Perfectness of Nikishin systems. Applications.

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STANDARD ORTHOGONALITY.

Let s be a finite Borel measure supported on an interval Δ. For each $n \in \mathbb{Z}_+$ there exists a monic polynomial $Q_n, \deg Q_n \leq n$, such that

$$0 = \int x^\nu Q_n(x) ds(x), \qquad \nu = 0, \ldots, n-1.$$

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TYPE II MULTIPLE ORTHOGONALITY.

Let $S = (s_0, \ldots, s_m)$ be a system of finite Borel measures with constant sign, $\operatorname{supp} s_k \subset \Delta \subset \mathbb{R}$. Assume that

$$ds_k = w_k ds_0, \quad w_k \in \mathrm{C}(\Delta), \quad k = 0, \ldots, m.$$

Fix

$$\mathbf{n} = (n_0, \ldots, n_m) \in \mathbb{Z}_+^{m+1}.$$

Definition 1. We say that $Q_{\mathbf{n}}$ is the $\mathbf{n}$-th type II multiple orthogonal polynomials with respect to S if

- $Q_{\mathbf{n}} \not\equiv 0, \deg Q_{\mathbf{n}} \leq |\mathbf{n}| = n_0 + \cdots + n_m,$
- $\int x^\nu Q_{\mathbf{n}}(x) ds_k(x) = 0, \quad \nu = 0, \ldots, n_k - 1, \quad k = 0, \ldots, m.$

Definition 2. If $\deg Q_{\mathbf{n}} = |\mathbf{n}|$, we say that $\mathbf{n}$ is **normal.** If all $\mathbf{n} \in \mathbb{Z}_+^{m+1}$ are normal S is type II **perfect.**

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AT SYSTEM.

Definition 3. $W = (1, w_1, \dots, w_m)$ is an AT system on Δ if for each $\mathbf{n} \in \mathbb{Z}_+^{m+1}$ the (real) linear form of degree $|\mathbf{n}|$

$$\mathcal{W}_{\mathbf{n}} = p_0 + p_1 w_1 + \cdots + p_m w_m, \quad \deg p_k \leq n_k - 1,$$

has at most $|\mathbf{n}| - 1$ zeros on Δ,

Theorem. If W is an AT system on Δ, then $\deg Q_{\mathbf{n}} = |\mathbf{n}|$ for all $\mathbf{n} \in \mathbb{Z}_+^{m+1}$; moreover, its zeros are all simple and lie in the interior of Δ.

Proof. The definition of $Q_{\mathbf{n}}$ implies that

$$\int (p_0(x) + \sum_{k=0}^{m} p_k(x) w_k(x)) Q_{\mathbf{n}}(x) ds_0(x) = 0$$

If $Q_{\mathbf{n}}$ had $\leq |\mathbf{n}| - 1$ sign changes inside $\mathrm{Co}(\operatorname{supp} \sigma_0)$, a contradiction is immediate.

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NIKISHIN SYSTEMS.

Given two measures $\sigma_\alpha, \sigma_\beta$ supported on disjoint intervals, set

$$d\langle \sigma_\alpha, \sigma_\beta \rangle(x) := \int \frac{d\sigma_\beta(t)}{x-t} d\sigma_\alpha(x) := \widehat{\sigma}_\beta(x) d\sigma_\alpha(x).$$

Definition 4. Let $(\sigma_0, \ldots, \sigma_m)$ be a system of finite measures,

$$\Delta_k = \mathrm{Co}(\mathrm{supp}(\sigma_k)), \qquad \Delta_k \cap \Delta_{k+1} = \emptyset, \quad k = 0, \ldots, m-1.$$

The system $S = (s_0, \ldots, s_m) = \mathcal{N}(\sigma_0, \ldots, \sigma_m)$ given by

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MAIN RESULTS.

Set

$$s_{j,k} = \langle \sigma_j, \ldots, \sigma_k \rangle, \quad 0 \leq j \leq k \leq m.$$

$$s_{j,k} = \langle \sigma_j, \sigma_{j-1} \ldots, \sigma_k \rangle, \quad 0 \leq k < j \leq m.$$

Theorem 1. The system of functions $(1, \widehat{s}_{1,1}, \ldots, \widehat{s}_{1,m})$ forms an AT system on any subinterval disjoint from $\mathrm{Co}(\operatorname{supp} \sigma_1)$.

Theorem 2. $S = \mathcal{N}(s_0, \ldots, s_m)$ is type II perfect. All the zeros of $Q_{\mathbf{n}}$ are simple and lie in the interior of $\mathrm{Co}(\operatorname{supp} \sigma_0)$.

Proof of Th. 2. Notice that

$$\int x^\nu Q_{\mathbf{n}}(x) \widehat{s}_{1,j}(x) d\sigma_0(x) = 0, \quad \nu = 0, \ldots, n_j - 1, \quad j = 0, \ldots, m.$$

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TYPE II HERMITE PADÉ APPROXIMATION.

Given $S = (s_0, \ldots, s_m) = \mathcal{N}(\sigma_0, \ldots, s_m), \mathbf{n} \in \mathbb{Z}_+^{m+1}$, there exist polynomials, $Q_{\mathbf{n}}, P_{\mathbf{n},j}$:

- $Q_{\mathbf{n}} \not\equiv 0, \deg Q_{\mathbf{n}} \leq |\mathbf{n}|$
- $(Q_{\mathbf{n}} \widehat{s}_j - P_{\mathbf{n},j})(z) = \mathcal{O}(1/z^{n_j+1}), z \to \infty, \quad j = 0, \ldots, m.$

The vector rational function $(P_{\mathbf{n},0}/Q_{\mathbf{n}}, \ldots, P_{\mathbf{n},m}/Q_{\mathbf{n}})$ is called type II Hermite-Padé approximant of $(\widehat{s}_0, \ldots, \widehat{s}_m)$.

Corollary 1. Let there be constants $c > 0, \kappa < 1$:

$$n_j \geq \frac{|\mathbf{n}|}{m+1} - c|\mathbf{n}|^{\kappa}, \quad \mathbf{n} \in \Lambda \subset \mathbb{Z}_+^{m+1}.$$

Then,

$$\lim_{\mathbf{n} \in \Lambda} P_{\mathbf{n},j}/Q_{\mathbf{n}} = \widehat{s}_j, \quad K \subset \overline{\mathbb{C}} \setminus \mathrm{Co}(\operatorname{supp} \sigma_0), \quad j = 0, \ldots, m.$$

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SIMULTANEOUS GAUSS-JACOBI TYPE FORMULAS.

The zeros $x_{\mathbf{n},j}, j = 1, \ldots, |\mathbf{n}|$, of $Q_{\mathbf{n}}$ are simple so

$$\frac{P_{\mathbf{n},k}(z)}{Q_{\mathbf{n}}(z)} = \sum_{i=1}^{|\mathbf{n}|} \frac{\lambda_{\mathbf{n},k,i}}{z - x_{\mathbf{n},i}},$$

where

$$\lambda_{\mathbf{n},k,i} = \lim_{z \to x_{\mathbf{n},i}} (z - x_{\mathbf{n},i}) \frac{P_{\mathbf{n},k}(z)}{Q_{\mathbf{n}}(z)} = \frac{P_{\mathbf{n},k}(x_{\mathbf{n},i})}{Q'_{\mathbf{n}}(x_{\mathbf{n},i})}.$$

Corollary 2. Let $(s_{0,0}, \ldots, s_{0,m}) = \mathcal{N}(\sigma_0, \ldots, \sigma_m)$ and $\mathbf{n} = \mathbb{Z}_+^{m+1}$. For each $k = 0, \ldots, m$ and $p, \deg p \leq |\mathbf{n}| + n_k - 1$

$$\int p(x) ds_{0,k}(x) = \sum_{i=1}^{|\mathbf{n}|} \lambda_{\mathbf{n},k,i} p(x_{\mathbf{n},i}).$$

If $\mathbf{n} = (n, n+1, \ldots, n+1)$, then

$$\operatorname{sign}(\lambda_{\mathbf{n},k,i}) = \operatorname{sign}(s_{0,k}), \qquad i = 1, \ldots, |\mathbf{n}|.$$

CONVERGE OF GAUSS-JACOBI TYPE FORMULAS.

Corollary 3. Let $(s_{0,0},\ldots,s_{0,m})=\mathcal{N}(\sigma_0,\ldots,\sigma_m)$ and $\mathbf{n}=\mathbb{Z}_+^{m+1}$ be given. Consider $\{(n,n+1,\ldots,n+1)\}_{n\in\mathbb{Z}_+}\subset\mathbb{Z}_+^{m+1}$. For any bounded Riemann-Stieltjes integrable function f on $\mathrm{Co}(\operatorname{supp}\sigma_0)$ and each $k=0,\ldots,m$

$$\int f(x)ds_{0,k}(x)=\lim_{n\to\infty}\sum_{i=1}^{|\mathbf{n}|}\lambda_{\mathbf{n},k,i}f(x_{\mathbf{n},i}).$$

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