Survey of Numerical Stability Issues in Convergence Acceleration

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Introduction

Notation: $\{A_m\}$; A limit or antilimit of $\{A_m\}$.
$\{A_n^{(j)}\}$: Obtained by applying an extrapolation method to $\{A_m\}$.

When computed numerically (i.e., in floating-point arithmetic), the A_i and hence the $A_n^{(j)}$ are in error. This may cause numerical instabilities, which eventually may destroy the accuracy of the *computed* $A_n^{(j)}$ completely.

For almost all methods,

$$A_n^{(j)} = \sum_{i=0}^{K_n} \gamma_{ni}^{(j)} A_{j+i}\,; \quad \sum_{i=0}^{K_n} \gamma_{ni}^{(j)} = 1.$$

The $\gamma_{ni}^{(j)}$ are functions of the ΔA_i.

If $\bar{A}_i = A_i + \epsilon_i$, $\bar{\gamma}_{ni}^{(j)} = \gamma_{ni}^{(j)} + \delta_{ni}^{(j)}$ are the computed quantities, then the computed $A_n^{(j)}$ is

$$\bar{A}_n^{(j)} = \sum_{i=0}^{K_n} \bar{\gamma}_{ni}^{(j)}\, \bar{A}_{j+i}.$$

Then

$$|\bar{A}_n^{(j)} - A| \le |A_n^{(j)} - A| + \sum_{i=0}^{K_n} |\gamma_{ni}^{(j)}||\epsilon_{j+i}| + \sum_{i=0}^{K_n} |\delta_{ni}^{(j)}||\bar{A}_{j+i}|.$$

If $|\epsilon_i| = |A_i||\rho_i|$ and $|\delta_{ni}^{(j)}| = |\gamma_{ni}^{(j)}||\eta_{ni}^{(j)}|$, where $|\rho_i|, |\eta_{ni}^{(j)}| \le \mathbf{u}$, then

$$|\bar{A}_n^{(j)} - A| \le |A_n^{(j)} - A| + \mathbf{u}\left[\sum_{i=0}^{K_n} |\gamma_{ni}^{(j)}||A_{j+i}| + \sum_{i=0}^{K_n} |\gamma_{ni}^{(j)}||\bar{A}_{j+i}|\right].$$

By $A_i \approx \bar{A}_i$ and $\mathbf{u}|\bar{A}_i| = \mathbf{u}|A_i| + O(\mathbf{u}^2)$, we have

$$|\bar{A}_n^{(j)} - A| \lessapprox |A_n^{(j)} - A| + 2\mathbf{u} \sum_{i=0}^{K_n} |\gamma_{ni}^{(j)}||A_{j+i}|.$$

In case $\{A_m\}$ converges, $A_i \approx A$ for all large i. Therefore,

$$|\bar{A}_n^{(j)} - A| \lessapprox |A_n^{(j)} - A| + 2\mathbf{u}|A| \sum_{i=0}^{K_n} |\gamma_{ni}^{(j)}|.$$

$$\frac{|\bar{A}_n^{(j)} - A|}{|A|} \lessapprox \frac{|A_n^{(j)} - A|}{|A|} + 2\mathbf{u} \sum_{i=0}^{K_n} |\gamma_{ni}^{(j)}|.$$

Now, the theoretical relative error $|A_n^{(j)} - A|/|A| \to 0$ as $j \to \infty$ or $n \to \infty$. Consequently, for all large j or n,

$$\frac{|\bar{A}_n^{(j)} - A|}{|A|} \lessapprox 2\mathbf{u} \sum_{i=0}^{K_n} |\gamma_{ni}^{(j)}|.$$

Thus, the quantity

$$\Gamma_n^{(j)} = \sum_{i=0}^{K_n} |\gamma_{ni}^{(j)}| \geq 1$$

plays a very important role when applying extrapolation methods in floating-point arithmetic. In case $\Gamma_n^{(j)}$ is of order 10^p and $\mathbf{u}$ is of order 10^{-s}, at most $s-p$ decimal digits of $\bar{A}_n^{(j)}$ can be trusted.

Note: $\mathbf{u}$ is of order 10^{-16} for double precision, and of order 10^{-35} for quadruple precision. Therefore, better accuracy can be obtained from $\bar{A}_n^{(j)}$ by increasing the precision of the floating-point arithmetic, provided the A_i are also computed in this arithmetic.

We want $\sup_j \Gamma_n^{(j)}$ or $\sup_n \Gamma_n^{(j)}$ to be finite and small. In case $\Gamma_n^{(j)}$ is unbounded in j or n, we want to be able to force it to grow as slowly as possible.

Example. Levin u transformation

Defined via the linear system

$$A_l = A_n^{(j)} + (l+1)\Delta A_l \sum_{i=0}^{n-1} \frac{\beta_i}{(l+1)^i}, \quad j \le l \le j+n.$$

Thus,

$$A_n^{(j)} = \frac{\sum_{i=0}^{n} (-1)^i \binom{n}{i} (j+i+1)^{n-2} (A_{j+i}/a_{j+i})}{\sum_{i=0}^{n} (-1)^i \binom{n}{i} (j+i+1)^{n-2} (1/a_{j+i})}; \quad a_i = A_i - A_{i-1}.$$

$$\gamma_{ni}^{(j)} = \frac{(-1)^i \binom{n}{i} (j+i+1)^{n-2} (1/a_{j+i})}{\sum_{r=0}^{n} (-1)^r \binom{n}{r} (j+r+1)^{n-2} (1/a_{j+r})}, \quad i = 0, 1, \ldots, n.$$

Best results are obtained from $\{A_n^{(j)}\}_{n=0}^{\infty}$ with j fixed (such as $j = 0$).

Example.

u transformation on $A_n = \sum_{i=1}^{n} 1/i^2$, $n = 1, 2, \ldots,$
$\lim_{n\to\infty} A_n = \pi^2/6$.

n	$\bar{E}_n^{(0)}$(d)	$\bar{E}_n^{(0)}$(q)	$\Gamma_n^{(0)}$
0	$3.92D-01$	$3.92D-01$	$1.00D+00$
2	$1.21D-02$	$1.21D-02$	$9.00D+00$
4	$1.90D-05$	$1.90D-05$	$9.17D+01$
6	$6.80D-07$	$6.80D-07$	$1.01D+03$
8	$1.56D-08$	$1.56D-08$	$1.15D+04$
10	$1.85D-10$	$1.83D-10$	$1.35D+05$
12	$1.09D-11$	$6.38D-13$	$1.60D+06$
14	$2.11D-10$	$2.38D-14$	$1.92D+07$
16	$7.99D-09$	$6.18D-16$	$2.33D+08$
18	$6.10D-08$	$7.78D-18$	$2.85D+09$
20	$1.06D-07$	$3.05D-20$	$3.50D+10$
22	$1.24D-05$	$1.03D-21$	$4.31D+11$
24	$3.10D-04$	$1.62D-22$	$5.33D+12$
26	$3.54D-03$	$4.33D-21$	$6.62D+13$
28	$1.80D-02$	$5.44D-20$	$8.24D+14$
30	$1.15D-01$	$4.74D-19$	$1.03D+16$

$\bar{E}_n^{(0)}$(d): relative error in $\bar{A}_n^{(0)}$ in double precision. ($\approx$ 16 decimal digits)
$\bar{E}_n^{(0)}$(q): relative error in $\bar{A}_n^{(0)}$ in quadruple precision. ($\approx$ 35 decimal digits)

Improving stability

Example. Levin–Sidi $d^{(1)}$ transformation

Defined via the linear system, with $1 \le R_0 < R_1 < R_2 < \cdots$,

$$A_{R_l} = A_n^{(j)} + R_l\, a_{R_l} \sum_{i=0}^{n-1} \frac{\beta_i}{R_l^i}, \quad j \le l \le j+n; \quad a_i = A_i - A_{i-1}.$$

The $A_n^{(j)}$ and $\Gamma_n^{(j)}$ can be computed via the W algorithm:

$M_0^{(j)} = \frac{A_{R_j}}{R_j\, a_{R_j}}, \;\; N_0^{(j)} = \frac{1}{R_j\, a_{R_j}}, \;\; H_0^{(j)} = (-1)^j |N_0^{(j)}|, \quad j \ge 0.$

For $n = 1, 2, \ldots$ do

$M_n^{(j)} = \frac{M_{n-1}^{(j+1)} - M_{n-1}^{(j)}}{R_{j+n}^{-1} - R_j^{-1}}, \; N_n^{(j)} = \frac{N_{n-1}^{(j+1)} - N_{n-1}^{(j)}}{R_{j+n}^{-1} - R_j^{-1}}, \; H_n^{(j)} = \frac{H_{n-1}^{(j+1)} - H_{n-1}^{(j)}}{R_{j+n}^{-1} - R_j^{-1}}, \; j \ge 0.$

$A_n^{(j)} = \frac{M_n^{(j)}}{N_n^{(j)}}, \; \Gamma_n^{(j)} = \left| \frac{H_n^{(j)}}{N_n^{(j)}} \right|, \quad j \ge 0.$

end do

Best results are obtained from $\{A_n^{(j)}\}_{n=0}^{\infty}$ with j fixed (such as $j = 0$).

Example: Logarithmic convergence

$d^{(1)}$ transformation on $A_n = \sum_{i=1}^{n} 1/i^2$, $n = 1, 2, \ldots$, $\lim_{n\to\infty} A_n = \pi^2/6$.
$R_0 = 1, R_l = \max\{R_{l-1} + 1, \lfloor \sigma R_{l-1} \rfloor\}$, with $\sigma = 1.3$ (GPS).

n	R_n	$\bar{E}_n^{(0)}$(d)	$\bar{E}_n^{(0)}$(q)	$\Gamma_n^{(0)}$
0	1	$3.92D-01$	$3.92D-01$	$1.00D+00$
2	3	$1.21D-02$	$1.21D-02$	$9.00D+00$
4	5	$1.90D-05$	$1.90D-05$	$9.17D+01$
6	7	$6.80D-07$	$6.80D-07$	$1.01D+03$
8	11	$1.14D-08$	$1.14D-08$	$3.04D+03$
10	18	$6.58D-11$	$6.59D-11$	$3.75D+03$
12	29	$1.58D-13$	$1.20D-13$	$3.36D+03$
14	48	$1.55D-15$	$4.05D-17$	$3.24D+03$
16	80	$7.11D-15$	$2.35D-19$	$2.76D+03$
18	135	$5.46D-14$	$1.43D-22$	$2.32D+03$
20	227	$8.22D-14$	$2.80D-26$	$2.09D+03$
22	383	$1.91D-13$	$2.02D-30$	$1.97D+03$
24	646	$1.00D-13$	$4.43D-32$	$1.90D+03$
26	1090	$4.21D-14$	$7.24D-32$	$1.86D+03$
28	1842	$6.07D-14$	$3.27D-31$	$1.82D+03$
30	3112	$1.24D-13$	$2.52D-31$	$1.79D+03$

$\bar{E}_n^{(0)}$(d): relative error in $\bar{A}_n^{(0)}$ in double precision. ($\approx$ 16 decimal digits)
$\bar{E}_n^{(0)}$(q): relative error in $\bar{A}_n^{(0)}$ in quadruple precision. ($\approx$ 35 decimal digits)

Example: Linear convergence

u and $d^{(1)}$ transformations on $A_n = \sum_{i=1}^{n} z^{i-1}/i$, $n = 1, 2, \ldots,$
$\lim_{n\to\infty} A_n = -\log(1-z)/z \equiv f(z)$; $z = 0.95$.
$R_l = \kappa(l+1)$, with $\kappa > 0$ integer (APS). ($\kappa = 1$ gives the u transformation.)

n	$\bar{E}_n^{(0)}(\kappa=1)$	$\Gamma_n^{(0)}(\kappa=1)$	$\bar{E}_n^{(0)}(\kappa=10)$	$\Gamma_n^{(0)}(\kappa=10)$
0	$6.83D-01$	$1.00D+00$	$1.72D-01$	$1.00D+00$
4	$1.70D-01$	$3.92D+02$	$1.09D-04$	$1.49D+01$
8	$9.65D-03$	$1.64D+04$	$2.36D-08$	$9.92D+01$
12	$6.69D-04$	$7.86D+05$	$5.45D-12$	$6.71D+02$
16	$4.88D-05$	$3.87D+07$	$1.28D-15$	$4.55D+03$
20	$3.63D-06$	$1.93D+09$	$3.04D-19$	$3.09D+04$
24	$2.73D-07$	$9.66D+10$	$7.24D-23$	$2.10D+05$
28	$2.06D-08$	$4.85D+12$	$1.73D-26$	$1.43D+06$
32	$1.56D-09$	$2.44D+14$	$1.92D-28$	$9.73D+06$
36	$1.19D-10$	$1.23D+16$	$3.97D-27$	$6.62D+07$
40	$9.02D-12$	$6.17D+17$	$1.34D-26$	$4.51D+08$
44	$6.87D-13$	$3.11D+19$	$3.30D-26$	$3.07D+09$
48	$4.47D-14$	$1.57D+21$	$3.05D-25$	$2.09D+10$
52	$1.45D-12$	$7.89D+22$	$5.19D-25$	$1.42D+11$
56	$1.45D-11$	$3.98D+24$	$2.43D-23$	$9.67D+11$
60	$1.75D-09$	$2.01D+26$	$1.39D-22$	$6.58D+12$

$\bar{E}_n^{(0)}$: relative error in $\bar{A}_n^{(0)}(z)$. $\bar{A}_n^{(j)}$: the computed $A_n^{(j)}(z)$.
(Computations in quadruple-precision.) Note: $\bar{E}_{24}^{(0)}(\kappa=20) = 2.21D-33$.

Example: Linear convergence (cont'd).

u and $d^{(1)}$ transformations on $A_n = \sum_{i=1}^{n} z^{i-1}/i$, $n = 1, 2, \ldots,$ $\lim_{n\to\infty} A_n = -\log(1-z)/z \equiv f(z)$.
$R_l = \kappa(l+1)$, with $\kappa > 0$ integer (APS). ($\kappa = 1$ gives the u transformation.)

s	z	$\bar{E}_n^{(0)}(\kappa = 1)$ "smallest" error	$\bar{E}_{28}^{(0)}(\kappa = 1)$	$\bar{E}_{28}^{(0)}(\kappa = s)$
1	0.5	$4.97D-30$ ($n = 28$)	$4.97D-30$	$4.97D-30$
2	$0.5^{1/2} \approx 0.707$	$1.39D-25$ ($n = 35$)	$5.11D-21$	$3.26D-31$
3	$0.5^{1/3} \approx 0.794$	$5.42D-23$ ($n = 38$)	$4.70D-17$	$4.79D-31$
4	$0.5^{1/4} \approx 0.841$	$1.41D-21$ ($n = 41$)	$1.02D-14$	$5.74D-30$
5	$0.5^{1/5} \approx 0.871$	$1.31D-19$ ($n = 43$)	$3.91D-13$	$1.59D-30$
6	$0.5^{1/6} \approx 0.891$	$1.69D-18$ ($n = 43$)	$5.72D-12$	$2.60D-30$
7	$0.5^{1/7} \approx 0.906$	$1.94D-17$ ($n = 44$)	$4.58D-11$	$4.72D-30$

$\bar{E}_n^{(0)} = |\bar{A}_n^{(0)}(z) - f(z)|$, and the $\bar{A}_n^{(j)}(z)$ are the computed $A_n^{(j)}(z)$.
(Computations in quadruple-precision arithmetic.)

Best results for z^κ *fixed* and away from 1, the point of singularity of $f(z)$. In our case, $z^\kappa = 0.5$ maintained for every $z > 0$ in the table.

Note that $R_l = \lfloor \kappa(l+1) \rfloor$ with non-integer $\kappa > 1$ also gives excellent results.

Theoretical treatment of stability

In most cases, $A_n^{(j)}$ is the solution to linear systems of the form

$$A(y_l) = A_n^{(j)} + \sum_{k=1}^{n} \alpha_k \phi_k(y_l), \quad j \le l \le j+n; \quad y_0 > y_1 > \cdots > 0.$$

Here $A(y)$ and $\phi_k(y)$ are known for $y > 0$. In matrix form, $Mx = b$, where

$$M = \begin{bmatrix} 1 & \phi_1(y_j) & \cdots & \phi_n(y_j) \\ 1 & \phi_1(y_{j+1}) & \cdots & \phi_n(y_{j+1}) \\ \vdots & \vdots & & \vdots \\ 1 & \phi_1(y_{j+n}) & \cdots & \phi_n(y_{j+n}) \end{bmatrix}, \quad x = \begin{bmatrix} A_n^{(j)} \\ \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}, \quad b = \begin{bmatrix} a(y_j) \\ a(y_{j+1}) \\ \vdots \\ a(y_{j+n}) \end{bmatrix}$$

If the first row of M^{-1} is $[m_0, m_1, \ldots, m_n]$, then

$$\gamma_{ni}^{(j)} = m_i, \quad i = 0, 1, \ldots, n.$$

In addition, the $\gamma_{ni}^{(j)}$ also satisfy the linear system

$$M^{\mathsf{T}} \overrightarrow{\gamma} = e_1,$$

where

$$\overrightarrow{\gamma} = [\gamma_{n0}^{(j)}, \gamma_{n1}^{(j)}, \ldots, \gamma_{nn}^{(j)}]^{\mathsf{T}}, \quad e_1 = [1, 0, \ldots, 0]^{\mathsf{T}}.$$

Finally, the $\gamma_{ni}^{(j)}$ satisfy

$$\sum_{i=0}^{n} \gamma_{ni}^{(j)} z^i = \frac{H_n^{(j)}(z)}{H_n^{(j)}(1)},$$

where

$$H_n^{(j)}(z) = \det \begin{bmatrix} z^0 & \phi_1(y_j) & \cdots & \phi_n(y_j) \\ z^1 & \phi_1(y_{j+1}) & \cdots & \phi_n(y_{j+1}) \\ \vdots & \vdots & & \vdots \\ z^n & \phi_1(y_{j+n}) & \cdots & \phi_n(y_{j+n}) \end{bmatrix}.$$

Theoretical studies of extrapolation methods are ultimately carried out by analyzing $H_n^{(j)}(z)$ asymptotically. This analysis seems to be possible for $j \to \infty$ (n fixed) in some cases. It is extremely difficult for $n \to \infty$ (j fixed). Nevertheless, conclusions drawn from $j \to \infty$ asymptotics and that are relevant for the sequences $\{A_n^{(j)}\}_{j=0}^{\infty}$ seem to be just as valid for the sequences $\{A_n^{(j)}\}_{n=0}^{\infty}$.

Examples

1. Classical Richardson extrapolation

For functions $A(y)$ of the form

$$A(y) \sim A + \sum_{k=1}^{\infty} \alpha_k y^{\sigma_k} \quad (y \to 0).$$

Defined via

$$A(y_l) = A_n^{(j)} + \sum_{k=1}^{n} \bar{\alpha}_k y_l^{\sigma_k}, \quad j \le l \le j+n.$$

Here $\Re\sigma_1 < \Re\sigma_2 < \cdots$, and $\lim_{k\to\infty} \Re\sigma_k = +\infty$.

(a) In case $y_l = y_0\omega^l$ for some fixed $\omega \in (0,1)$,

$$\sum_{i=0}^{n} \gamma_{ni}^{(j)} z^i = \prod_{k=1}^{n} \frac{z - c_k}{1 - c_k}, \quad \Gamma_n^{(j)} \le \prod_{k=1}^{n} \frac{1+|c_k|}{|1-c_k|}; \quad c_k = \omega^{\sigma_k}.$$

$\Gamma_n^{(j)}$ bounded in j always.
$\Gamma_n^{(j)}$ bounded in n provided $\Re\sigma_{k+1} - \Re\sigma_k \ge d > 0$ for all k.

(b) In case $\lim_{l\to\infty}(y_{l+1}/y_l) = \omega \in (0,1)$,

$$\lim_{j\to\infty} \sum_{i=0}^{n} \gamma_{ni}^{(j)} z^i = \prod_{k=1}^{n} \frac{z - c_k}{1 - c_k}, \quad \lim_{j\to\infty} \Gamma_n^{(j)} \leq \prod_{k=1}^{n} \frac{1+|c_k|}{|1-c_k|}; \quad c_k = \omega^{\sigma_k}.$$

$\Gamma_n^{(j)}$ bounded in j always.

(c) In case $y_l = c/(l+\eta)^q$, $c, \eta, q > 0$,

$$\Gamma_n^{(j)} \sim \left(\prod_{i=1}^{n} |\sigma_i| \right)^{-1} \left(\frac{2j}{q} \right)^n \quad (j \to \infty).$$

That is, $\lim_{j\to\infty} \Gamma_n^{(j)} = \infty$.
In addition, when $\sigma_k = k$, there holds

$$\Gamma_n^{(j)} = O(n^\mu) \quad (n \to \infty)\ \forall \mu > 0; \quad \lim_{n\to\infty} \Gamma_n^{(j)} = \infty.$$

2. GREP$^{(1)}$

For functions $A(y)$ of the form

$$A(y) \sim A + \phi(y)\sum_{k=1}^{\infty}\beta_k y^k \quad (y \to 0).$$

Defined via

$$A(y_l) = A_n^{(j)} + \phi(y_l)\sum_{k=1}^{\infty}\bar{\beta}_k y_l^k, \quad j \le l \le j+n.$$

(I) For $\phi(y) \sim y^{\delta}H(y),\ H \in \mathbb{C}^{\infty}[0,a]$ for some $a > 0$,

(a) In case $\lim_{l\to\infty}(y_{l+1}/y_l) = \omega \in (0,1)$,

$$\lim_{j\to\infty}\sum_{i=0}^{n}\gamma_{ni}^{(j)}z^i = \prod_{k=1}^{n}\frac{z-c_k}{1-c_k}, \quad \lim_{j\to\infty}\Gamma_n^{(j)} = \prod_{k=1}^{n}\frac{1+|c_k|}{|1-c_k|}; \quad c_k = \omega^{\delta+k-1}.$$

(b) In case $y_{l+1}/y_l = \omega \in (0,1)$,

$$\lim_{j\to\infty}\Gamma_n^{(j)} = \prod_{k=1}^{n}\frac{1+|c_k|}{|1-c_k|}; \quad \lim_{n\to\infty}\Gamma_n^{(j)} = \prod_{k=1}^{\infty}\frac{1+|c_k|}{|1-c_k|}.$$

(c) In case $y_l = c/(l+\eta)^q$, $c, \eta, q > 0$,

$$\Gamma_n^{(j)} \sim \frac{1}{|(\delta)_n|}\left(\frac{2j}{q}\right)^n \quad (j \to \infty); \quad \lim_{n\to\infty}\Gamma_n^{(j)} = \infty.$$

(II) For $\phi(y) = e^{u(y)}h(y)$, with

$$u(y) \sim \sum_{i=0}^{\infty} u_k y^{-s+k} \quad (y \to 0); \quad h(y) \sim h_0 y^{\delta} \quad (y \to 0).$$

(a) In case $y_l \sim cl^{-q}$ and $y_l - y_{l+1} \sim cpl^{-q-1}$ as $l \to \infty$, with $q = 1/s$,

$$\Gamma_n^{(j)} \sim \left(\frac{1 + |\xi|}{|1 - \xi|}\right)^n \quad (j \to \infty); \quad \xi = \exp(-spu_0/c^s).$$

(b) In case $\lim_{l\to\infty} \operatorname{sign}[\phi(y_{l+1})/\phi(y_l)] = -1$,

$$\Gamma_n^{(j)} \sim 1 \quad (n \to \infty).$$

Definition: Let $A_n = \sum_{i=0}^{n} a_i$, $n = 0, 1, \ldots$.
$\{a_n\} \in \mathbf{b}^{(1)}$/LOG if $a_n \sim \sum_{i=0}^{\infty} \epsilon_i n^{\gamma - i} \quad (n \to \infty), \quad \gamma \neq 0, 1, \ldots$.
$\{a_n\} \in \mathbf{b}^{(1)}$/LIN if $a_n \sim \zeta^n \sum_{i=0}^{\infty} \epsilon_i n^{\gamma - i} \quad (n \to \infty), \quad \zeta \neq 1$.

3. Iterated Lubkin, Brezinski θ algorithm, Levin u transformation

(a) In case $\{a_n\} \in \mathbf{b}^{(1)}$/LOG,

$$\sum_{i=0}^{n} \gamma_{ni}^{(j)} z^i \sim C_n (1 - z)^n j^n \quad (j \to \infty); \quad \Gamma_n^{(j)} \sim C_n (2j)^n \quad (j \to \infty).$$

(b) In case $\{a_n\} \in \mathbf{b}^{(1)}$/LIN,

$$\lim_{j \to \infty} \sum_{i=0}^{n} \gamma_{ni}^{(j)} z^i = \left(\frac{z - \zeta}{1 - \zeta} \right)^n; \quad \lim_{j \to \infty} \Gamma_n^{(j)} = \left(\frac{1 + |\zeta|}{|1 - \zeta|} \right)^n.$$

For $\{a_n\} \in \mathbf{b}^{(1)}$/LOG, nothing can be done to improve stability.
For $\{a_n\} \in \mathbf{b}^{(1)}$/LIN, when $\zeta \approx 1$, apply methods to the sequence $\{A_{\kappa n}\}$ with some integer $\kappa \geq 2$, depending on how close ζ is to 1. We then have

$$\lim_{j \to \infty} \Gamma_n^{(j)} = \left(\frac{1 + |\zeta^\kappa|}{|1 - \zeta^\kappa|} \right)^n.$$

4. Iterated Aitken, Shanks transformation

(a) In case $\{a_n\} \in \mathbf{b}^{(1)}$/LOG, no convergence acceleration.

(b) In case $\{a_n\} \in \mathbf{b}^{(1)}$/LIN,

$$\lim_{j\to\infty} \sum_{i=0}^{n} \gamma_{ni}^{(j)} z^i = \left(\frac{z-\zeta}{1-\zeta}\right)^n; \quad \lim_{j\to\infty} \Gamma_n^{(j)} = \left(\frac{1+|\zeta|}{|1-\zeta|}\right)^n.$$

When $\zeta \approx 1$, apply methods to the sequence $\{A_{\kappa n}\}$ with some integer $\kappa \geq 2$, depending on how close ζ is to 1. We then have

$$\lim_{j\to\infty} \Gamma_n^{(j)} = \left(\frac{1+|\zeta^\kappa|}{|1-\zeta^\kappa|}\right)^n.$$

(c)(Shanks) In case $A_n \sim A + \sum_{k=1}^{\infty} \alpha_k \lambda_k^n \quad (n\to\infty), \quad |\lambda_1| \geq |\lambda_2| \geq \cdots,$

$$\lim_{j\to\infty} \sum_{i=0}^{n} \gamma_{ni}^{(j)} z^i = \prod_{i=1}^{n} \frac{z-\lambda_i}{1-\lambda_i}; \quad \lim_{j\to\infty} \Gamma_n^{(j)} \leq \prod_{i=1}^{n} \frac{1+|\lambda_i|}{|1-\lambda_i|} \quad (|\lambda_n| > |\lambda_{n+1}|).$$

When $\lambda_1 \approx 1$, apply methods to the sequence $\{A_{\kappa n}\}$ with some integer $\kappa \geq 2$, depending on how close λ_1 is to 1. We then have

$$\lim_{j\to\infty} \Gamma_n^{(j)} \leq \prod_{i=1}^{n} \frac{1+|\lambda_i^\kappa|}{|1-\lambda_i^\kappa|}.$$

(d) (Shanks) In case

$$A_n \sim A + \sum_{k=1}^{\infty} P_k(n)\lambda_k^n \quad (n \to \infty), \quad |\lambda_1| \geq |\lambda_2| \geq \cdots, \quad P_k \in \pi_{\omega_k - 1}.$$

Let $|\lambda_t| > |\lambda_{t+1}|$ and set $n = \sum_{i=1}^{t} \omega_i$. Then

$$\lim_{j\to\infty} \sum_{i=0}^{n} \gamma_{ni}^{(j)} z^i = \prod_{i=1}^{n} \left(\frac{z-\lambda_i}{1-\lambda_i}\right)^{\omega_i}; \ \lim_{j\to\infty} \Gamma_n^{(j)} \leq \prod_{i=1}^{n} \left(\frac{1+|\lambda_i|}{|1-\lambda_i|}\right)^{\omega_i}.$$

When $\lambda_1 \approx 1$, apply methods to the sequence $\{A_{\kappa n}\}$ with some integer $\kappa \geq 2$, depending on how close λ_1 is to 1. We then have

$$\lim_{j\to\infty} \Gamma_n^{(j)} \leq \prod_{i=1}^{n} \left(\frac{1+|\lambda_i^\kappa|}{|1-\lambda_i^\kappa|}\right)^{\omega_i}.$$

General linear sequences and GREP$^{(m)}$

$$A_n \sim A + \sum_{k=1}^{m} \zeta_k^n \sum_{i=0}^{\infty} \beta_{ki} n^{\gamma_k - i} \quad (n \to \infty),$$

$$\zeta_k \neq 1 \quad \text{distinct}, \quad \gamma_k \quad \text{arbitrary}.$$

Define GREP$^{(m)}$ on $\{A_n\}$ via the equations

$$A_l = A_n^{(j)} + \sum_{k=1}^{m} \zeta_k^l \sum_{i=0}^{\nu-1} \bar{\beta}_{ki} (l+1)^{\gamma_k - i}, \quad j \leq l \leq j+n, \quad n = m\nu.$$

Then

$$A_n^{(j)} - A = \sum_{k=1}^{m} O(\zeta_k^j j^{\gamma_k - 2\nu}) \quad (j \to \infty).$$

$$\lim_{j\to\infty} \sum_{i=0}^{n} \gamma_{ni}^{(j)} z^i = \prod_{k=1}^{m} \left(\frac{z - \zeta_k}{1 - \zeta_k} \right)^{\nu}; \ \lim_{j\to\infty} \Gamma_n^{(j)} \leq \prod_{i=1}^{n} \left(\frac{1 + |\zeta_k|}{|1 - \zeta_k|} \right)^{\nu}.$$

This means that if $\zeta_k \approx 1$ for some k, then apply GREP$^{(m)}$ to $\{A_{\kappa n}\}$ with some $\kappa > 1$ integer. Make sure that all ζ_k^κ stay away from 1.

This strategy can be applied with the GREP$^{(m)}$ above replaced by the $d^{(m)}$ transformation:

$$A_{R_l} = A_n^{(j)} + \sum_{k=1}^{m} (\Delta^{k-1} a_{R_l}) \sum_{i=0}^{\nu-1} \bar{\beta}_{ki} (l+1)^{-i}, \quad j \leq l \leq j+n, \quad n = m\nu.$$

This time we can also choose $R_l = \kappa(l+1)$ with integer $\kappa > 1$. (APS)

The choice $R_l = \lfloor \kappa(l+1) \rfloor$ with non-integer $\kappa > 1$ also gives excellent results.

Example. $f(x) = \sum_{k=1}^{\infty} (\cos kx)/k = -\log|2\sin(x/2)|; \quad |x| \leq \pi.$
$A_n = \sum_{k=1}^{n} (\cos kx)/k, \quad n = 1, 2, \ldots .$

$$A_n \sim f(x) + a_n \sum_{i=0}^{\infty} \beta_{1i} n^{-i} + \Delta a_n \sum_{i=0}^{\infty} \beta_{2i} n^{-i} \quad (n \to \infty).$$

$$A_n \sim f(x) + a_n \sum_{i=0}^{\infty} \beta_{1i} n^{-i} + a_{n+1} \sum_{i=0}^{\infty} \beta_{2i} n^{-i} \quad (n \to \infty).$$

$$A_n \sim f(x) + \frac{\cos nx}{n} \sum_{i=0}^{\infty} \beta_{1i} n^{-i} + \frac{\sin nx}{n} \sum_{i=0}^{\infty} \beta_{2i} n^{-i} \quad (n \to \infty).$$

$$A_n \sim f(x) + \frac{e^{\mathrm{i}nx}}{n} \sum_{i=0}^{\infty} \beta_{1i} n^{-i} + \frac{e^{-\mathrm{i}nx}}{n} \sum_{i=0}^{\infty} \beta_{2i} n^{-i} \quad (n \to \infty).$$

The $d^{(2)}$ transformation applies. For $x \approx 0$ [point od singularity of f(x)] use APS.

Example $\sum_{k=1}^{\infty} \frac{1}{k} \cos kx \quad x = \pi/3,\ \kappa = 1,\ m = 2$

L	R_L	ERR(L,0)	ERR(0,L)
0	1	5.000D-01	5.000D-01
4	5	1.083D-01	6.547D-02
8	9	4.385D-02	2.099D-03
12	13	7.340D-02	1.747D-05
16	17	3.082D-02	3.759D-09
20	21	2.157D-02	3.283D-09
24	25	3.911D-02	1.354D-10
28	29	1.777D-02	1.124D-13
32	33	1.424D-02	2.162D-15
36	37	2.663D-02	4.667D-15
40	41	1.247D-02	1.279D-18
44	45	1.062D-02	7.907D-20
48	49	2.019D-02	1.926D-23
52	53	9.602D-03	3.359D-24
56	57	8.465D-03	1.381D-23
60	61	1.625D-02	5.347D-27

Example $\sum_{k=1}^{\infty} \frac{1}{k} \cos kx \quad x = \pi/6,\ \kappa = 1,\ m = 2$

L	R_L	ERR(L,0)	ERR(0,L)
0	1	2.075D-01	2.075D-01
4	5	1.593D-01	2.259D-01
8	9	1.935D-01	6.296D-02
12	13	8.514D-02	1.818D-03
16	17	3.866D-02	3.074D-05
20	21	8.748D-02	5.019D-05
24	25	4.921D-02	2.162D-06
28	29	2.072D-02	2.549D-08
32	33	5.617D-02	1.940D-08
36	37	3.446D-02	1.414D-09
40	41	1.400D-02	4.955D-11
44	45	4.132D-02	6.086D-12
48	49	2.649D-02	7.793D-14
52	53	1.053D-02	1.422D-13
56	57	3.266D-02	3.547D-11
60	61	2.150D-02	5.102D-09

Example $\sum_{k=1}^{\infty} \frac{1}{k} \cos kx$ $x = \pi/6$, $\kappa = 2$, $m = 2$

L	R_L	ERR(L,0)	ERR(0,L)
0	2	4.575D-01	4.575D-01
4	10	1.435D-01	3.238D-02
8	18	1.690D-02	9.352D-03
12	26	6.844D-02	1.643D-05
16	34	4.147D-02	7.244D-08
20	42	9.814D-03	4.987D-09
24	50	3.649D-02	9.177D-11
28	58	2.404D-02	3.643D-13
32	66	6.723D-03	7.402D-15
36	74	2.485D-02	7.319D-16
40	82	1.691D-02	1.426D-18
44	90	5.096D-03	6.869D-20
48	98	1.883D-02	7.365D-23
52	106	1.304D-02	7.144D-24
56	114	4.099D-03	4.611D-24
60	122	1.516D-02	3.836D-24

Example. $f(x) = \sum_{k=0}^{\infty} c_k \cos kx = H(h - |x|) \;\; |x| \le \pi$. $(0 < h < \pi)$

$$c_0 = h/\pi,\; c_k = (2/\pi)(\sin kh)/k,\; k \ge 1;\;\; A_n = \sum_{k=0}^{n} c_k(\cos kx),\;\; n = 1, 2, \ldots$$

$$A_n \sim f(x) + \sum_{k=1}^{4} \Delta^{k-1} a_n \sum_{i=0}^{\infty} \beta_{ki} n^{-i} \;\; (n \to \infty).$$

$$A_n \sim f(x) + \sum_{k=1}^{4} a_{n+k-1} \sum_{i=0}^{\infty} \beta_{ki} n^{-i} \;\; (n \to \infty).$$

$$A_n \sim f(x) + \frac{\cos n(x-h)}{n} \sum_{i=0}^{\infty} \beta_{1i} n^{-i} + \frac{\sin n(x-h)}{n} \sum_{i=0}^{\infty} \beta_{2i} n^{-i}$$
$$+ \frac{\cos n(x+h)}{n} \sum_{i=0}^{\infty} \beta_{3i} n^{-i} + \frac{\sin n(x+h)}{n} \sum_{i=0}^{\infty} \beta_{4i} n^{-i} \quad (n \to \infty).$$

$$
\begin{aligned}
A_n \sim f(x) &+ \frac{e^{\mathrm{i}n(x-h)}}{n} \sum_{i=0}^{\infty} \beta_{1i} n^{-i} + \frac{e^{-\mathrm{i}n(x-h)}}{n} \sum_{i=0}^{\infty} \beta_{2i} n^{-i} \\
&+ \frac{e^{\mathrm{i}n(x+h)}}{n} \sum_{i=0}^{\infty} \beta_{3i} n^{-i} + \frac{e^{-\mathrm{i}n(x+h)}}{n} \sum_{i=0}^{\infty} \beta_{4i} n^{-i} \quad (n \to \infty).
\end{aligned}
$$

The $d^{(4)}$ transformation applies. For $x \approx \pm h$ [points of singularity of $f(x)$], use APS.

Example $\frac{2h}{\pi}\left[\frac{1}{2}+\sum_{k=1}^{\infty}\frac{\sin kh}{kh}\cos kx\right]$ $h=1,\ x=0.9, \kappa=1, m=4$

L	R_L	ERR(L,0)	ERR(0,L)
0	1	3.487D-01	3.487D-01
8	9	2.261D-01	4.651D-01
16	17	2.672D-02	2.147D-01
24	25	7.178D-02	2.199D-01
32	33	8.339D-02	1.592D-01
40	41	5.476D-02	2.191D-01
48	49	7.433D-03	1.634D-02
56	57	3.858D-02	2.119D-03
64	65	4.789D-02	2.925D-04
72	73	2.540D-02	2.227D-03
80	81	8.328D-03	5.825D-05
88	89	2.723D-02	1.377D-05
96	97	3.337D-02	7.025D-05
104	105	1.374D-02	1.301D-04
112	113	6.630D-03	1.283D-03
120	121	2.273D-02	6.202D-04

Example $\frac{2h}{\pi}\left[\frac{1}{2} + \sum_{k=1}^{\infty} \frac{\sin kh}{kh} \cos kx\right]$ $h = 1,\ x = 0.9,$
$\kappa = 1.65 \approx \pi/(x+h), m = 4$

L	R_L	ERR(L,0)	ERR(0,L)
0	1	3.487D-01	3.487D-01
8	14	7.917D-02	6.248D-01
16	28	9.009D-02	1.724D-01
24	41	5.476D-02	7.503D-02
32	54	2.685D-02	6.222D-03
40	67	4.098D-02	1.164D-04
48	80	4.367D-03	2.256D-04
56	94	3.494D-02	9.736D-05
64	107	1.146D-02	3.533D-05
72	120	2.014D-02	2.927D-06
80	133	1.694D-02	2.405D-07
88	146	9.728D-03	2.950D-07
96	160	2.015D-02	1.205D-07
104	173	8.372D-04	2.457D-08
112	186	1.563D-02	1.288D-07
120	199	7.443D-03	4.328D-05

<u>Example</u> $\frac{2h}{\pi}\Big[\frac{1}{2}+\sum_{k=1}^{\infty}\frac{\sin kh}{kh}\cos kx\Big]$ $h=1,\ x=0.9,$
$\kappa = 4.95 \approx 3\pi/(x+h)$, $m=4$

L	R_L	ERR(L,0)	ERR(0,L)
0	4	3.335D-01	3.335D-01
8	44	3.421D-02	3.136D-02
16	84	1.939D-02	7.537D-03
24	123	2.355D-02	4.848D-05
32	163	1.730D-02	2.307D-06
40	202	3.501D-03	1.072D-08
48	242	7.507D-03	5.011D-10
56	282	1.185D-02	1.123D-10
64	321	7.660D-03	6.826D-14
72	361	2.679D-04	1.066D-16
80	400	5.066D-03	1.677D-17
88	440	7.341D-03	5.461D-19
96	480	4.216D-03	9.233D-22
104	519	2.651D-04	2.424D-23
112	559	4.771D-03	5.558D-22

Example $\frac{2h}{\pi}\left[\frac{1}{2}+\sum_{k=1}^{\infty}\frac{\sin kh}{kh}\cos kx\right]$ $h=1,\ x=0.9,$
$\kappa=8.27\approx 5\pi/(x+h)$, $m=4$

L	R_L	ERR(L,0)	ERR(0,L)
0	8	2.190D-01	2.190D-01
8	74	1.853D-02	2.602D-02
16	140	2.128D-03	8.088D-04
24	206	3.667D-03	3.599D-07
32	272	6.280D-03	3.501D-10
40	339	7.832D-03	1.606D-12
48	405	7.809D-03	6.545D-15
56	471	7.114D-03	6.913D-18
64	537	5.967D-03	4.099D-20
72	603	4.545D-03	3.288D-22
80	669	3.000D-03	1.549D-25
88	736	8.811D-04	9.767D-26
96	802	3.867D-04	3.517D-31
104	868	1.422D-03	2.379D-31
112	934	2.174D-03	2.889D-33
120	1000	2.619D-03	2.504D-33

Example $\frac{2h}{\pi}\Big[\frac{1}{2}+\sum_{k=1}^{\infty}\frac{\sin kh}{kh}\cos kx\Big]$ $h=1,\ x=0.9,$
$\kappa=11.55\approx 7\pi/(x+h)$, $m=4$

L	R_L	ERR(L,0)	ERR(0,L)
0	11	1.430D-01	1.430D-01
8	103	2.101D-02	4.003D-03
16	196	1.094D-02	1.107D-02
24	288	9.404D-03	3.645D-06
32	381	7.262D-03	4.402D-08
40	473	6.413D-03	1.017D-12
48	565	5.974D-03	1.570D-15
56	658	4.580D-03	6.195D-18
64	750	4.214D-03	4.546D-21
72	843	3.129D-03	2.110D-23
80	935	2.745D-03	7.352D-26
88	1027	1.894D-03	9.682D-30
96	1120	1.485D-03	4.391D-32
104	1212	8.443D-04	2.793D-33
112	1305	4.279D-04	3.178D-33

<u>Example</u> $\frac{2h}{\pi}[\frac{1}{2}+\sum_{k=1}^{\infty}\frac{\sin kh}{kh}\cos kx]$ $h=1,\ x=0.9,$
$\kappa = 3.3 \approx 2\pi/(x+h), m=4$

L	R_L	ERR(L,0)	ERR(0,L)
0	3	4.415D-01	4.415D-01
8	29	8.182D-02	1.269D+00
16	56	4.107D-02	6.744D-02
24	82	8.261D-03	2.886D-04
32	108	6.091D-03	1.551D-04
40	135	1.571D-02	1.405D-06
48	161	1.758D-02	2.765D-08
56	188	1.784D-02	5.609D-09
64	214	1.172D-02	2.981D-11
72	240	5.583D-03	3.143D-12
80	267	4.536D-04	2.825D-12
88	293	5.456D-03	2.957D-14
96	320	8.701D-03	1.384D-15
104	346	9.056D-03	9.901D-17
112	372	7.261D-03	3.330D-16