Convergence of sequences of linear fractional transformations

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$$f = \lim S_n(0), \qquad S_n \to S.$$

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One way to transform a sequence of numbers or functions to a sequence of linear fractional transformations:

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Sylvester 1869: *I think a substantial difference does arise in favor of the continued fraction form, inasmuch as it indicates a certain obvious correction to be applied in order that the convergence may become more exact.*

A second method to transform a power series $\sum c_n z^n$ (or Fourier series or...) to a sequence of linear fractional transformations: Choose the coefficients of $\tau_n(w)$ to be polynomials in z (or other functions of z) such that the series expansion of $\tau_n(z)$ coincides with the given series as far out as possible.

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EXAMPLE: With

$$a_0(z) := z^a e^{-z}, \quad a_{2n+1}(z) := -(a+n)z, \quad a_{2n}(z) := nz, \quad b_n := a+n$$

the continued fraction

$$K(a,z) := \cfrac{a_0(z)}{b_0 + \cfrac{a_1(z)}{b_1(z) + \cfrac{a_2(z)}{b_2(z) + \cdots}}}$$

converges to the incomplete gamma function $\gamma(a,z)$.

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The approximants of $\frac{K(a,z)}{a_0(z)}$ are diagonal Padé approximants to $\gamma(a,z)/a_0(z)$. The convergence acceleration we do is therefore convergence acceleration of Padé approximants.

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OBS! Let $\tau_n \to \gamma$ and $w_n := \tau_n^{-1}(\gamma^\dagger)$ for some $\gamma^\dagger \neq \gamma$. Then $\tau_n(w_n) \not\to \gamma$.

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Continued fractions:

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More generally: Fore given $\{\varphi_n\}$ from $\mathcal{M}$:

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Find sufficient conditions on $\{\varphi_n\}$ to conclude convergence of $\{\tau_n\}$.

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THEOREM: (L 1994) Let $\varphi_n(\mathbb{D}) \subseteq \mathbb{D} \setminus B(z_n \varepsilon)$ for all n. If $\exists \{w_n\}$ bounded away from $\partial\mathbb{D}$ such that $\{\varphi_n(w_n)\}$ is bounded away from $\partial\mathbb{D}$, then $\tau_n \to \gamma \in \overline{\mathbb{D}}$.

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THEOREM: (L 2009) $\mathrm{K}(a_n/1)$ with $a_n \to -\frac{1}{4}$ converges generally if $\Gamma_\varepsilon := \partial V \cap B(-\frac{1}{2}, \varepsilon)$ is simple arc with uniformly non-horizontal tangent at every point.

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$\partial^* V := \partial V \cap \partial(-1-V)$ is the only part where they can meet.

$$s_n(w) = \frac{a_n}{1+w} \quad \Rightarrow \quad s_n(\infty) = 0, \;\; s_n(-1) = \infty$$

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LEMMA: Let $S_{n_{2k}} \to \lambda[\gamma_0, w_0^\dagger]$ and $S_{n_{2k+1}} \to \lambda[\gamma_1, w_1^\dagger]$ where $\gamma_1 \neq \gamma_0$, and let $\psi_k := S_{n_{k-1}}^{-1} \circ S_{n_k}$ for all k. Then $w_1^\dagger, w_2^\dagger \in \partial^* V$ and $\psi_{2k} \to \lambda[w_1^\dagger, w_2^\dagger]$, $\psi_{2k+1} \to \lambda[w_0^\dagger, w_1^\dagger]$.

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LEMMA: (Cordova 1992) If

$$\left|\frac{\zeta_{n-1}^2}{a_n}\right| \cdot \frac{d(\zeta_n,\partial V)}{d(\zeta_{n-1},\partial V)} \geq 1 + C \cdot d(\zeta_{n-1},\partial V)$$

then $\mathrm{K}(a_n/1)$ converges.

THE END