Method of summation of some slowly convergent series

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Approximation and extrapolation of convergent and divergent sequences and series

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Outline

Notations and definitions

Series, partial sums, remainders

- Consider an infinite series

$$s = \sum_{n=0}^{\infty} a_n$$

with terms a_n, partial sums

$$s_n = \sum_{j=0}^{n-1} a_n, \qquad n \in \mathbb{N},$$

and remainders

$$r_n = \sum_{j=0}^{\infty} a_{n+j}, \qquad n \in \mathbb{N} \cup \{0\}.$$

- Thus

$$s = s_n + r_n$$

Notations and definitions

Linear Difference Operators

- Identity operator

$$\mathbb{I}\, x_n = x_n$$

- Shift operator

$$\mathbb{E}\, x_n = x_{n+1}, \qquad \mathbb{E}^k\, x_n = x_{n+k}, \qquad k \in \mathbb{Z}$$

- Linear difference operator $\mathbb{L}$ of order $\operatorname{ord}\mathbb{L} = \ell$

$$\mathbb{L} = \sum_{k=k_0}^{k_0+\ell} \lambda_k(n) \cdot \mathbb{E}^k, \qquad \lambda_{k_0}(n),\, \lambda_{k_0+\ell}(n) \neq 0$$

- Example — forward difference operator:

$$\Delta := \mathbb{E} - \mathbb{I},$$

$$\Delta\, s_n = s_{n+1} - s_n = a_n \tag{1}$$

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- Multiplication

$$\mathbb{P}x_n = y_n, \qquad \mathbb{Q}y_n = z_n \quad \Longrightarrow \quad (\mathbb{Q}\cdot\mathbb{P})\,x_n = z_n$$

- Powers

$$\mathbb{L}^0 \coloneqq \mathbb{I}, \qquad \mathbb{L}^{k+1} \coloneqq \mathbb{L}\cdot\mathbb{L}^k$$

- Operator $\mathbb{L}$ annihilates the sequence x_n, if

$$\mathbb{L}x_n = 0$$

Example

If x_n is a polynomial in n of degree k, then

$$\Delta^{k+1}x_n = 0.$$

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Motivation

We have

$$s = s_n + r_n. \tag{2}$$

Let the linear difference operator $\mathbb{L}^{(\infty)}$ annihilate the remainder r_n. Then

$$\mathbb{L}^{(\infty)}(s) = \mathbb{L}^{(\infty)}(s_n) + \mathbb{L}^{(\infty)}(r_n),$$
$$s \cdot \mathbb{L}^{(\infty)}(1) = \mathbb{L}^{(\infty)}(s_n),$$

and consequently

$$s = \frac{\mathbb{L}^{(\infty)}(s_n)}{\mathbb{L}^{(\infty)}(1)}, \tag{3}$$

if $\mathbb{L}^{(\infty)}(1) \neq 0$.

Problems

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Levin-Type Sequence Transformation

- Let $\mathbb{L}^{(m)}$, $m \in \mathbb{N}$, be an approximation of $\mathbb{L}^{(\infty)}$ in the sense that

$$\begin{cases} \mathbb{L}^{(m)}(r_n^{(m)}) = 0, \\ \mathbb{L}^{(m)}(1) \neq 0, \end{cases} \tag{4}$$

where

$$r_n^{(m)} = r_n - r_{n+m} = a_n + a_{n+1} + \cdots + a_{n+m-1}.$$

- Since

$$s \approx s_n + r_n^{(m)},$$

we can expect that

$$Q_n^{(m)} := \frac{\mathbb{L}^{(m)}(s_n)}{\mathbb{L}^{(m)}(1)} \tag{5}$$

gives an approximation of s, of accuracy growing when $m \to \infty$.

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Method Of Obtaining The Annihilators $\mathbb{L}^{(m)}$

Recurrent Construction

P. Wozny, R. Nowak,
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Applied Mathematics and Computation (2009), accepted.

- According to

$$\mathbb{L}^{(m)}(\underbrace{a_n + a_{n+1} + \ldots + a_{n+m-1}}_{r_n^{(m)}}) = 0, \tag{6}$$

operator $\mathbb{L}^{(1)}$ should annihilate a_n.

- A possible choice is the first-order operator

$$\mathbb{L}^{(1)} := \Delta \cdot \left(\frac{1}{a_n}\mathbb{I}\right) = \frac{1}{a_{n+1}}\mathbb{E} - \frac{1}{a_n}\mathbb{I}. \tag{7}$$

- The operators $\mathbb{L}^{(2)}, \mathbb{L}^{(3)}, \ldots$ are constructed recursively by

$$\mathbb{L}^{(m)} = \mathbb{P}^{(m)}\mathbb{L}^{(m-1)}, \qquad m \geqslant 2. \tag{8}$$

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Step Of Construction

$$r_n^{(m)} = r_n^{(m-1)} + a_{n+m-1}$$

$$\mathbb{L}^{(m-1)}(r_n^{(m-1)}) = 0$$

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Assume that operators $\mathbb{P}^{(m)}$ and $\mathbb{R}^{(m)}$ are such that

$$\mathbb{P}^{(m)}\mathbb{L}^{(m-1)} = \mathbb{R}^{(m)}\widetilde{\mathbb{L}}^{(m-1)}. \tag{9}$$

Then

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Proof:

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Example (A)

$$2\ln 2 = \sum_{n=0}^{\infty} a_n, \qquad a_n := \frac{1}{(n+1)\,2^n} \tag{10}$$

$$\mathbb{L}^{(1)} := (n+1)\,\mathbb{I} - (2n+4)\,\mathbb{E} \quad\Longrightarrow\quad \mathbb{L}^{(1)}(a_n) = 0 \tag{11}$$

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$$\Downarrow$$

$$\mathbb{P}^{(m)} = (n+m)\,\mathbb{I} - (2n+4m)\,\mathbb{E}, \qquad m \in \mathbb{N}\ ?$$

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$$\mathbb{P}^{(m)} = (n+m)\,\mathbb{I} - (2n+4m)\,\mathbb{E}, \qquad m \in \mathbb{N}\ ?$$

$$\mathbb{L}^{(m)} = \mathbb{P}^{(m)}\mathbb{P}^{(m-1)}\cdots\mathbb{P}^{(1)}$$

Example (A (cont.))

$$\mathbb{L}^{(2)} = \boxplus\,\mathbb{I} + \boxminus\,\mathbb{E} + \boxtimes\,\mathbb{E}^2, \qquad \mathbb{L}^{(2)}(a_n + a_{n+1}) = 0$$

$$\widetilde{\mathbb{L}}^{(2)} = \mathbb{E}^2\,\mathbb{L}^{(1)}\,\mathbb{E}^{-2} = (n+3)\,\mathbb{I} - (2n+8)\,\mathbb{E}$$

$$\mathbb{P}^{(3)}\mathbb{L}^{(2)} = \mathbb{R}^{(3)}\widetilde{\mathbb{L}}^{(2)} \tag{13}$$

$$\mathbb{P}^{(3)} = (n+3)\,\mathbb{I} - (2n+12)\,\mathbb{E}, \qquad \mathbb{R}^{(3)} = \rho_0^{(3)}(n)\,\mathbb{I} + \rho_1^{(3)}(n)\,\mathbb{E} + \rho_2^{(3)}(n)\,\mathbb{E}^2,$$

$$\mathbb{L}^{(3)} := \mathbb{P}^{(3)}\mathbb{L}^{(2)} = \boxplus\,\mathbb{I} + \boxminus\,\mathbb{E} + \boxtimes\,\mathbb{E}^2 + \boxdot\,\mathbb{E}^3$$

$$\vdots$$

$$\Downarrow$$

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$$\Downarrow$$

$$\mathbb{P}^{(m)} = (n+m)\,\mathbb{I} - (2n+4m)\,\mathbb{E}, \qquad m \in \mathbb{N}\ ?$$

$$\mathbb{L}^{(m)} = \mathbb{P}^{(m)}\mathbb{P}^{(m-1)}\ldots\mathbb{P}^{(1)}$$

Transformation $\mathcal{Q}_n^{(m)}$

$$\mathbb{L}^{(m)}(a_n + a_{n+1} + \cdots + a_{n+m-1}) = 0, \qquad m \in \mathbb{N}$$
$$\mathbb{L}^{(m)} = \mathbb{P}^{(m)}\mathbb{L}^{(m-1)} = \mathbb{P}^{(m)}\mathbb{P}^{(m-1)} \dots \mathbb{P}^{(1)}$$

$$\mathcal{Q}_n^{(m)} := \frac{\mathbb{L}^{(m)}(s_n)}{\mathbb{L}^{(m)}(1)} =: \frac{N_n^{(m)}}{D_n^{(m)}} \tag{14}$$

$$N_n^{(m)} = \mathbb{P}^{(m)}(N_n^{(m-1)}), \qquad D_n^{(m)} = \mathbb{P}^{(m)}(D_n^{(m-1)})$$

$$\mathbb{P}^{(m)} = ? \qquad \text{for } m \in \mathbb{N}$$

Transformation $\mathcal{Q}_n^{(m)}$

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$$\mathbb{P}^{(m)} = ? \qquad \text{for } m \in \mathbb{N}$$

Algorithm

Let

$$\mathbb{L}^{(0)} \coloneqq \mathbb{I}, \qquad \mathbb{L}^{(1)}(a_n) = 0, \qquad \mathbb{P}^{(1)} \coloneqq \mathbb{L}^{(1)},$$

$$N_n^{(0)} \coloneqq s_n, \qquad D_n^{(0)} \coloneqq 1.$$

For $k = 1, 2, \dots$ do

1. If $k \geqslant 2$, then determine such operators $\mathbb{P}^{(k)}$ and $\mathbb{R}^{(k)}$ that

$$\mathbb{P}^{(k)}\mathbb{L}^{(k-1)} = \mathbb{R}^{(k)}\widetilde{\mathbb{L}}^{(k-1)}, \tag{15}$$

   where

$$\mathbb{L}^{(k-1)} = \mathbb{P}^{(k-1)}\mathbb{P}^{(k-2)} \dots \mathbb{P}^{(1)},$$

$$\widetilde{\mathbb{L}}^{(k-1)} = \mathbb{E}^{k-1}\,\mathbb{L}^{(1)}\,\mathbb{E}^{1-k}$$

2. Compute

$$N_n^{(k)} \coloneqq \mathbb{P}^{(k)}(N_n^{(k-1)}), \qquad D_n^{(k)} \coloneqq \mathbb{P}^{(k)}(D_n^{(k-1)})$$

$$Q_n^{(k)} \coloneqq \frac{N_n^{(k)}}{D_n^{(k)}}$$

Example A ...

$$s_n = \sum_{j=0}^{n-1} a_n, \qquad a_n = \frac{1}{(n+1)2^n}$$

$$\mathbb{P}^{(m)} = (n+m)\,\mathbb{I} - (2n+4m)\,\mathbb{E}, \qquad m \in \mathbb{N}$$

$$N_n^{(0)} \coloneqq s_n = \sum_{j=0}^{n-1} a_j, \qquad D_n^{(0)} \coloneqq 1,$$

$$N_n^{(m)} \coloneqq \mathbb{P}^{(m)}(N_n^{(m-1)}) = (n+m)N_n^{(m-1)} - (2n+4m)N_{n+1}^{(m-1)}, \qquad m \geqslant 1,$$

$$D_n^{(m)} \coloneqq \mathbb{P}^{(m)}(D_n^{(m-1)}) = (n+m)D_n^{(m-1)} - (2n+4m)D_{n+1}^{(m-1)}, \qquad m \geqslant 1,$$

$$\mathcal{Q}_n^{(m)} \coloneqq \frac{N_n^{(m)}}{D_n^{(m)}}.$$

Outline

Theorem

If ord $\mathbb{P}^{(m)} = 1$, *then*

$$\mathcal{Q}_n^{(m)} = \varepsilon_{2m}^{(n)}, \qquad m \in \mathbb{N}.$$

Example A ...

Given the partial sums $s_1, s_2, \ldots, s_5$, one can obtain the following array of quantities $\mathcal{Q}_n^{(m)}$ (underlined digits are exact):

$\mathcal{Q}_1^{(0)} = \underline{1.}000$	$\mathcal{Q}_1^{(1)} = \underline{1.3}750$	$\mathcal{Q}_1^{(2)} = \underline{1.38}596$	$\mathcal{Q}_1^{(3)} = \underline{1.3862}85$	$\mathcal{Q}_1^{(4)} = \underline{1.386294}08$
$\mathcal{Q}_2^{(0)} = \underline{1.}250$	$\mathcal{Q}_2^{(1)} = \underline{1.38}33$	$\mathcal{Q}_2^{(2)} = \underline{1.3862}2$	$\mathcal{Q}_2^{(3)} = \underline{1.38629}2$	
$\mathcal{Q}_3^{(0)} = \underline{1.3}33$	$\mathcal{Q}_3^{(1)} = \underline{1.38}54$	$\mathcal{Q}_3^{(2)} = \underline{1.3862}7$		
$\mathcal{Q}_4^{(0)} = \underline{1.3}65$	$\mathcal{Q}_4^{(1)} = \underline{1.386}0$			
$\mathcal{Q}_5^{(0)} = \underline{1.3}77$				

Using Wynn's ε algorithm and Aitken's iterated Δ^2 process one obtains

$$\varepsilon_4^{(1)} = \underline{1.38}596, \qquad \mathcal{A}_2^{(1)} = \underline{1.386}11.$$

Theorem

If $\operatorname{ord} \mathbb{P}^{(m)} = 1$, *then*

$$\mathcal{Q}_n^{(m)} = \varepsilon_{2m}^{(n)}, \qquad m \in \mathbb{N}.$$

Example A ...

Given the partial sums $s_1, s_2, \ldots, s_5$, one can obtain the following array of quantities $\mathcal{Q}_n^{(m)}$ (underlined digits are exact):

$$\begin{array}{lllll}
\mathcal{Q}_1^{(0)} = \underline{1.}000 & \mathcal{Q}_1^{(1)} = \underline{1.37}50 & \color{red}{\mathcal{Q}_1^{(2)}} = \underline{1.38}596 & \mathcal{Q}_1^{(3)} = \underline{1.3862}85 & \mathcal{Q}_1^{(4)} = \underline{1.386294}08 \\
\mathcal{Q}_2^{(0)} = \underline{1.}250 & \mathcal{Q}_2^{(1)} = \underline{1.383}3 & \mathcal{Q}_2^{(2)} = \underline{1.3862}2 & \mathcal{Q}_2^{(3)} = \underline{1.38629}2 & \\
\mathcal{Q}_3^{(0)} = \underline{1.33}3 & \mathcal{Q}_3^{(1)} = \underline{1.385}4 & \mathcal{Q}_3^{(2)} = \underline{1.3862}7 & & \\
\mathcal{Q}_4^{(0)} = \underline{1.36}5 & \mathcal{Q}_4^{(1)} = \underline{1.386}0 & & & \\
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\end{array}$$

Using Wynn's ε algorithm and Aitken's iterated Δ^2 process one obtains

$$\color{red}{\varepsilon_4^{(1)}} = \underline{1.38}596, \qquad \mathcal{A}_2^{(1)} = \underline{1.386}11.$$

Series ${}_{p+1}F_p(1, \boldsymbol{\alpha};\, \boldsymbol{\beta};\, x)$, $p \geqslant 1$

$$
{}_{p+1}F_p\left(\begin{matrix} 1,\ \alpha_1,\ \alpha_2, \ldots,\ \alpha_p \\ \beta_1,\ \beta_2, \ldots,\ \beta_p \end{matrix} \,\middle|\, x\right) = \sum_{n=0}^{\infty} a_n \quad \text{with } a_n \coloneqq \frac{(\alpha_1)_n(\alpha_2)_n \cdots (\alpha_p)_n}{(\beta_1)_n(\beta_2)_n \cdots (\beta_p)_n}\, x^n.
$$

Operators $\mathbb{P}^{(m)}$

$$
\mathbb{P}^{(1)} \coloneqq \Delta^p \left(\frac{1}{a_n}\mathbb{I}\right) \quad \Longrightarrow \quad \mathbb{P}^{(1)}(a_n) = 0,
$$

$$
\mathbb{P}^{(m)} \coloneqq \sum_{j=0}^{p} \binom{mp}{j} \left[\Delta^j \left(\prod_{i=1}^{p}(\beta_i + n + m(p+1) - j - 2)\right)\right] \Delta^{p-j}, \quad m = 2, 3 \ldots
$$

$$
\mathbb{L}^{(m)} \coloneqq \mathbb{P}^{(m)}\mathbb{P}^{(m-1)} \cdots \mathbb{P}^{(1)} \implies \mathbb{L}^{(m)} = \Delta^{mp}\left(\prod_{i=1}^{p} \frac{(\beta_i)_{n+m-1}}{(\alpha_i)_n} \cdot x^{-n}\, \mathbb{I}\right)
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\mathbb{L}^{(m)}\left(a_n + a_{n+1} + \ldots + a_{n+m-1}\right) = 0
$$

Series ${}_{p+1}F_p(1, \boldsymbol{\alpha};\ \boldsymbol{\beta};\ x)$, $p \geqslant 1$

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{}_{p+1}F_p\left(\begin{matrix}1,\ \alpha_1,\ \alpha_2,\dots,\ \alpha_p\\ \beta_1,\ \beta_2,\dots,\ \beta_p\end{matrix}\ \middle|\ x\right) = \sum_{n=0}^{\infty} a_n \quad \text{with } a_n \coloneqq \frac{(\alpha_1)_n(\alpha_2)_n\cdots(\alpha_p)_n}{(\beta_1)_n(\beta_2)_n\cdots(\beta_p)_n}\,x^n.
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$$

$$
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$$

Series ${}_{p+1}F_p(1, \boldsymbol{\alpha};\ \boldsymbol{\beta};\ x)$, $p \geqslant 1$

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$$ \mathbb{L}^{(m)}\left(a_n + a_{n+1} + \ldots + a_{n+m-1}\right) = 0 $$

Example (Lemniscate constant)

$$A \coloneqq {}_2F_1\left(\begin{matrix}\frac{1}{4},\frac{1}{2}\\ \frac{5}{4}\end{matrix}\,\middle|\,1\right) = {}_3F_2\left(\begin{matrix}1,\frac{1}{4},\frac{1}{2}\\ 1,\frac{5}{4}\end{matrix}\,\middle|\,1\right) = \sum_{n=0}^{\infty} a_n, \qquad a_n \coloneqq \frac{\left(\frac{1}{4}\right)_n\left(\frac{1}{2}\right)_n}{(1)_n\left(\frac{5}{4}\right)_n}$$

$$\mathrm{acc}(\sigma) \coloneqq -\log_{10}\left|\frac{\sigma}{s}-1\right| \quad \text{— accuracy of } \sigma,$$

- $\mathrm{acc}(s_{15}) = 1.25, \qquad \mathrm{acc}(s_{10^3}) = 2.17, \quad \mathrm{acc}(s_{10^6}) = 3.67$
- $\mathrm{acc}(\varepsilon_{14}^{(1)}) = 1.61$
- $\mathrm{acc}({}^{u}\mathscr{L}_1^{(14)}(1,\{s_n\})) = 11.50$
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$$A := {}_2F_1\left(\begin{matrix}\frac{1}{4}, \frac{1}{2} \\ \frac{5}{4}\end{matrix} \middle| \, 1\right) = {}_3F_2\left(\begin{matrix}1, \frac{1}{4}, \frac{1}{2} \\ 1, \frac{5}{4}\end{matrix} \middle| \, 1\right) = \sum_{n=0}^{\infty} a_n, \qquad a_n := \frac{(\frac{1}{4})_n (\frac{1}{2})_n}{(1)_n (\frac{5}{4})_n}$$

$$\mathrm{acc}(\sigma) := -\log_{10}\left|\frac{\sigma}{s} - 1\right| \quad \text{— accuracy of } \sigma,$$

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Some Theoretical Results

Theorem

The $\mathcal{Q}$ *transformation, applied to the series* ${}_{p+1}F_p(1,\alpha;\beta;x)$ *with* $x \neq 1$*, is regular, i.e.,*

$$\lim_{n\to\infty} \mathcal{Q}_n^{(m)}(s_n) = {}_{p+1}F_p\left(\begin{matrix} 1,\ \alpha_1,\ \alpha_2,\ldots,\ \alpha_p \\ \beta_1,\ \beta_2,\ldots,\ \beta_p \end{matrix} \,\middle|\, x\right) \quad \textit{for all } m \in \mathbb{N}.$$

Theorem

$${}_{p+1}F_p\left(\begin{matrix} 1,\ \alpha_1,\ \alpha_2,\ldots,\ \alpha_p \\ \beta_1,\ \beta_2,\ldots,\ \beta_p \end{matrix} \,\middle|\, x\right) - \mathcal{Q}_n^{(m)}(s_n) = \mathcal{O}\left(x^{n+m(p+1)}\right), \qquad x \to 0.$$

Cížek, Zamastil and Skála transformation $\mathscr{G}_n^{(m)}$

$$\mathscr{G}_n^{(m)}\left(\{\!\{q_k\}\!\}_{k=1}^{m-1}, \{\!\{s_n\}\!\}, \{\!\{\omega_n\}\!\}\right) := \frac{\Delta^m \left[\prod_{k=1}^{m-1}(n+q_k)\,\frac{s_n}{\omega_n}\right]}{\Delta^m \left[\prod_{k=1}^{m-1}(n+q_k)\,\frac{1}{\omega_n}\right]} \tag{16}$$

$\{\!\{q_k\}\!\}_{k=1}^{m-1}$ set of parameters,
$\{\!\{\omega_n\}\!\}$ remainder estimates (see [1] or [4, (2.13)])

Theorem

$$Q_n^{(m)} = \mathscr{G}_n^{(m^*)}\left(\{\!\{q_k^*\}\!\}_{k=1}^{m^*-1}, \{\!\{s_n\}\!\}, \{\!\{\omega_n^*\}\!\}\right),$$

$$m^* := mp, \qquad \omega_n^* := (n+1)^{p-1}a_n,$$

$$\{\!\{q_k^*\}\!\}_{k=1}^{m^*-1} := \{\underbrace{1, 1, \ldots, 1}_{p-1 \text{ times}};\ \beta_1, \beta_2, \ldots, \beta_p;\ \beta_1+1, \beta_2+1, \ldots, \beta_p+1;\ \ldots;\ \beta_1+m-2, \beta_2+m-2, \ldots, \beta_p+m-2\}. \tag{17}$$

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$\{\!\{q_k\}\!\}_{k=1}^{m-1}$	set of parameters,	(see [1] or [4, (2.13)])
$\{\!\{\omega_n\}\!\}$	remainder estimates	

$q_k \equiv 1 \implies$ Levin $\mathcal{L}$ transformation

$q_k = k \implies$ Weniger $\mathscr{S}$ transformation (see [4, §8.2])

Theorem

$$\mathcal{Q}_n^{(m)} = \mathscr{G}_n^{(m^*)}\left(\{\!\{q_k^*\}\!\}_{k=1}^{m^*-1}, \{\!\{s_n\}\!\}, \{\!\{\omega_n^*\}\!\}\right),$$

$$m^* := mp, \qquad \omega_n^* := (n+1)^{p-1}a_n,$$

$$\{\!\{q_k^*\}\!\}_{k=1}^{m^*-1} := \{\underbrace{1, 1, \ldots, 1}_{p-1 \text{ times}};\ \beta_1, \beta_2, \ldots, \beta_p;\ \beta_1+1, \beta_2+1, \ldots, \beta_p+1;\ \ldots;$$

Cížek, Zamastil and Skála transformation $\mathscr{G}_n^{(m)}$

$$\mathscr{G}_n^{(m)}\left(\{q_k\}_{k=1}^{m-1},\{s_n\},\{\omega_n\}\right) := \frac{\Delta^m\left[\prod_{k=1}^{m-1}(n+q_k)\,\frac{s_n}{\omega_n}\right]}{\Delta^m\left[\prod_{k=1}^{m-1}(n+q_k)\,\frac{1}{\omega_n}\right]} \tag{16}$$

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Example

$$s = {}_5F_4\left(\begin{matrix}1,1,2,2,7\\3,5,6,9\end{matrix}\;\middle|\;\frac{99}{100}\right) \approx 1.0376328566238592296948,$$

$$\mathcal{Q}_1^{(5)} = \mathscr{G}_1^{(20)}(\{q_k^*\},\{s_n\},\{\omega_n^*\}) \approx \underline{1.03763285662385922}975208,$$

$\mathrm{acc}(\mathscr{G}_n^{(m)}(\{q_k\}_{k=1}^{m-1},\{s_n\},\{\omega_n\}))$: remainder estimates ω_n

parameters q_k	ω_n^*	t-variant	u-variant	v-variant
q_k^*	**19.26**	15.46	16.78	17.67
1 (Levin $\mathcal{L}$ [4, (7.1-7), $\beta = 1$])	18.44	14.46	15.83	16.69
k (Weniger $\mathcal{S}$ [4, (8.2-7), $\beta = 1$])	14.20	16.04	17.33	18.24
k^2 (cf. [1, Tab. 1, 2])	1.51	10.60	8.39	9.73

q-hypergeometric series

- Consider the series

$$
{}_{p+1}\phi_p\left(\begin{matrix} q, \alpha_1, \ldots, \alpha_p \\ \beta_1, \beta_2, \ldots, \beta_p \end{matrix} \;\middle|\; q;\, x\right) = \sum_{n=0}^{\infty} a_n
$$

$$
\text{with } a_n \coloneqq \frac{(\alpha_1;\, q)_n(\alpha_2;\, q)_n \cdots (\alpha_p;\, q)_n}{(\beta_1;\, q)_n(\beta_2;\, q)_n \cdots (\beta_p;\, q)_n} x^n, \qquad (18)
$$

where $(z;\, q)_k$ is a q-Pochhammer symbol defined by:

$$
(z;\, q)_k \coloneqq \begin{cases} 1, & k = 0, \\ \prod_{j=0}^{k-1}(1 - zq^j), & k > 0, \\ \prod_{j=0}^{\infty}(1 - zq^j), & k = \infty, \end{cases}
$$

Notations

$$\triangle_q^{(0;i)} := \mathbb{I}, \qquad \triangle_q^{(m;i)} := \left(\mathbb{E} - q^{m+i-1}\,\mathbb{I}\right)\triangle_q^{(m-1;i)}, \qquad m \in \mathbb{N},\ i \in \mathbb{Z}$$

Operators $\mathbb{P}^{(m)}$

$$\mathbb{P}^{(1)} := \triangle_q^{(p;0)}\left(\frac{1}{a_n}\,\mathbb{I}\right),$$

$$\mathbb{P}^{(m)} := \sum_{j=0}^{p} \begin{bmatrix} mp \\ j \end{bmatrix}_q \cdot \left[\triangle_q^{(j;0)}\left(\prod_{i=1}^{p}(1-\beta_i\, q^{n+m(p+1)-j-2})\right)\right]\triangle_q^{(p-j;\,mp-m)}, \qquad m \geqslant 2$$

where

$$\begin{bmatrix} r \\ j \end{bmatrix}_q := \frac{(q;\,q)_r}{(q;\,q)_j\,(q;\,q)_{r-j}}, \qquad r, j \in \mathbb{N}_0, \quad j \leqslant r.$$

Notations

$$\triangle_q{}^{(0;i)} \coloneqq \mathbb{I}, \qquad \triangle_q{}^{(m;i)} \coloneqq \left(\mathbb{E} - q^{m+i-1}\,\mathbb{I}\right) \triangle_q{}^{(m-1;i)}, \qquad m \in \mathbb{N},\ i \in \mathbb{Z}$$

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Example

$$ {}_3\phi_2\left(\begin{matrix} q, a, b \\ q, c \end{matrix} \middle| \; q; \; c/ab\right) = {}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| \; q; \; c/ab\right) = \frac{(c/a;\, q)_\infty (c/b;\, q)_\infty}{(c;\, q)_\infty (c/ab;\, q)_\infty} $$

Table: Values of $\mathtt{acc}(\mathcal{Q}_n^{(m)})$ with $a = 9/10$, $b = 5/8$, $c = 1/2$ and $q = 1/5$.

$n \diagdown m$	0	1	2	3	4	5	6
1	0.42	3.18	8.75	14.77	22.17	30.97	41.15
2	0.48	4.67	11.64	18.36	26.46	35.96	
3	0.53	6.13	14.50	21.92	30.72	40.91	
4	0.59	7.58	17.35	25.47	34.97		
5	0.64	9.03	20.19	29.01	39.21		
6	0.69	10.48	23.04	32.56			
7	0.74	11.93	25.89	36.11			
8	0.79	13.38	28.73				
9	0.84	14.83	31.58				
10	0.89	16.28					
11	0.94	17.73					
12	1.00						
13	1.05						

$$ (z;\, q)_\infty = \prod_{j=0}^{\infty}(1 - zq^j) $$

Chebyshev polynomials

$$s = \sum_{n=1}^{\infty} a_n, \qquad a_n \coloneqq \frac{1}{z^n \, (n+t)_\nu} \, f_n, \qquad f_n \in \{T_n(x), U_n(x)\}$$

$$\hat{\mathbb{P}}^{(m)} \coloneqq (n+m+t+\nu-2)\,\mathbb{I} - 2zx(n+2m+t+\nu-2)\,\mathbb{E} + z^2(n+3m+t+\nu-2)\,\mathbb{E}^2$$

$$\mathbb{L}^{(m)} \coloneqq \hat{\mathbb{P}}^{(m)}\hat{\mathbb{P}}^{(m-1)} \dots \hat{\mathbb{P}}^{(1)} \left((n+t)_{\nu-1}\,\mathbb{I}\right)$$

$$\Downarrow$$

$$\mathbb{L}^{(m)}(\underbrace{a_n + a_{n+1} + \ldots + a_{n+m-1}}_{r_n^{(m)}}) = 0 \quad \Longrightarrow \quad \mathcal{Q}_n^{(m)} \coloneqq \frac{\mathbb{L}^{(m)}(s_n)}{\mathbb{L}^{(m)}(1)}$$

Example

$$-2 - \frac{\pi}{2}\sqrt{2+2x} = \sum_{n=0}^{\infty} \underbrace{b_n T_n(x)}_{a_n}, \qquad b_n \coloneqq \frac{(-1)^n}{n^2 - 1/4} \tag{19}$$

$$\hat{\mathbb{P}}^{(m)} \coloneqq (n+m-1/2)\,\mathbb{I} + 2x(n+2m-1/2)\,\mathbb{E} + (n+3m-1/2)\,\mathbb{E}^2,$$

Chebyshev polynomials

$$s = \sum_{n=1}^{\infty} a_n, \qquad a_n := \frac{1}{z^n\,(n+t)_v}\, f_n, \qquad f_n \in \{T_n(x), U_n(x)\}$$

$$\hat{\mathbb{P}}^{(m)} := (n+m+t+v-2)\,\mathbb{I} - 2zx(n+2m+t+v-2)\,\mathbb{E} + z^2(n+3m+t+v-2)\,\mathbb{E}^2$$

$$\mathbb{L}^{(m)} := \hat{\mathbb{P}}^{(m)}\hat{\mathbb{P}}^{(m-1)}\ldots\hat{\mathbb{P}}^{(1)}\left((n+t)_{v-1}\,\mathbb{I}\right)$$

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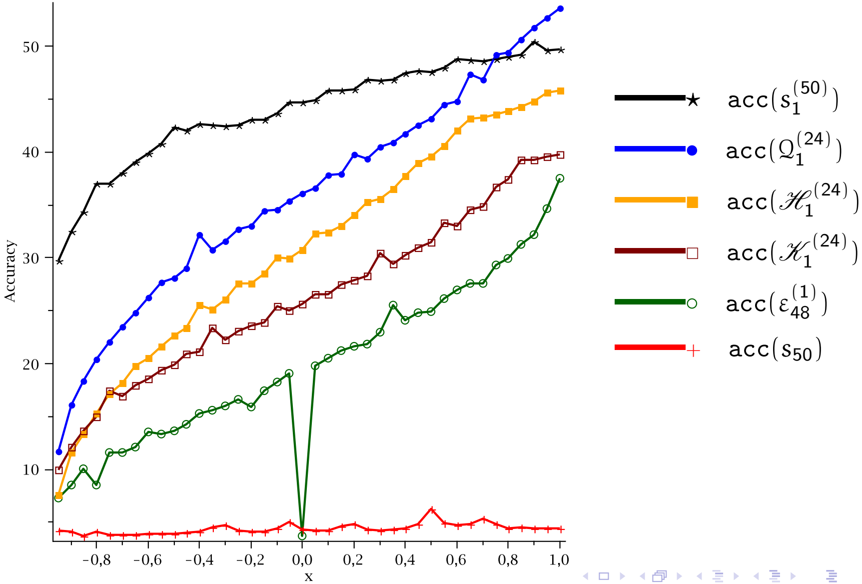
Figure: Accuracy of the summation of the series (19) for $x = -0.95, -0.9, \ldots, 1$.



Summary

- The method for summation of some slowly convergent series was proposed
- Method may be successfully applied to the summation of generalized and basic hypergeometric series, as well as some classical orthogonal polynomial series expansions
- In some special cases, our algorithm is equivalent to Wynn's epsilon algorithm, Weniger $\mathscr{S}$ transformation or the technique recently introduced by Čížek, Zamastil and Skála [1].
- In the case of trigonometric series, our method is very similar to the Homeier's $\mathscr{H}$ transformation, while in the case of orthogonal series — to the $\mathscr{K}$ transformation.

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For Further Reading

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