PADES and LARGE-N_c QCD

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Luminy, Sep. 2009

Introduction

★**Q**uantum **C**hromo **D**ynamics is a Yang-Mills theory with fermions, based on local $SU(N_c)$:

$$\mathcal{L} = i\,\overline{\psi}_a\ \not{D}_{ab}\,\psi_b - \frac{1}{4g^2}F^A_{\mu\nu}F^{A\,\mu\nu}$$

- ψ_b : Dirac field, $b = 1, ..., N_c$
- $F^A_{\mu\nu}$: Yang-Mills field strength tensor, $A = 1, ..., N_c^2 - 1$
- g: coupling constant
- $N_c = 3$

★ 1 of 7 Millennium Problems worth $ 1M Prize from the Clay Math Inst.

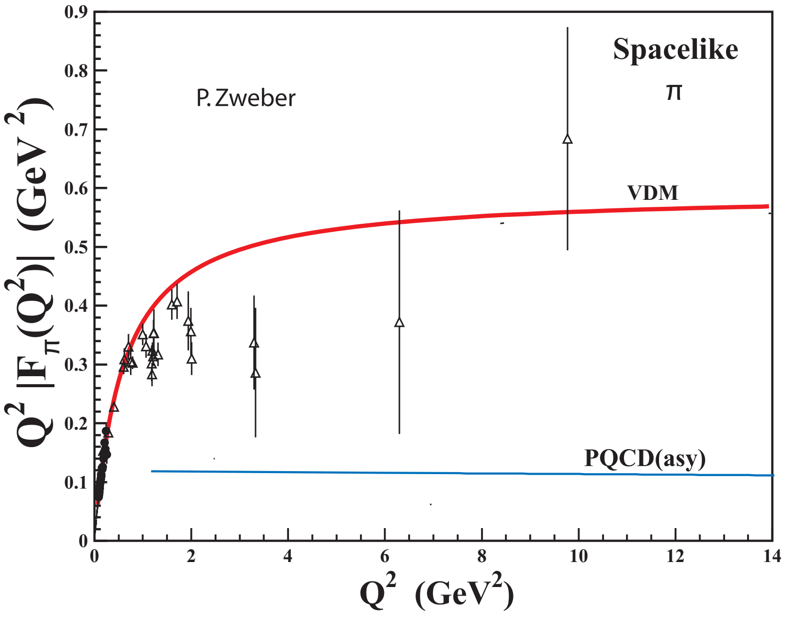
★ Although Nature has $N_c = 3$, the limit $\underline{N_c \to \infty}$ ($g^2 N_c \to$ constant) is very interesting. ('t Hooft '74; Witten '79)

- In this limit, all functions are meromorphic.

Resonance Saturation

$$< \pi(p')|V_\mu|\pi(p) >= F(-q^2)\ (p'+p)_\mu \quad , \quad q^2 = (p'-p)^2$$

$$\begin{aligned}
F(-q^2) \quad &\simeq \quad \frac{M_V^2}{-q^2+M_V^2} \qquad (VMD \quad '60s) \\
&\approx \quad 1 + q^2 \sum_R^{\infty} \frac{C_R^2}{-q^2+M_R^2} \qquad (\underline{\text{meromorphic}}, \quad \mathrm{N_c} \to \infty) \\
&\approx \quad -16\pi F_\pi^2 \, \frac{\alpha_s(\mu)}{q^2} \left(1 + \sharp\ \alpha_s(\mu)\ \log \frac{-q^2}{\mu^2} + \ldots \right) + \ldots \qquad (q^2 \to -\infty) \\
&\approx \quad 1 + a\ q^2 + \ldots \qquad (q^2 \to 0) \quad \Rightarrow \quad a = \frac{1}{M_V^2} \simeq \frac{1}{(0.74\ \mathrm{GeV})^2} \quad \equiv \frac{2L_9}{F_\pi^2}
\end{aligned}$$



$$(Q^2 = -q^2)$$

Sakurai ’69
Ecker et al. ’89
Donoghue et al. ’89
Moussallam ’97
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Knecht, de Rafael ’98
Perrottet, de Rafael, S.P. ’98

$$F(-q^2) = 1 + q^2 \sum_R^{\infty} \frac{C_R^2}{-q^2 + M_R^2}$$

$$\overset{(\star)}{\approx} \frac{M_V^2}{-q^2 + M_V^2}$$

- What is this approximation $(\star)$? (1,2,....∞)
- Does it work for all functions ?
- Where in the complex Q^2 plane does $(\star)$ converge ?
- How are the poles/residues of the approx. $(\star)$ related to the physical counterparts?

High Energy: Weinberg SRs

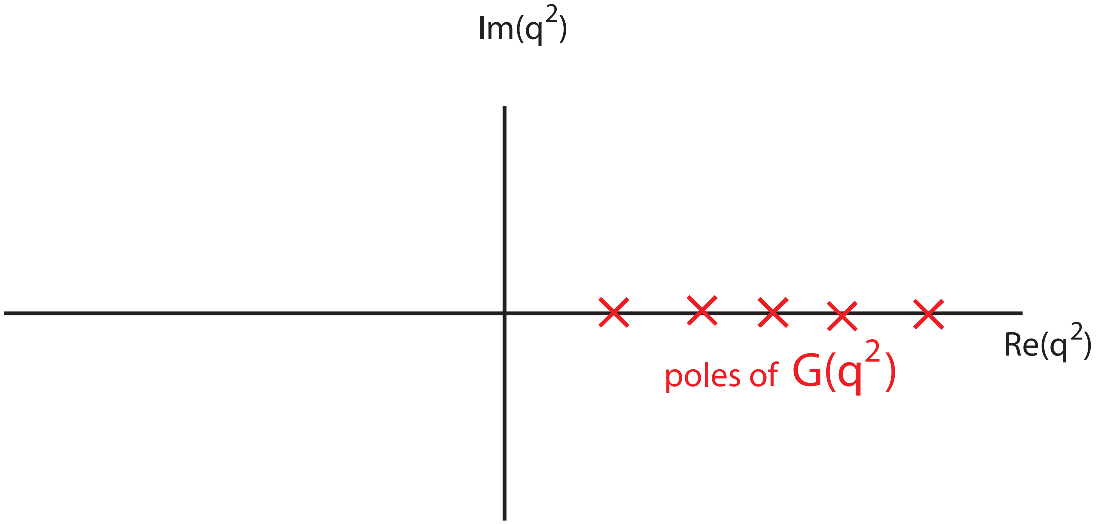
★$< VV - AA >$ with “Regge” spectrum (large n)

- $M^2_{A_n} \sim M^2_{V_n} \sim n$, for poles $n \gg 1$.
- $F_{A_n} \sim F_{V_n} \sim \mathrm{const.}$, for residues $n \gg 1$.

$$q^2\Pi(-q^2) = \lim_{N\to\infty}\left\{F^2 - q^2\sum_n^N \frac{F^2_{A_n}}{-q^2+M^2_{A_n}} + q^2\sum_n^{N+c}\frac{F^2_{V_n}}{-q^2+M^2_{V_n}}\right\}$$

★$\Pi(-q^2)$ <u>independent</u> of **c**.

★Analyticity:



High Energy: WSRs *(and II)*

Imposing that $q^2\Pi(-q^2)|_{q^2\to-\infty} \sim \frac{1}{(q^2)^2}$ for N finite:

$$\lim_{N\to\infty}\left[-F^2-\sum_n^N F_{A_n}^2+\sum_n^{N+\mathbf{c}} F_{V_n}^2\right]=0 \qquad ??$$

$$\lim_{N\to\infty}\left[\sum_n^N F_{A_n}^2 M_{A_n}^2-\sum_n^{N+\mathbf{c}} F_{V_n}^2 M_{V_n}^2\right]=0 \qquad ??$$

dependent on c !! (Golterman, S.P. '03)

★ Physical poles and residues do <u>not</u> obey WSRs.

What is resonance saturation ?

- It's a Pade Approximant to a meromorphic function

  ▶ $F(-q^2) \approx \frac{M_V^2}{-q^2+M_V^2}$ is the PA $P_1^0(-q^2)$ to $F(-q^2)$.

- $N_{A,V}$ resonances in

$$q^2\Pi(-q^2) = F^2 - q^2 \sum_A^{N_A} \frac{F_A^2}{-q^2+M_A^2} + q^2 \sum_V^{N_V} \frac{F_V^2}{-q^2+M_V^2}$$

$\Longrightarrow P_N^N(-q^2)$ with $N = N_A + N_V$

$\oplus\ 1/(q^2)^2$ fall-off $\Longrightarrow P_N^{N-2}(-q^2)$

▶ WSRs are obeyed by PA's parameters.

Parameters (residues + poles) of Pade Approx.	$\neq$	Residues and poles of physical functions

Pade Approximants

(Physics $\Leftrightarrow z \equiv -q^2$)

Let $\quad G(z)|_{z\to 0} \approx G_0 + G_1\ z + G_2\ z^2 + G_3\ z^3 + ...$

Define rational function $P_N^M(z)$ such that

$$P_N^M(z) \equiv \frac{Q_M(z)}{R_N(z)} \approx G_0 + G_1 z + G_2 z^2 + ... + G_{M+N} z^{M+N} + \mathcal{O}(z^{M+N+1})$$

If $G(z) \sim 1/z^K$, choose $P_{M+K}^M(z)$.

(Pommerenke '73)

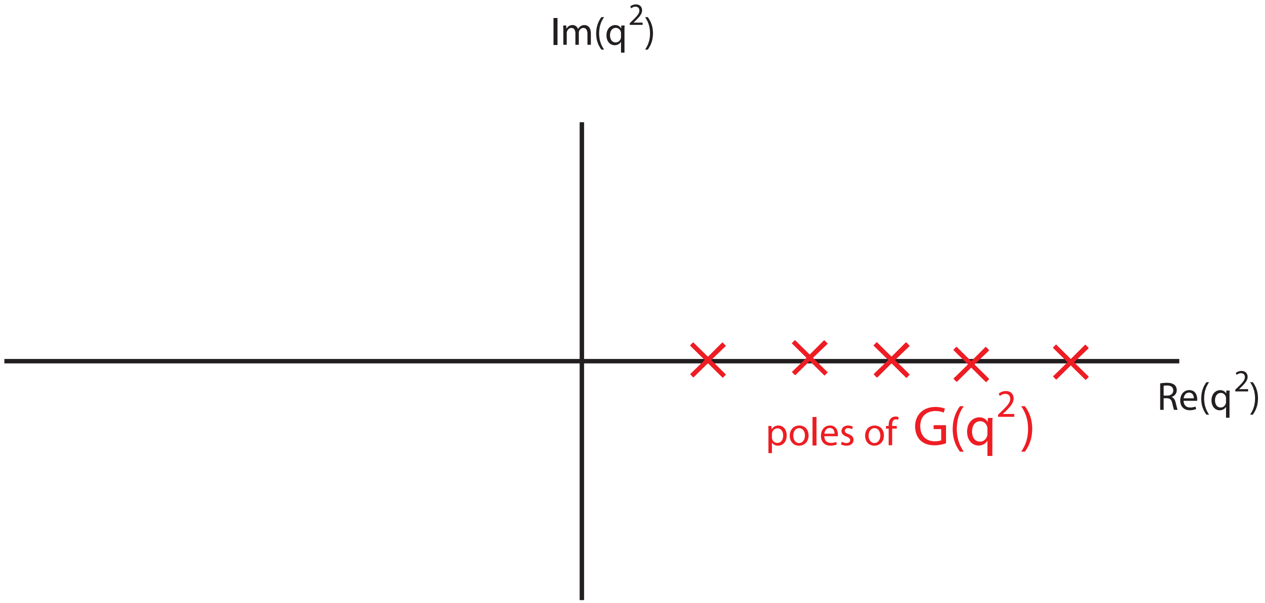
Convergence Theorem

Let $G(z)$ be meromorphic and analytic at the origin. Then,

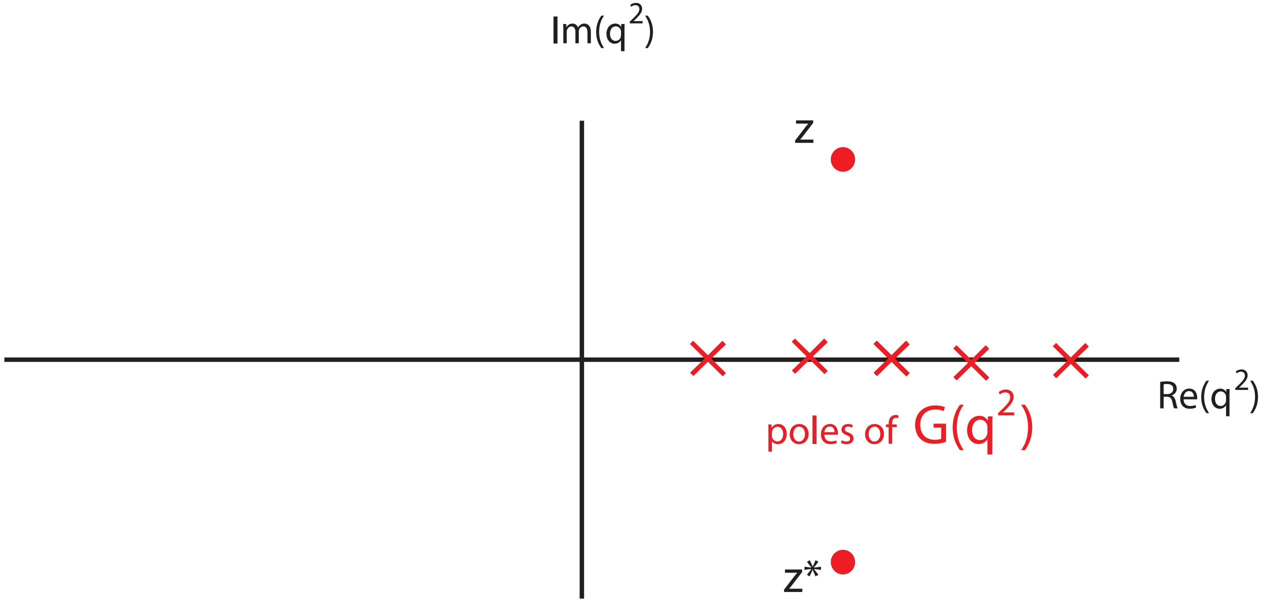
$$\lim_{M\to\infty} P_{M+K}^M(z) = G(z)$$

for $z \in$ compact set in $\mathbb{C}$, <u>except</u> on isolated points.

Convergence Map



Convergence Map



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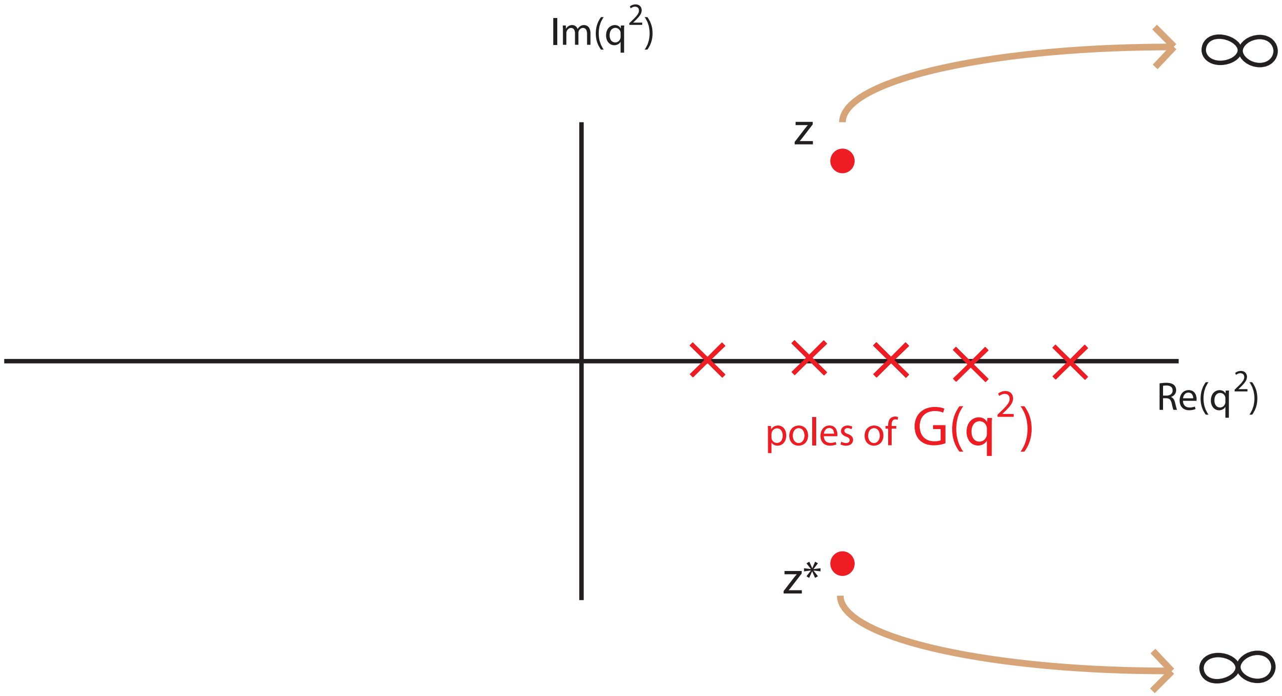
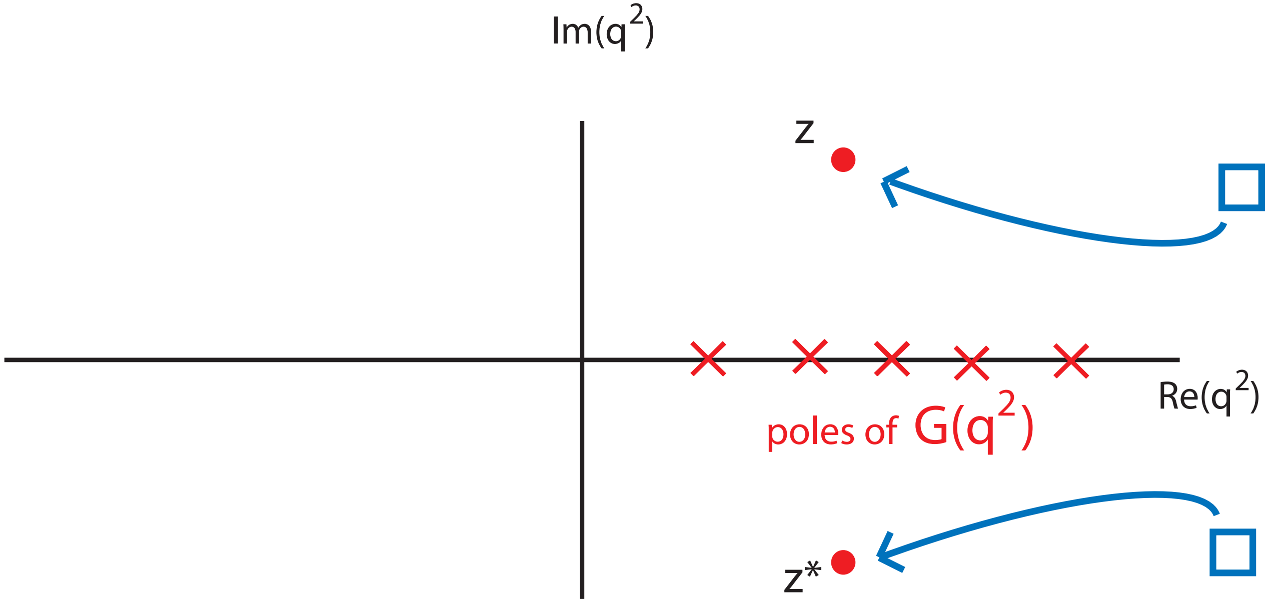


Convergence Map

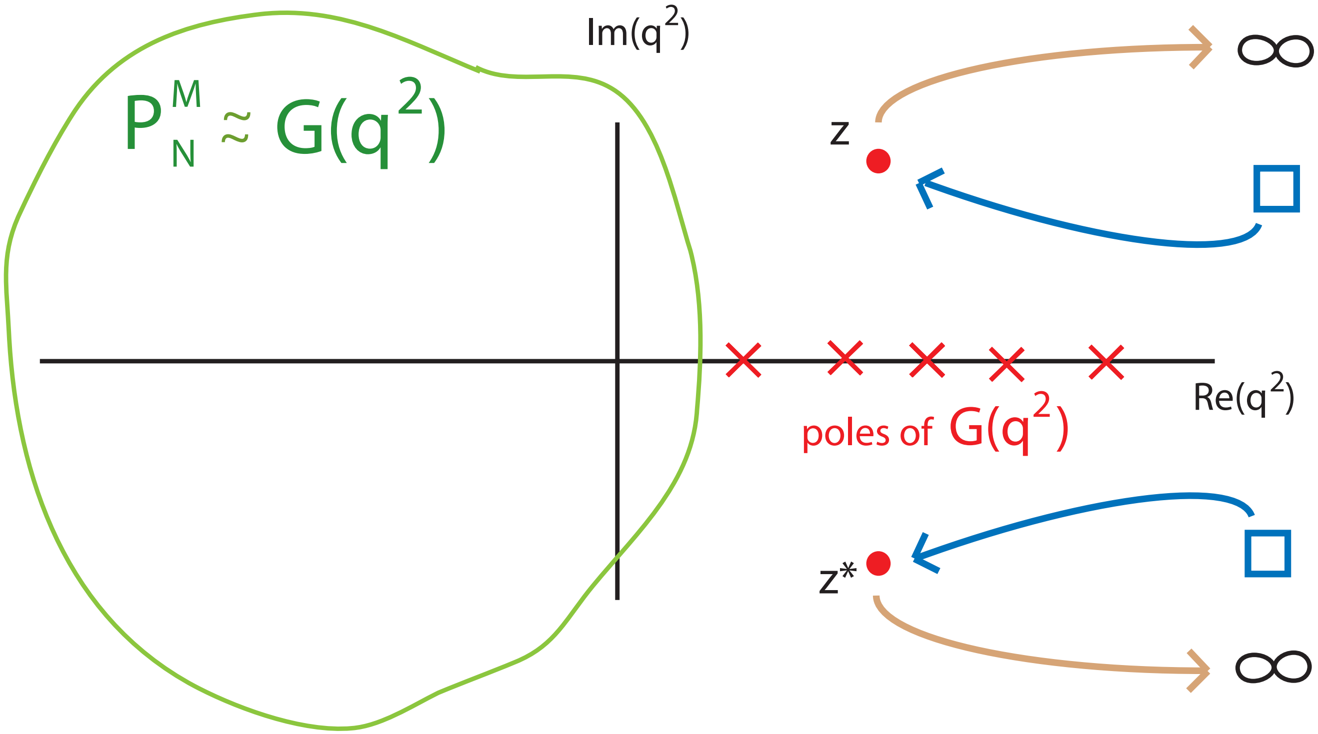


▣ = pole U zero ≡ ``defect´´

N.B. This is why sometimes residues turn out to be “unexpectedly” small.

(Friot, Greynat, de Rafael ’04)

Convergence Map



Pommerenke '73

▣ = pole ∪ zero ≡ ``defect´´

Toy Model for VV-AA

- Not Stieltjes.
- Meromorphic.
- Spectrum (Regge-like): $M^2_{V,A}(n) = m^2_{V,A} + \ n\ \Lambda^2_{QCD}$

Shifman et al. '98
Golterman, S.P., '01

$$q^2\Pi(-q^2) = F^2 + q^2 \frac{F_\rho^2}{-q^2+M_\rho^2} + q^2 \sum_{n=0}^{\infty} \left(\frac{F^2}{-q^2+M_V^2(n)} - \frac{F^2}{-q^2+M_A^2(n)} \right)$$

where $\sum$'s can be written in terms of $\psi(z) = \Gamma'(z)/\Gamma(z)$.

Can choose realistic numbers so that

$$\begin{aligned} -\ q^2\Pi(-q^2)|_{q^2\to 0} \quad &\approx \quad C_0 - C_2\ q^2 + C_4\ (q^2)^2 + \ldots \quad (\text{ finite radius conv.}) \\ -q^2\Pi(-q^2)|_{q^2\to -\ \infty} \quad &\approx \quad 0 + \frac{0}{q^2} + \frac{C_{-4}}{(q^2)^2} - \frac{C_{-6}}{(q^2)^3} + \ldots \quad (\text{ no logs},\ \text{asymptotic}) \end{aligned}$$

with $C's$ which are calculable !

PAs to VV-AA model: Poles and Residues

PAs work beautifully.

- $\exists$ Convergence in complex Q^2 plane, away from singularities.
- Prediction of physical residues and poles good near the origin but deteriorates very quickly as you move away, eventually becoming complex. Last pole always off.
- PAs approximate original function at the expense of altering residues and poles hierarchically (more the farther away from the origin).
- Prediction of a global quantity such as

$$\int_{-\infty}^{0} dq^2\ q^2\ \Pi(-q^2) \sim (m_{\pi^+} - m_{\pi^0})_{EM}$$

very good. It can even be used as input.

PAs: predicting the next coeff's

Only with C_0, C_2, C_4:

$$P_2^0 = \frac{-r^2}{(-q^2+z_R)(-q^2+z_R^*)} \ , \quad r^2 = 3.379 \times 10^{-3} \ , \quad z_R = 0.6550 + \underline{i}\ 0.1732 \ .$$

(natural units: GeV=1)

- Poles are complex , i.e. not physical.
- One can predict next terms in Taylor expansion at $q^2 = 0$ and at $-\infty$.

  Let's call $X_i \equiv \frac{C_i(predicted)}{C_i(real)}$.

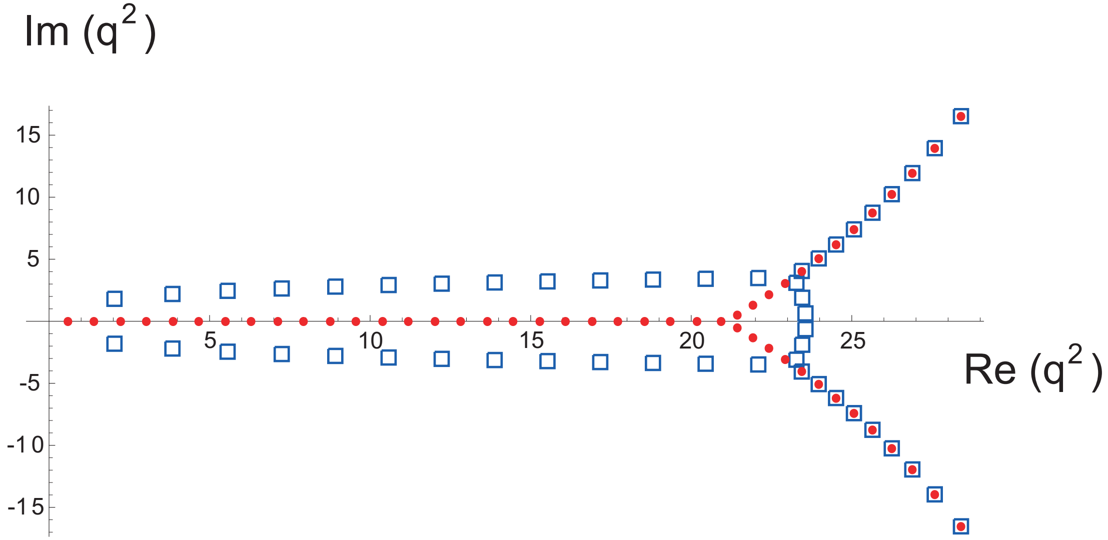
  With $P_2^0 \Longrightarrow X_{-4} = 1.3 \ , \ X_6 = 0.97$ (not bad !).
- Gone up to P_{52}^{50} (with 103 parameters):

$$X_{-4,-6,-8} = 1 + \mathcal{O}(10^{-52,-48,-45}) \quad , \quad X_{206} = 1 + \mathcal{O}(10^{-192}) \quad !!$$

Could one always assure this for a Pade ?

PAs: poles and zeros

E.g. Analytic structure of P^{50}_{52}:



(Masjuan, S.P., '07)

Other kind of PAs: Pade-Type Approx.

Denominator is fixed with poles at physical masses.

Simplest one (3 inputs, $C_0 = -F_0^2, M_\rho^2, M_A^2$) :

$$T_2^0 \, \frac{-F_0^2 \; M_\rho^2 M_A^2}{(-q^2 + M_\rho^2)(-q^2 + M_A^2)}$$

- Low-q^2 expansion and integrals over negative q^2 not bad.
- Low-order PTAs tend to be worse than PAs, in particular the high-q^2 expansion. Gone up to T_9^7.
- Since poles are predetermined, residues pay full price:

$\Rightarrow$ Residues deteriorate hierarchically (worse the farther from origin).
$\Rightarrow$ Last residue considered, completely off.

Other kind of PAs: Partial-Pade Approx.

- Denominator with some poles fixed and some free.
- Interesting example:

$$\mathbb{P}^0_{1,1} = \frac{-\, r_R^2}{(-q^2 + M_\rho^2)(-q^2 + z_R)} \;, \text{ with } \quad r_R^2 = 3.75 \times 10^{-3} \;, \quad z_R = 0.8665 \;.$$

$$\Longrightarrow M_A|_{\text{Pade}} = \sqrt{z_R} = 0.930 \quad \text{while} \quad M_A|_{\text{exact}} = 1.18$$

(exactly the same as what is often found in the literature, e.g. Ecker et al. ’89; Friot, Greynat and de Rafael ’04)

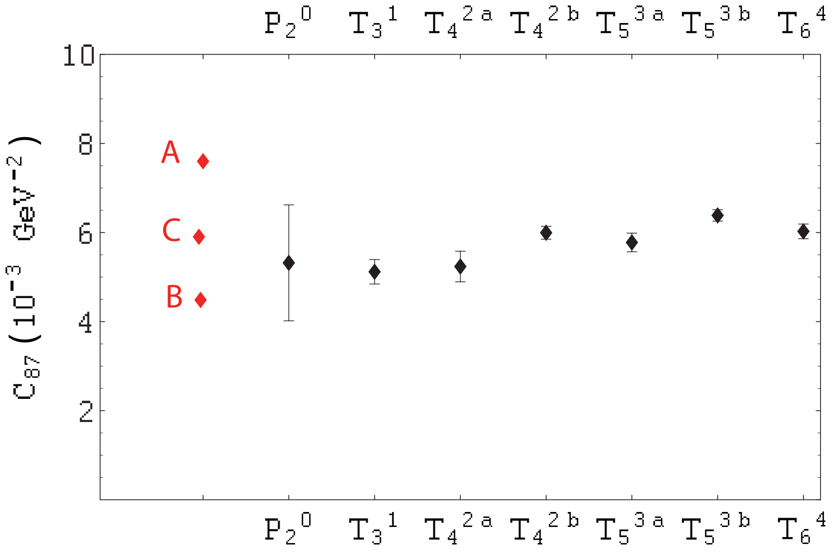
- In general, PPAs are an intermediate situation between PAs and PTAs.

Insert: PT prediction in QCD($N_c \to \infty$)

Assume physical masses from PDG.

$$q^2\Pi \approx f_0^2 + 4L_{10}\, q^2 - 8\, C_{87}\, (q^2)^2 + \dots$$

$\mathbb{T}^n_m$	inputs
$\mathbb{T}^1_3$	$f_0,\, L_{10}\,;\, m_\rho,\, m_a,\, m_{\rho'}$
$\mathbb{T}^{2\,(a)}_4$	$f_0,\, L_{10},\, \delta M_\pi\,;\, m_\rho,\, m_a,\, m_{\rho'},\, m_{a'}$
$\mathbb{T}^{2\,(b)}_4$	$f_0,\, L_{10},\, F_\rho\,;\, m_\rho,\, m_a,\, m_{\rho'},\, m_{a'}$
$\mathbb{T}^{3\,(a)}_5$	$f_0,\, L_{10},\, F_\rho,\, \delta M_\pi\,;\, m_\rho,\, m_a,\, m_{\rho'},\, m_{a'},\, m_{\rho''}$
$\mathbb{T}^{3\,(b)}_5$	$f_0,\, L_{10},\, F_\rho,\, F_a\,;\, m_\rho,\, m_a,\, m_{\rho'},\, m_{a'},\, m_{\rho''}$
$\mathbb{T}^4_6$	$f_0,\, L_{10},\, F_\rho,\, F_a,\, \delta M_\pi\,;\, m_\rho,\, m_a,\, m_{\rho'},\, m_{a'},\, m_{\rho''},\, m_{\rho'''}$



A ♦ Amoros et al. '00

B ♦ Knecht et al. '01

C ♦ Mateu et al. '07

● Masjuan, S.P., '08

$\longleftarrow\ 5.7(5)\cdot 10^{-3}\ \mathrm{GeV}^{-2} \quad (+1/N_c\mathrm{corr's})$

(G'lez-Alonso et al. '08)

τ decay $\Rightarrow 4.9(2)\cdot 10^{-3}\ \mathrm{GeV}^{-2}$

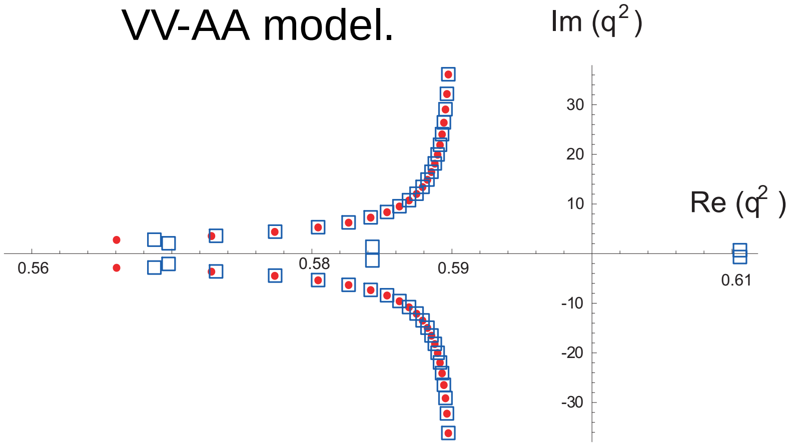
PAs at $-q^2 \to \infty$

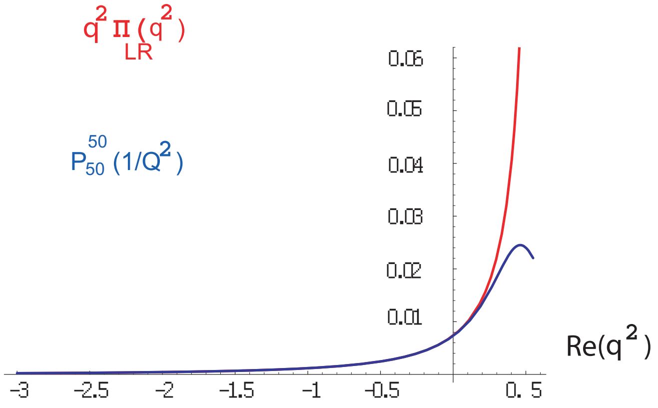
Our VV-AA model has asymptotic expansion in $1/q^2$ with coeffs. given by Bernoulli polynom. (i.e. radius of convergence is zero).
Can Pades at ∞ reproduce the spectrum ?

- If function is Stieltjes and coeff. grow $\leq (2n)!K^{2n}$ (Carleman's cond.), PAs constructed in $1/q^2$ converge.

  But I know of no QCD function which is Stieltjes in $1/q^2$!

- If Green's function is not Stieltjes, then PAs do not yield the spectrum. Ex: Our VV-AA model.





- You cannot get the spectrum out of resumming the OPE.

AdS/QCD ??

Conclusions and Outlook

- Resonance saturation at large-N_c can be understood from the theory of Pade Approximants to meromorphic functions.

- Expansion about $q^2 = 0$ allows to construct rational approx. at finite q^2 in region free of poles.

- For the last poles, the approximation is unreliable:

  ▶ Last Residues/poles in rational approx. not physical.

  E.g., form factors not to be extracted from rational approx. to 3-point functions (Bijnens et al. ’03).

- Weinberg-type Sum rules obeyed by PA’s parameters, not by physical masses and decay constants.

- Poles and residues not constrained by chiral symmetry : Lagrangian ?

Conclusions and Outlook (II)

★ May PAs (or PTAs) reliably predict Taylor coeffs at $q^2 = 0$?
(and at $-q^2 \to \infty$?)

Errors ?

★ and integrals of the function, e.g. $\int_{-\infty}^{0} dq^2 \; \Pi(q^2)$?

★ what to do if $\exists \log$'s ?