphi_{n,\sigma,\tau}(t);$$

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- as the eigenfunctions of a differential operator arising from a Helmholtz equation on a prolate spheroid:

$$(\tau^2 - t^2)\varphi''_{n,\sigma,\tau} - 2t\varphi'_{n,\sigma,\tau} - \sigma^2 t^2\varphi_{n,\sigma,\tau} = \mu_{n,\sigma,\tau}\varphi_{n,\sigma,\tau}.$$

Expansion of the solutions in the complex plane (1)

$$L_{\sigma,\tau} = (\tau^2 - t^2)\frac{d^2}{dt^2} - 2t\frac{d}{dt} - \sigma^2 t^2$$

Solutions of the equation $L_{\sigma,\tau}(\varphi) - \mu\varphi = 0$.
Two finite singularities: $\tau, -\tau$.
Both singularities are regular one.
In the neighborhood of τ, a basis of solutions constituted of:

- a regular function f
- a solution of the form $f\log(t-\tau)+g$, g is also regular.

Expansion of the solutions in the complex plane (2)

The point at infinity is also a singularity.
This is an irregular one.
We obtain the following basis of solutions, with $x = \frac{1}{t}$:

$$y_1(x) = e^{-\frac{I\sigma}{x}}\, x \left(1 - \frac{1}{2}\, \frac{I\sigma\,(\mu - \sigma^2\,\tau^2)\,x}{\sigma^2} + \mathrm{O}(x^2)\right)$$

$$y_2(x) = e^{\frac{I\sigma}{x}}\, x \left(1 + \frac{1}{2}\, \frac{I\sigma\,(\mu - \sigma^2\,\tau^2)\,x}{\sigma^2} + \mathrm{O}(x^2)\right)$$

There is no ramification. The series are *a priori* divergent, but 1-summable in each direction but $\pm I\sigma\mathbb{R}^+$.

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From now on:

$\sigma = \tau = 1$ and $D_\mu : y \longmapsto (t^2 - 1)y'' + 2ty' + (t^2 - \mu)y$.

Neighborhood of 1

$$f_\mu(t), \qquad f_\mu(t)\ln(t-1)+\varphi_\mu(t),$$

where

$$f_\mu(t) = 1+(-\frac{1}{2}+\frac{\mu}{2})\,(t-1)+(\frac{1}{16}\,\mu^2-\frac{1}{4}\,\mu-\frac{1}{16})\,(t-1)^2+\mathrm{O}((t-1)^3),$$

and

$$\varphi_\mu(t) = (\frac{1}{2}-\mu)\,(t-1)+(\frac{1}{2}\,\mu+\frac{1}{16}-\frac{3}{16}\,\mu^2)\,(t-1)^2+\mathrm{O}((t-1)^3).$$

Neighborhood of -1

$$g_\mu(t), \qquad g_\mu(t)\ln(t+1)+\psi_\mu(t),$$

where

$$g_\mu(t)=1+(\frac{1}{2}-\frac{\mu}{2})\,(t+1)+(\frac{1}{16}\,\mu^2-\frac{1}{4}\,\mu-\frac{1}{16})\,(t+1)^2+\mathrm{O}((t+1)^3),$$

and

$$\psi_\mu(t)=(-\frac{1}{2}+\mu)\,(t+1)+(\frac{1}{2}\,\mu+\frac{1}{16}-\frac{3}{16}\,\mu^2)\,(t+1)^2+\mathrm{O}((t+1)^3)$$

All the series $f_\mu, \varphi_\mu, g_\mu, \psi_\mu$ are convergent, with a radius of convergence at least 2.

Neighborhood of ∞

$$\hat{y}_1(x) = e^{\frac{-I}{x}} x \Big(1 - \frac{1}{2}\, I\,(-1+\mu)x + (-\frac{1}{8}\,\mu^2 + \frac{1}{2}\,\mu + \frac{1}{8})x^2$$

$$+\mathrm{O}(x^3)\Big)$$

$$\hat{y}_2(x) = e^{\frac{I}{x}} x \Big(1 + \frac{1}{2}\, I\,(-1+\mu)x + (-\frac{1}{8}\,\mu^2 + \frac{1}{2}\,\mu + \frac{1}{8})x^2$$

$$+\mathrm{O}(x^3)\Big)$$

Stokes phenomenon (1)

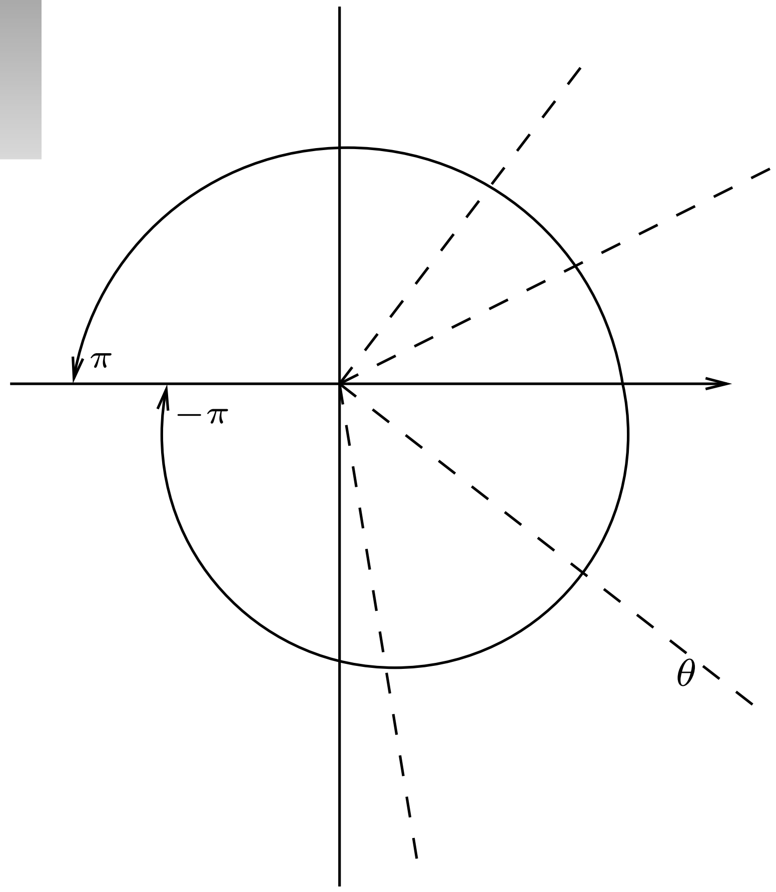
Let $\hat{f}_1 = \frac{e^{\frac{I}{x}}}{x}\hat{y}_1$ and $\hat{f}_2 = \frac{e^{\frac{-I}{x}}}{x}\hat{y}_2$.
The series $\hat{f}_1$ is 1-summable in all directions but $-I\mathbb{R}^+$.
The series $\hat{f}_2$ is 1-summable in all directions but $I\mathbb{R}^+$.

$$f_i^-(x) = \int_{d_\theta} e^{\frac{-\zeta}{t}} \mathcal{B}\hat{f}_i(\zeta)d\zeta, \qquad \theta \in]-\frac{\pi}{2}, \frac{\pi}{2}[$$

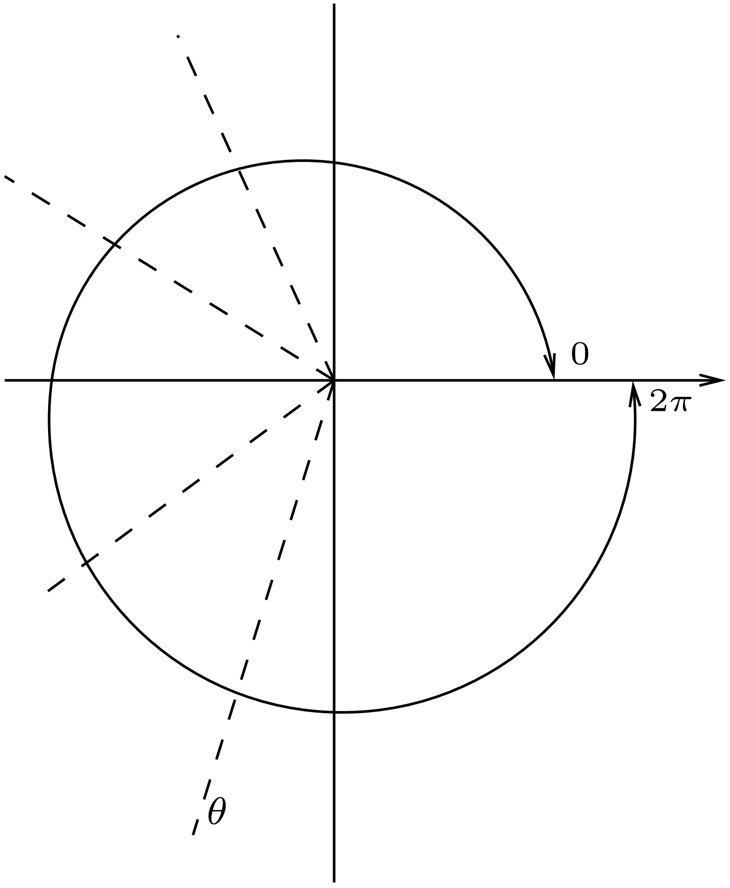
$$f_i^+(x) = \int_{d_\theta} e^{\frac{-\zeta}{t}} \mathcal{B}\hat{f}_i(\zeta)d\zeta, \qquad \theta \in]\frac{\pi}{2}, \frac{3\pi}{2}[$$

$$f_1^- = f_1^+$$

Stokes phenomenon (2)



Domain of y_i^-



Domain of y_i^+

Stokes phenomenon (3)

The functions y_i^- and y_i^+ are defined together on the sector $]-\pi,\pi[$.
The Stokes matrix associated to the Stokes ray $\frac{\pi}{2}$ is:

$$(y_1^- \quad y_2^-) = (y_1^+ \quad y_2^+) S^{\frac{\pi}{2}}.$$

As $y_1^- = y_1^+$, $S^{\frac{\pi}{2}} = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}.$

For similar reasons:

$$S^{\frac{-\pi}{2}} = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}.$$

Stokes matrices, in practice

It is possible to compute:

1. a linear ODE satisfied by $\hat{f}_1$ (formal)
2. a linear ODE satisfied by $\mathcal{B}\hat{f}_1$ (formal)
3. a basis of formal solutions of the previous ODE near each singularity ω, on the example $\omega = -2I$ (formal)
4. the expression of $\mathcal{B}\hat{f}_1$ in this basis (numerical)

We deduce a numerical value of α.

Stokes matrices: for which equations ?

- of single rank, $k \neq 1$;
- the Borel transform of the divergent series can have any polar, ramified or logarithmic singularities;
- the Borel transform of the divergent series don’t have irregular singularities;
- the Borel transform of the divergent series don’t have many singularities aligned on a half line issued from the origin.

A first Stokes matrix

For the first eigenvalue $\mu_0 = 0.319$:

$>$

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StokesMatrices(subs(mu=319/1000,resu),
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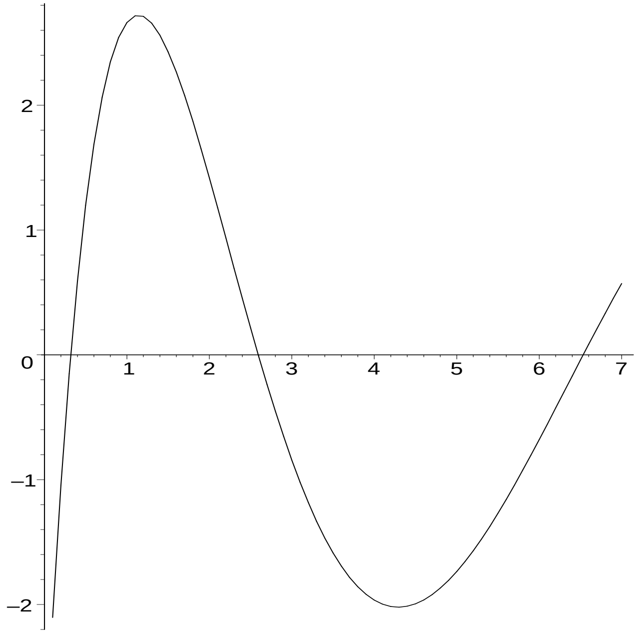
$[8, 8]);$

$$[[-\frac{\pi}{2}, \left[\begin{array}{cc} 1 & 0 \\ 0.1095087321\,10^{-23} - 0.4342357172\,10^{-6}\,I & 1 \end{array}\right]],$$

$$[\frac{\pi}{2}, \left[\begin{array}{cc} 1 & -0.1095087321\,10^{-23} - 0.4342357172\,10^{-6}\,I \\ 0 & 1 \end{array}\right]]]$$

A curve $\alpha(\mu)$

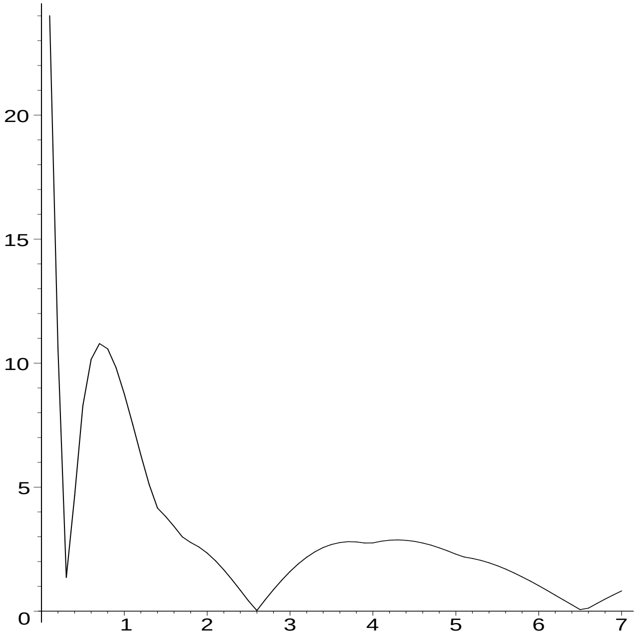
Numerically : $\alpha \in I\mathbb{R}$.
We draw the curve $\Im(\alpha(\mu)), \quad \mu \in]0,7]$.



$\mu_0 = 0.319$, $\mu_1 = 2.593084$, $\mu_2 = 6.533471$.

Monodromy around $[-1, 1]$

We draw $||M_1 M_{-1} - Id||_\infty$:

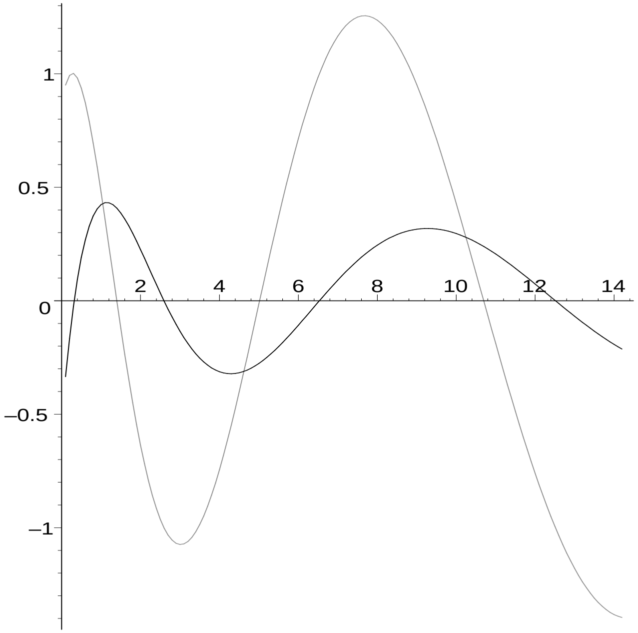


Again, it seems that the monodromy is trivial for the first three eigenvalues.

Connection between 1 ***and*** -1

Basis $[f_\mu, f_\mu \log(t-1)+\varphi_\mu]$ in the neighborhood of 1.
Basis $[g_\mu, g_\mu \log(t+1)+\psi_\mu]$ in the neighborhood of -1.

$$f_\mu = a_\mu g_\mu + b_\mu (g_\mu \log(t+1) + \psi_\mu).$$



For the first four eigenvalues: $b_\mu = 0$ and $a_\mu = \pm 1$.

Conjecture

The following properties are equivalent:

1. μ is an eigenvalue of the differential operator
   $L_{1,1} = (t^2 - 1)\dfrac{d^2}{dt^2} + 2t\dfrac{d}{dt} + t^2$;
2. the series f_μ and g_μ are entire functions (and so, eigenfunctions) ;
3. the series $\hat{f}_1$ and $\hat{f}_2$ (at ∞) are convergent ;
4. the Stokes phenomenon is trivial (the Stokes matrices in the basis $[\hat{y}_1, \hat{y}_2]$ are identity) ;
5. the monodromy around $[-1, 1]$ is trivial.

Proof of the conjecture

- $(4) \iff (5)$
  The monodromy around the segment is the product of the two Stokes matrices $S^{\frac{\pi}{2}}$ and $S^{\frac{-\pi}{2}}$.

$$S^{\frac{\pi}{2}} S^{\frac{-\pi}{2}} = \begin{pmatrix} 1+\alpha^2 & \alpha \\ \alpha & 1 \end{pmatrix}$$

- $(5) \implies (2)$
  f_μ can be continued at -1, and it has no other singularity, thus is entire.
- $(2) \implies (1)$
  Asymptotic behavior at $\pm\infty$ of type $\frac{e^{\pm It}}{t}$: f_μ is an eigenfunction.

Proof of the conjecture

- $(1) \Longrightarrow (5)$ D_μ admits a solution which is an entire function f. Then f is a multiple of f_μ and a multiple of g_μ. And:
  - f is odd or even;
  - $h_1 = f\log(t+1) + u$, u holomorphic at -1;
  - $u = \lambda f\log(t-1) + v$, v is an entire function;
  - $h_1(-t)$ is also a solution, then (f even) $h_2 = f\log(t-1) + u(-t)$ is a solution;
  - $h = f\log(\frac{t+1}{t-1}) + g$, g is an entire function.

  The monodromy around $[-1,1]$ in the basis $[f,h]$ is trivial.

Conclusion

- Hermite equation $y''(t) - ty(t) + ay(t) = 0$: a series solution becomes a polynomial for particular values of the parameter (a positive integer);
- Prolate Spheroidal equation: the divergent series become convergent for particular values of the parameter μ;
- ... and more information on the DESIR package:

`http://www-ljk.imag.fr/CASYS/LOGICIELS/desir2009.html`