Strong coupling asymptotics of the β-function in φ^4 theory and QED

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The problem

Divergent series

$$W(g) = \sum_{N=0}^{\infty} W_N (-g)^N$$

where positive W_N are given numerically and have the

factorial asymptotics

$$W_N^{as} = c a^N \Gamma(N+b) \qquad (N \to \infty)$$

given by the Lipatov method.

How to find W(g) at arbitrary g ?

The standard summation procedure

Borel transformation

$$W(g)=\sum_{N=0}^{\infty}W_N(-g)^N=\sum_{N=0}^{\infty}\frac{W_N}{N!}N!\,(-g)^N=$$

$$=\sum_{N=0}^{\infty}\frac{W_N}{N!}\int_0^{\infty}dx\,e^{-x}x^N(-g)^N=$$

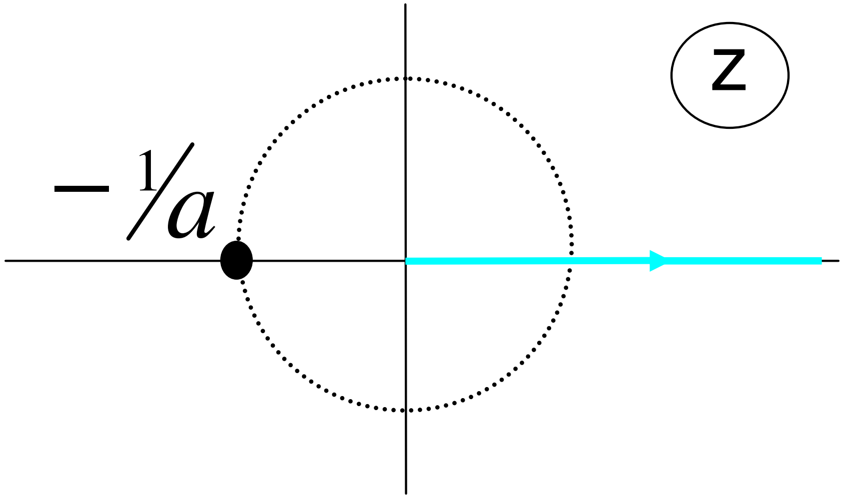
$$=\int_0^{\infty}dx\,e^{-x}\sum_{N=0}^{\infty}\frac{W_N}{N!}(-gx)^N$$

We can also use $\Gamma(N+b_0)$ instead N!

General Borel transformation

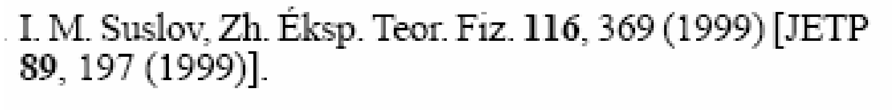
$$W(g) = \int_0^\infty dx\ e^{-x} x^{b_0-1} B(gx)$$

$$B(z) = \sum_{N=0}^\infty B_N (-z)^N \qquad B_N = \frac{W_N}{\Gamma(N+b_0)}$$



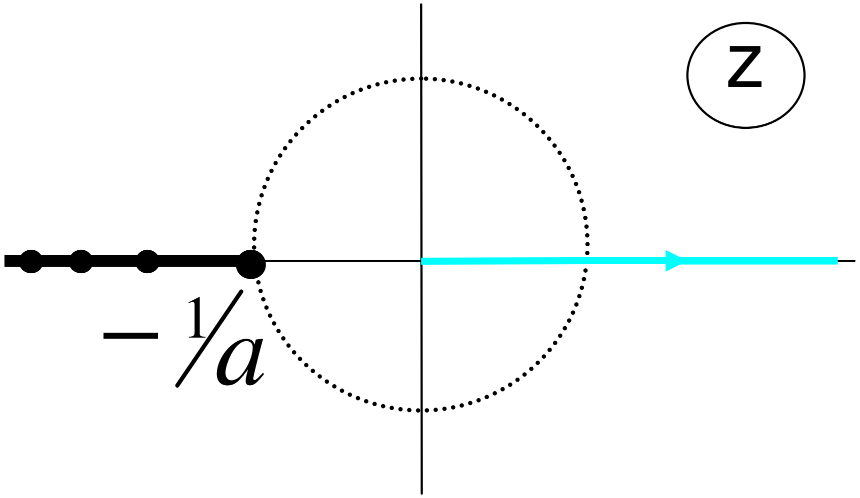
General Borel transformation

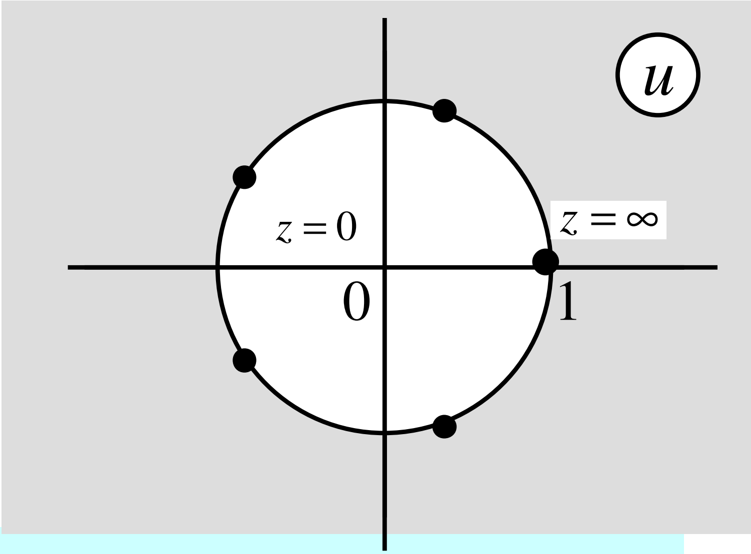
$$W(g)=\int_0^\infty dx\; e^{-x}x^{b_0-1}B(gx)$$


I. M. Suslov, Zh. Éksp. Teor. Fiz. **116**, 369 (1999) [JETP **89**, 197 (1999)].

$$B(z)=\sum_{N=0}^{\infty}B_N(-z)^N \qquad B_N=\frac{W_N}{\Gamma(N+b_0)}$$

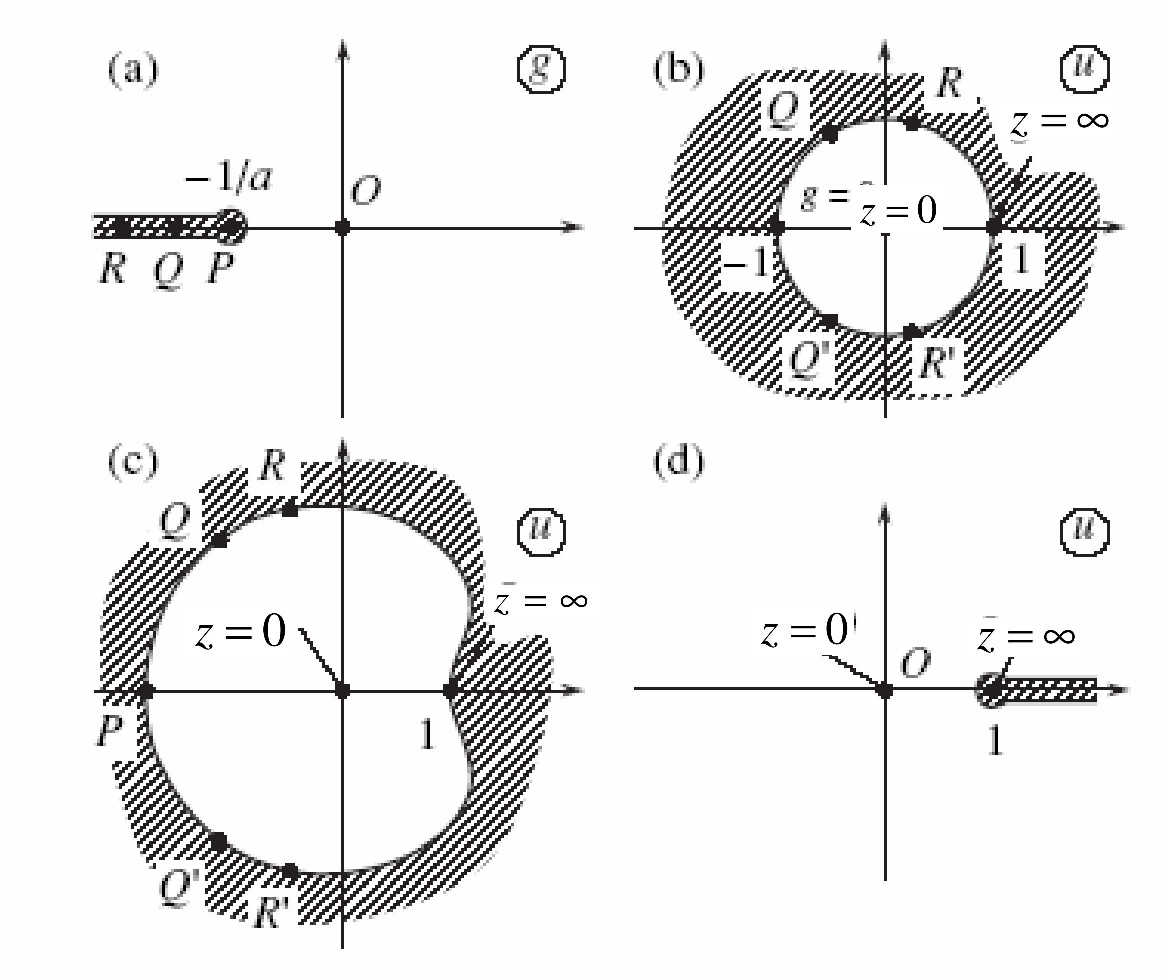
J.C.Le Guillou,
J.Zinn-Justin,
PRL 39, 55 (1977)





$$B(z)\xrightarrow{\;z=f(u)\;}B(u)=\sum_{N=0}^{\infty}U_N u^N$$

Summation in the strong coupling limit



$$z = \frac{u}{a(1-u)}$$

$$W(g) = W_\infty g^\alpha \qquad (g \to \infty)$$

$$U_N = \frac{W_\infty}{a^\alpha\ \Gamma(\alpha)\ \Gamma(b_0+\alpha)}\ N^{\alpha-1} \qquad (N \to \infty)$$

Resuling algorithm

The initial series

$$W(g) = \sum_{N=0}^{\infty} W_N (-g)^N$$

defines the coefficients of re-expanded series

$$U_N = \sum_{K=1}^{N} \frac{W_K}{a^K \Gamma(K + b_0)} (-1)^K C_{N-1}^{K-1} \qquad (1)$$

while their behavior at large N

$$U_N = U_\infty N^{\alpha - 1} \qquad (N \to \infty)$$

$$U_\infty = \frac{W_\infty}{a^\alpha \Gamma(\alpha) \Gamma(b_0 + \alpha)}$$

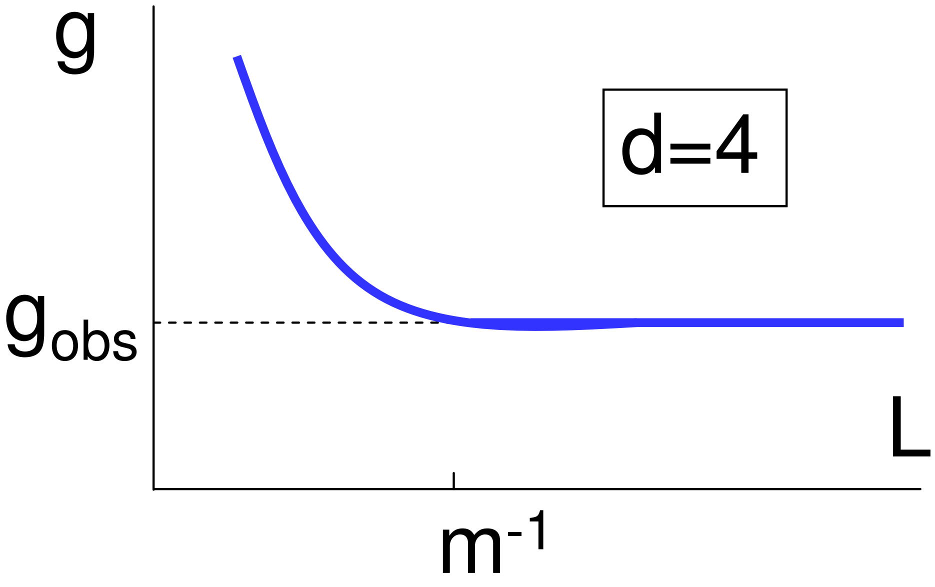
is related with the strong coupling asymptotics of W(g)

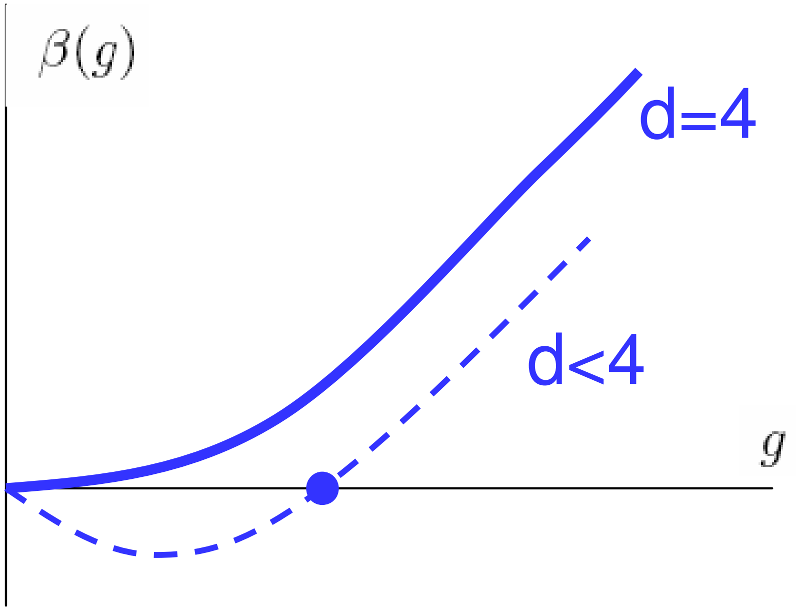
$$W(g) = W_\infty g^\alpha \qquad (g \to \infty)$$

Summation at arbitrary g presents no problem

The Gell-Mann -- Low function

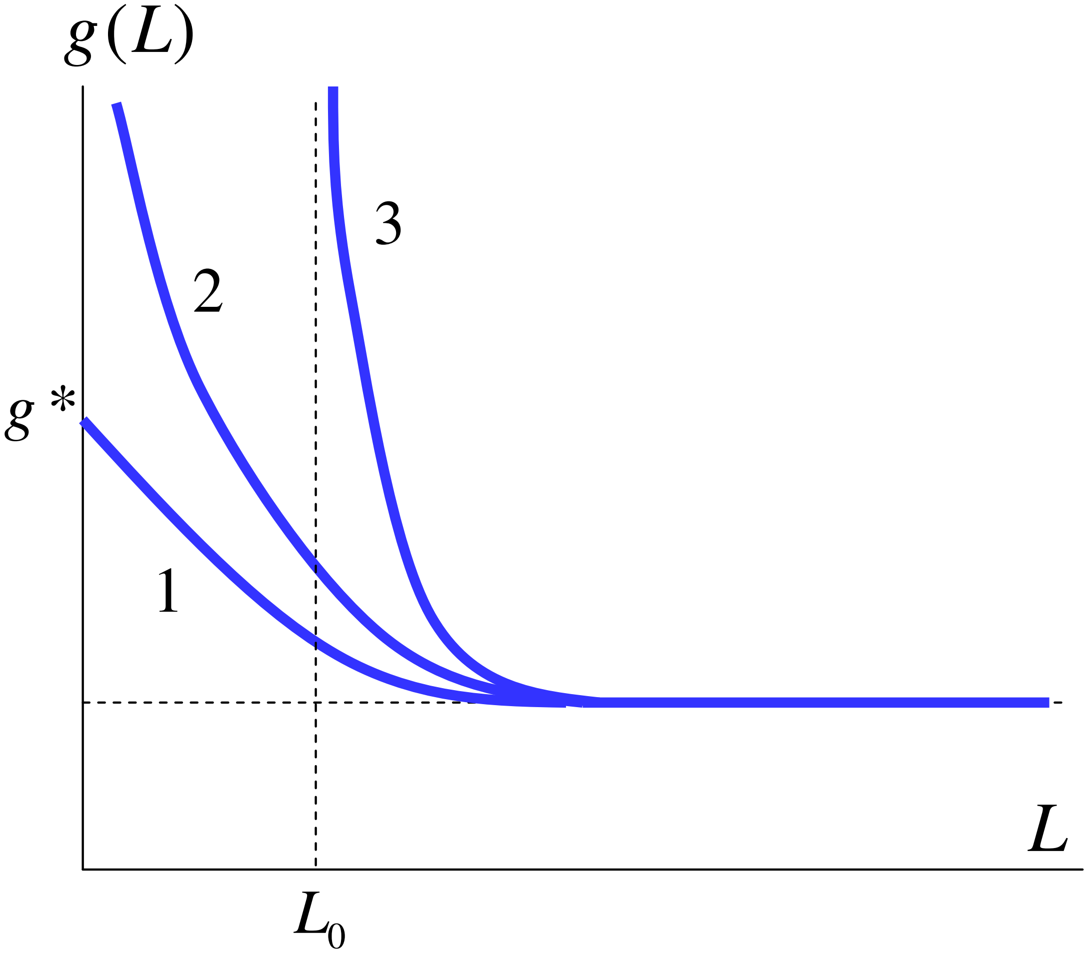
$$-\frac{d\ g}{d\ln L^2} = \beta(g)$$

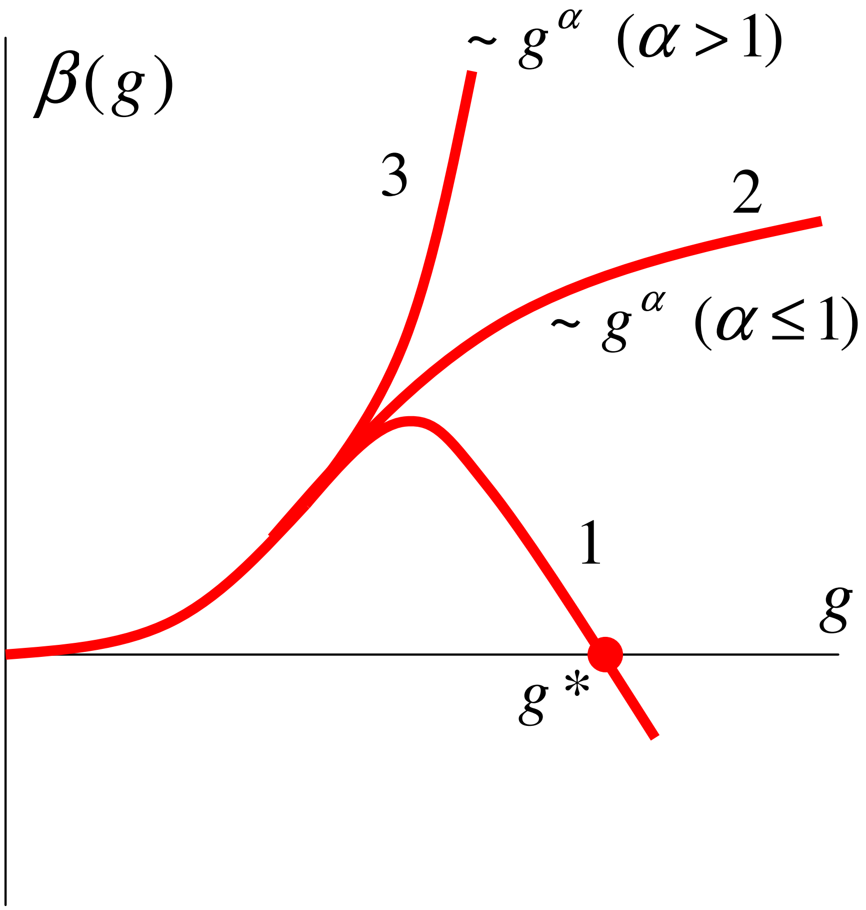




Classification by Bogolyubov and Shirkov

$$-\frac{d\ g}{d\ln L^2}=\beta(g)=\beta_2 g^2+\beta_3 g^3+\ldots$$





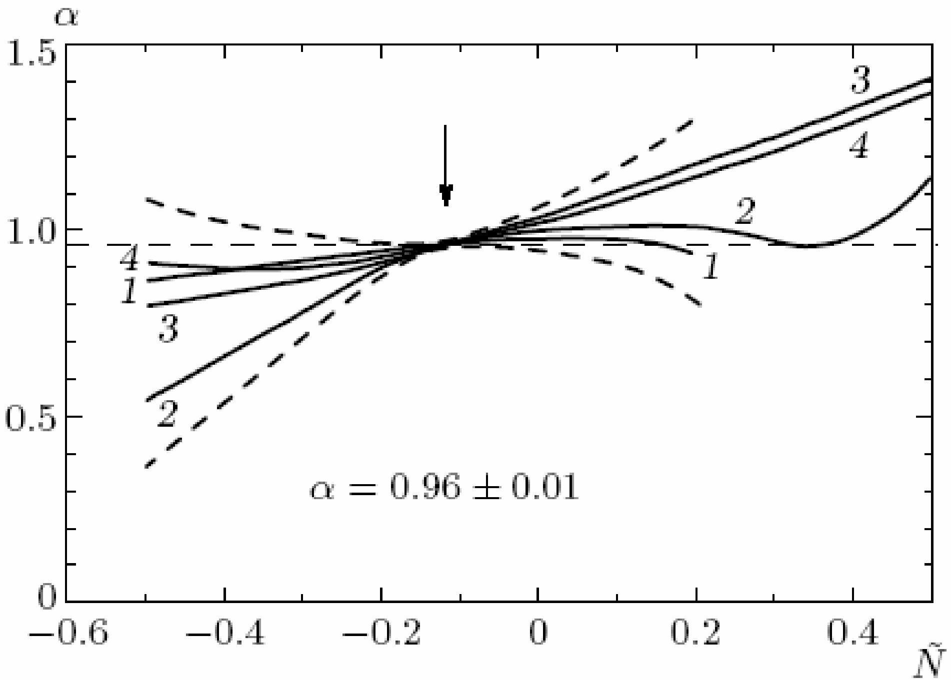
Summation results for the β-function of φ⁴ theory

$$\beta(g) = \beta_2 g^2 + \beta_3 g^3 + \ldots + \beta_L g^L + \ldots + c a^N \Gamma(N+b)\; g^N + \ldots$$

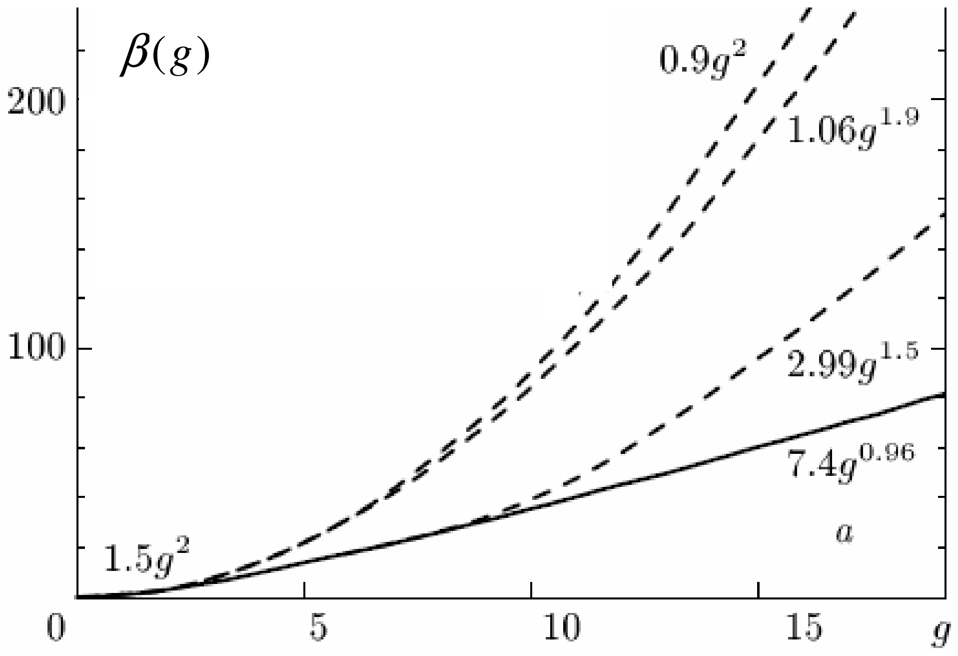
$$\beta_N = \beta_N^{as} \left(1 + \frac{A_1}{N} + \frac{A_2}{N^2} + \frac{A_3}{N^3} + \ldots\right)$$

$$\beta_N = \beta_N^{as} \left(1 + \frac{A_1}{(N-\tilde{N})} + \frac{A_2}{(N-\tilde{N})^2} + \ldots\right)$$

d=4



I.M.Suslov,
JETP 93, 1 (2001)



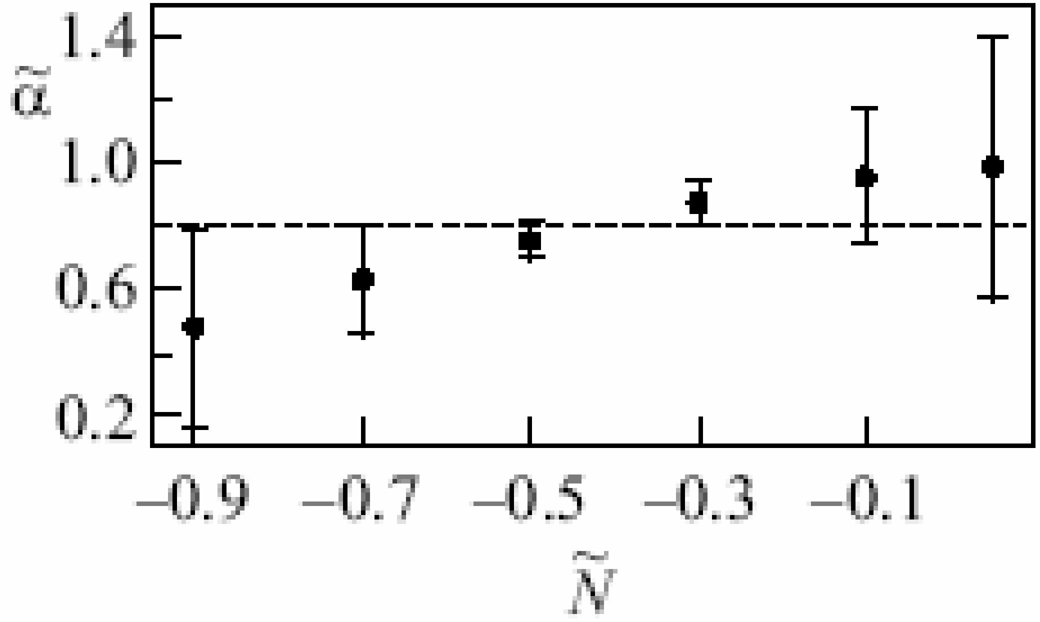
D. I. Kazakov, O. V. Tarasov, and D. V. Shirkov, Teor. Mat. Fiz. **38**, 15 (1979).

Yu. A. Kubyshin, Teor. Mat. Fiz. **58**, 137 (1984).

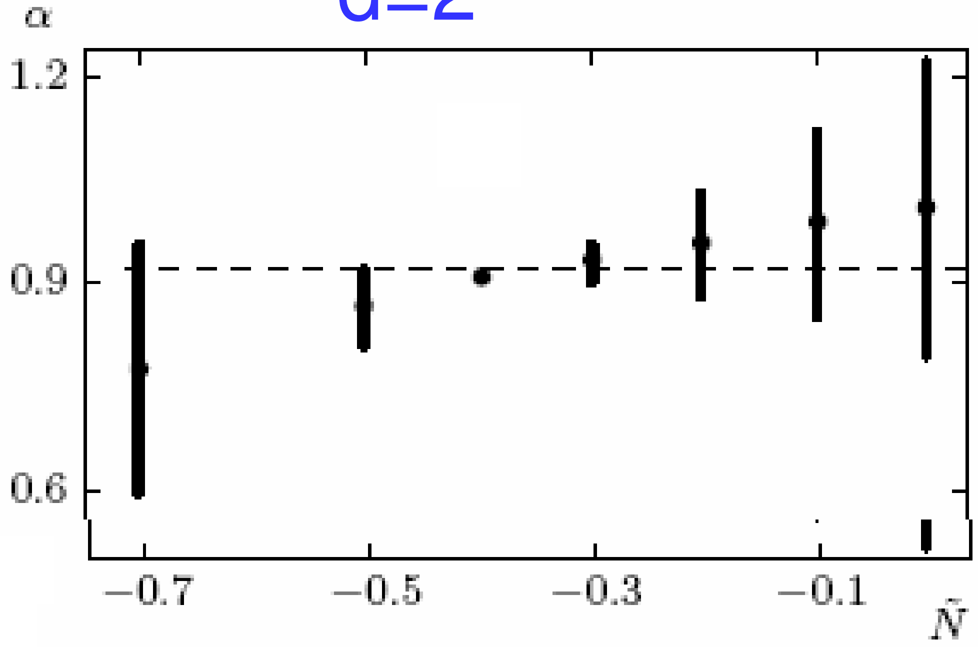
A. N. Sissakian et al., Phys. Lett. B **321**, 381 (1994).

Other dimensionalities

d=3 (n=2)



d=2



A.A.Pogorelov,
I.M.Suslov,
JETP 105, 360 (2007);
JETP Letters 86, 39 (2007)

The natural hypothesis:

$$\beta(g) \sim g \qquad for \qquad g \to \infty$$

at any d.

The natural strategy:

(a) To test for d=0
(b) To find out the mechanism
(c) To generalize at arbitrary d

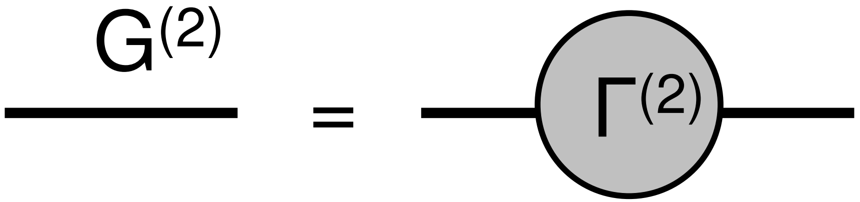
Expression of the β-function in terms of functional integrals

$$Z^{(M)}_{\alpha_1\ldots\alpha_M}(x_1,\ldots,x_M)=\int D\varphi\,\varphi_{\alpha_1}(x_1)\varphi_{\alpha_2}(x_2)\ldots\varphi_{\alpha_M}(x_M)\exp\left(-S\{\varphi\}\right)$$

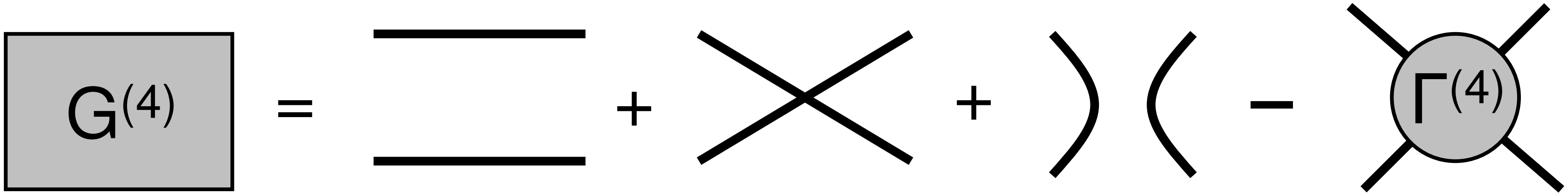
$$S\{\varphi\}=\int d^dx\,\left\{\tfrac{1}{2}(\nabla\varphi)^2+\tfrac{1}{2}m_0^2\varphi^2+\tfrac{1}{8}u\varphi^4\right\}$$

$$G^{(M)}=Z^{(M)}/Z^{(0)}$$

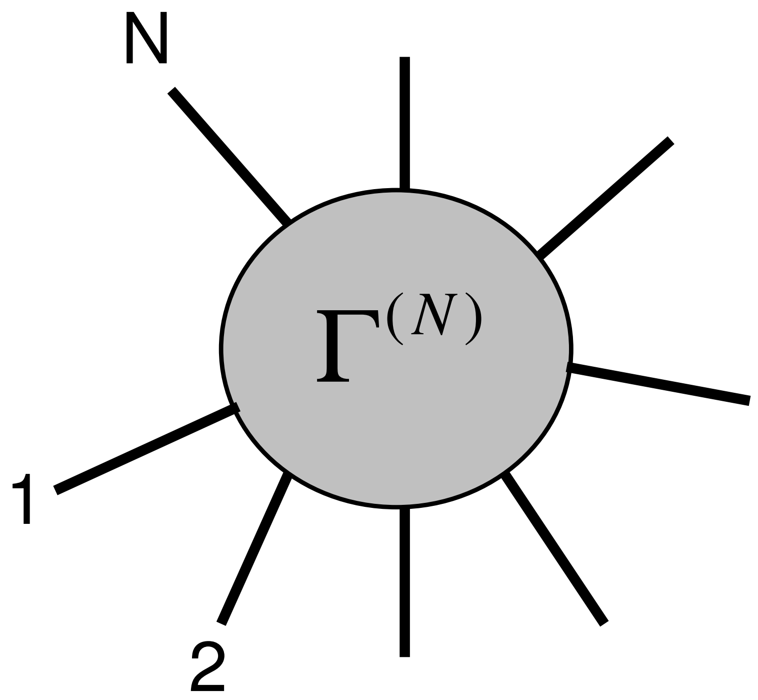
$$u=g_0\Lambda^{-\varepsilon},\qquad \varepsilon=4-d$$



$$\Gamma^{(2)}=1/G^{(2)}$$

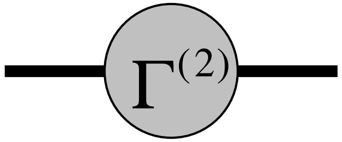


Multiplicative renormalizability



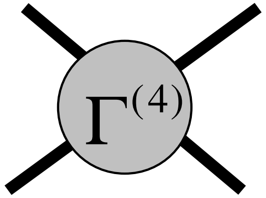
$$\Gamma^{(N)}(p_i; g_0, m_0, \Lambda) = Z^{-N/2}\, \Gamma_R^{(N)}(p_i; g, m)$$

Renormalizability conditions



$$\Gamma_R^{(2)}(p;g,m)\Big|_{p\to 0} = m^2 + p^2 + O(p^4)$$

$m_0^2 + p^2$



$$\Gamma_R^{(4)}(p_i;g,m)\Big|_{p_i=0} = g m^{\varepsilon}\,, \qquad \varepsilon = 4-d$$

$g_0\Lambda^{\varepsilon}$

Definition of the β-function

$$\beta(g) = \frac{d\,g}{d\ln m}\Bigg|_{g_0,\Lambda=const}$$

$$Z(g_0,m_0,\Lambda) = \left(\frac{\partial}{\partial p^2}\Gamma^{(2)}(p;g_0,m_0,\Lambda)\Big|_{p=0}\right)^{-1}$$

$$m^2 = Z(g_0,m_0,\Lambda)\;\Gamma^{(2)}(p;g_0,m_0,\Lambda)\Big|_{p=0}$$

$$g m^{\varepsilon} = Z^2(g_0,m_0,\Lambda)\;\Gamma^{(4)}(p_i;g_0,m_0,\Lambda)\Big|_{p_i=0}$$

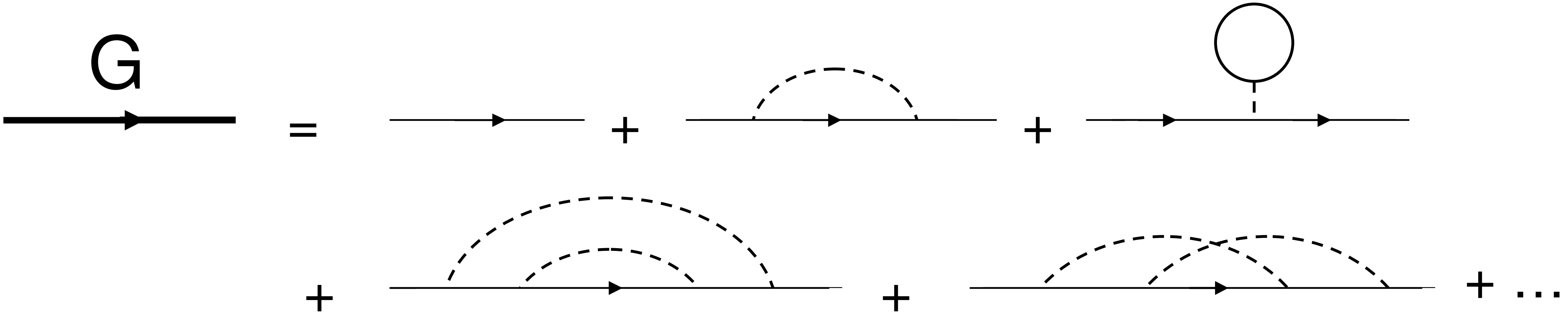
«Naive» zero-dimensional limit

$$Z^{(M)}_{\alpha_1\ldots\alpha_M}(x_1,\ldots,x_M)=\int D\varphi\,\varphi_{\alpha_1}(x_1)\varphi_{\alpha_2}(x_2)\ldots\varphi_{\alpha_M}(x_M)\exp\left(-S\{\varphi\}\right)$$

$$S\{\varphi\}=\int d^d x\,\left\{\,\tfrac{1}{2}(\nabla\varphi)^2+\tfrac{1}{2}m_0^2\varphi^2+\tfrac{1}{8}u\varphi^4\,\right\}$$

$$u=g_0\Lambda^{\varepsilon}\,,\qquad \varepsilon=4-d$$

$$Z^{(M)}_{\alpha_1\ldots\alpha_M}=\frac{m_0^n}{(2\pi)^{n/2}}\int d^n\varphi\,\varphi_{\alpha_1}\ldots\varphi_{\alpha_M}\exp\left(-\tfrac{1}{2}m_0^2\varphi^2-\tfrac{1}{8}u\varphi^4\right)$$



Definition of Z–factor:

$$G_2^{(0)}(p) = \frac{1}{p^2 + m_0^2}$$

$$G_2(r) = \left\langle \varphi(x)\ \ \varphi(x+r) \right\rangle$$

$$G_2(p) = \frac{1}{p^2 + m_0^2 + \Sigma(p, m_0)} =$$

$$= \frac{1}{p^2 + m_0^2 + a_0(m_0) + a_2(m_0)p^2 + a_4(m_0)p^4 + \ldots} =$$

$$= \frac{Z}{p^2 + m^2 + O(p^4)}$$

In the «naive» zero-dimensional limit we set

Z=1

Parametric representation for the β-function

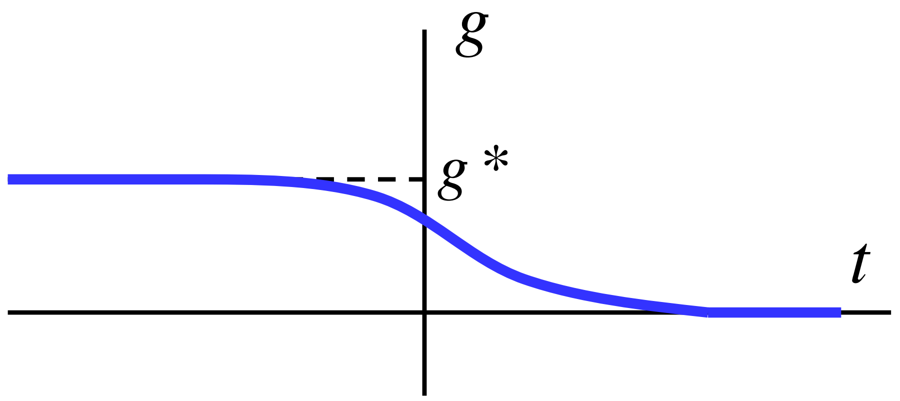
$$g = 1 - \frac{n}{n+2}\frac{K_4K_0}{K_2^2}$$

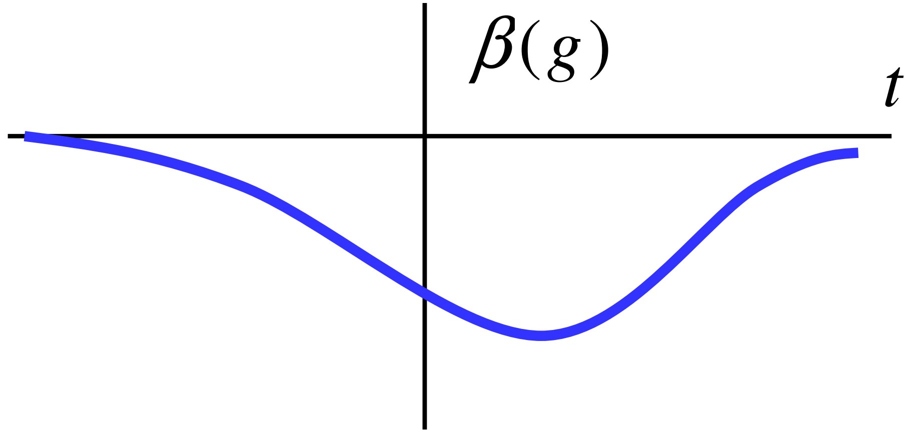
$$\beta(g) = -\frac{2n}{n+2}\frac{K_4K_0}{K_2^2}\left[2 + \frac{\frac{K_6K_0}{K_4K_2} - 1}{1 - \frac{K_4K_0}{K_2^2}}\right]$$

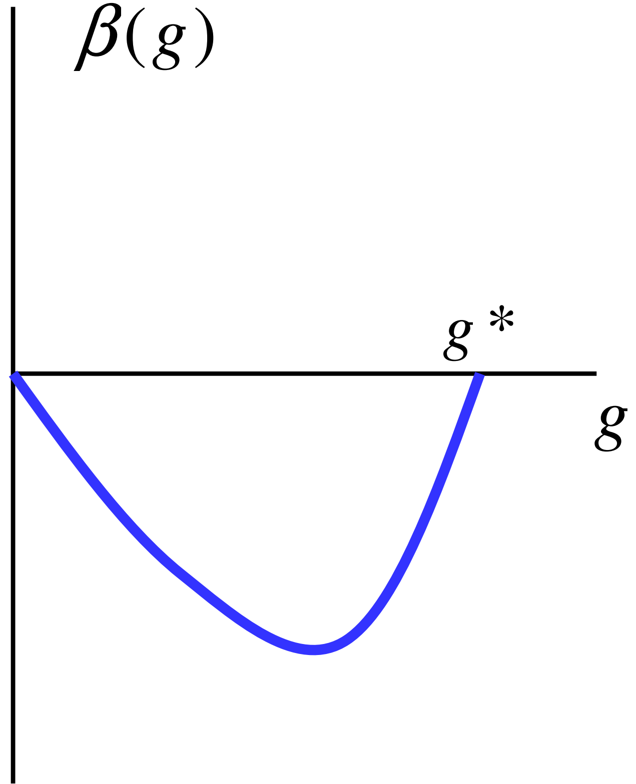
$$K_M(t) = \int_0^\infty \varphi^{M+n-1} d\varphi \, \exp\left(-t\varphi^2 - \varphi^4\right)$$

$$K_M(t) = \int_0^\infty \varphi^{M+n-1} d\varphi \, \exp\left(-t\varphi^2 - \varphi^4\right) =$$

$$= \begin{cases} \frac{1}{\sqrt{2}} t^{-(M+n)/2} \Gamma\left(\frac{M+n}{2}\right) \left[1 - \frac{(M+n)(M+n+2)}{4t^2} + \ldots\right], & t \to \infty \\ \frac{1}{4} \left[\Gamma\left(\frac{M+n}{4}\right) - t\Gamma\left(\frac{M+n+2}{4}\right) + \ldots\right], & t \to 0 \\ \frac{\sqrt{\pi}}{2} \mathrm{e}^{t^2/4} \left(\frac{|t|}{2}\right)^{(M+n-2)/2} \left[1 + \frac{(M+n-2)(M+n-4)}{4t^2} + \ldots\right], & t \to -\infty \end{cases}$$







Zeroes of integrals $K_M(t)$

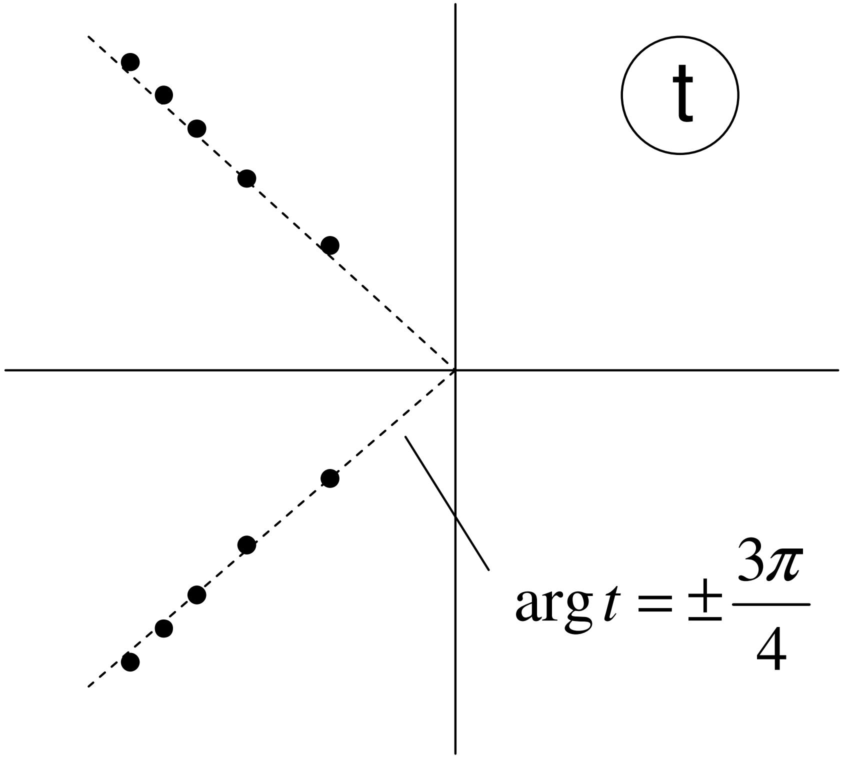
There are two saddle-points in $K_M(t)$

$$\varphi_{c1} = 0 \qquad \varphi_{c2} = \sqrt{-t/2}$$

If their contributions compensate each other

$$K_M(t) = A\mathrm{e}^{i\psi} + A_1\mathrm{e}^{i\psi_1} = A\mathrm{e}^{i\psi}\left(1 + a\mathrm{e}^{i\Delta}\right)$$

the integral can turn to zero



$$g = 1 - \frac{n}{n+2}\frac{K_4 K_0}{K_2^2}$$

$$\beta(g) = -\frac{2n}{n+2}\frac{K_4 K_0}{K_2^2}\left[2 + \frac{\frac{K_6 K_0}{K_4 K_2} - 1}{1 - \frac{K_4 K_0}{K_2^2}}\right]$$

For $K_2 \to 0$ we have

$$g \approx -\frac{n}{n+2}\frac{K_4 K_0}{K_2^2}\,, \qquad \beta(g) \approx -\frac{4n}{n+2}\frac{K_4 K_0}{K_2^2}$$

and parametric representation is resolved:

$$\boxed{\beta(g) = 4g\,, \qquad g \to \infty}$$

General d-dimensional case

$$Z^{(M)}_{\alpha_1\ldots\alpha_M}(x_1,\ldots,x_M) = \int D\varphi\,\varphi_{\alpha_1}(x_1)\varphi_{\alpha_2}(x_2)\ldots\varphi_{\alpha_M}(x_M)\exp\left(-S\{\varphi\}\right)$$

$$Z^{(M)}_{\alpha_1\ldots\alpha_M}(p_1,\ldots,p_M) = \sum_{x_1,\ldots,x_M} Z^{(M)}_{\alpha_1\ldots\alpha_M}(x_1,\ldots,x_M)\mathrm{e}^{ip_1x_1+\ldots+ip_Mx_M}$$

$$Z^{(M)}_{\alpha_1\ldots\alpha_M}\{p_i\} = K_M\{p_i\}\; I_{\alpha_1\ldots\alpha_M}\; N\delta_{p_1+\ldots+p_M}$$

where $I_{\alpha_1\ldots\alpha_M}$ is the sum of terms like $\delta_{\alpha_1\alpha_2}\delta_{\alpha_3\alpha_4}\ldots$ with all possible pairings

$$K_2(p) = K_2 - \tilde{K}_2 p^2 + \ldots$$

Parametric representation

$$g = -\left(\frac{K_2}{\tilde{K}_2}\right)^{d/2}\frac{K_4K_0}{K_2^2}$$

$$\beta(g) = \left(\frac{K_2}{\tilde{K}_2}\right)^{d/2}\left\{-d\frac{K_4K_0}{K_2^2} + 2\frac{(K_4'K_0 + K_4K_0')K_2 - 2K_4K_0K_2'}{K_2^2}\frac{\tilde{K}_2}{K_2\tilde{K}_2' - K_2'\tilde{K}_2}\right\}$$

For $\tilde{K}_2 \to 0$,

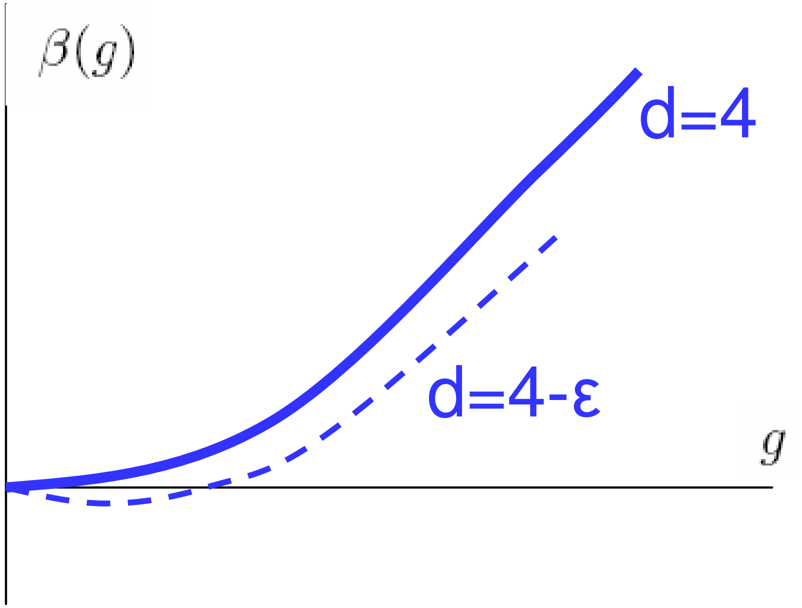
$$g = -\left(\frac{K_2}{\tilde{K}_2}\right)^{d/2} \frac{K_4 K_0}{K_2^2} \qquad \beta(g) = -d\left(\frac{K_2}{\tilde{K}_2}\right)^{d/2} \frac{K_4 K_0}{K_2^2}$$

and the parametric representation is resolved in the form

$$\boxed{\beta(g) = dg\,, \qquad g \to \infty}$$

For $K_2 \to 0$, the limit $g \to \infty$
can be achieved only for $d < 4$:

$$\beta(g) = (d-4)g\,, \qquad g \to \infty$$



Exact result for asymptotics of β(g) from the duality relation for the 2D Ising model

$$\beta(g)=2g$$

G. Jug, B.N. Shalaev, J.Phys. A **32**, 7249 (1999)

Zeroes of functional integrals

Instanton contribution

$$\left[Z^{(M)}_{\alpha_1\ldots\alpha_M}(p_1,\ldots,p_M)\right]^{inst} = ic_M(-g_0)^{-(M+r)/2}\mathrm{e}^{-S_0/g_0}\langle\phi_c\rangle_{p1}\ldots\langle\phi_c\rangle_{pM}\, I_{\alpha_1\ldots\alpha_M}$$

Contribution of two saddle-points for M=0,2,…

$$Z_0 = 1 + ic_0(-g_0)^{-r/2}\mathrm{e}^{-S_0/g_0}\,,$$

$$Z^{(2)}_{\alpha\beta}(p,p') = \frac{\delta_{\alpha\beta}}{p^2+m_0^2} + ic_2(-g_0)^{-(r+2)/2}\mathrm{e}^{-S_0/g_0}\langle\phi_c\rangle^2_p\,\delta_{\alpha\beta}\,, \qquad \text{etc.}$$

Setting

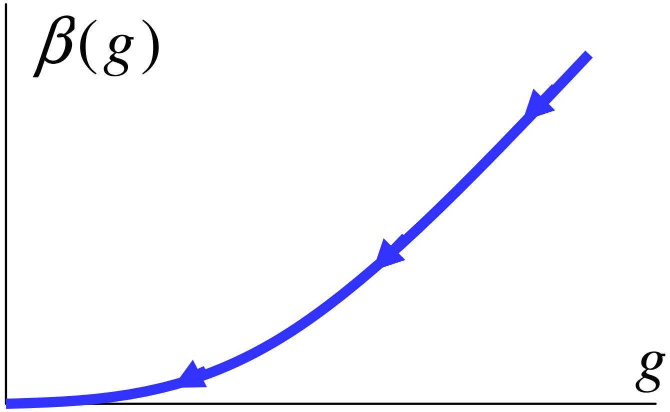
$$t^2 = -S_0/g_0$$

we come to expressions analogous to d=0

Is φ^4 theory trivial ?

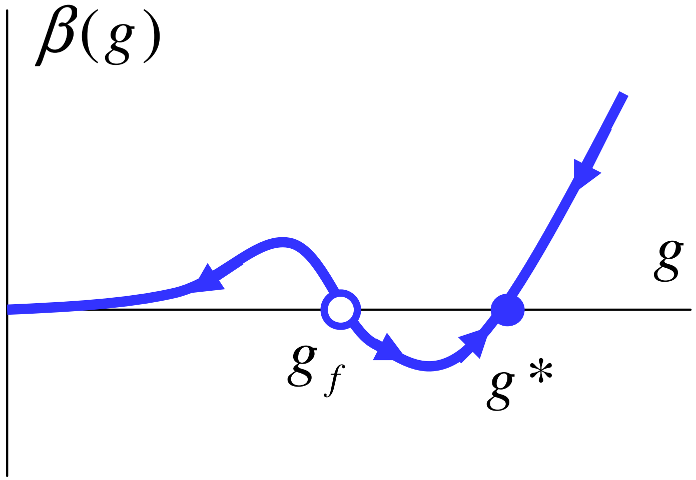
Triviality in Wilson’s sense

$$g = \frac{g_0}{1 + \beta_2 g_0 \ln \Lambda^2/m^2}$$



g_0 and Λ are fixed parameters having a physical sense.
For $m \to 0$ we have $g \to 0$, which is a real “zero charge”.

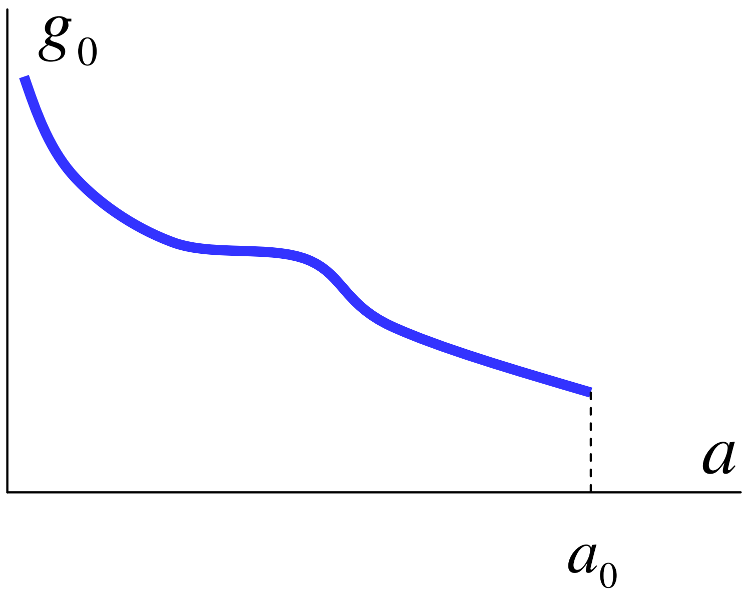
K.G.Wilson, J.Kogut,
Phys.Rep. 12C, 75 (1974)



No indications for g_f are found.

Triviality in mathematical sense

It is equivalent to internal inconsistency in the Bogolyubov and Shirkov sense



J. P. Eckmann, R. Epstein, Commun. Math. Soc. **64**, 95 (1979).

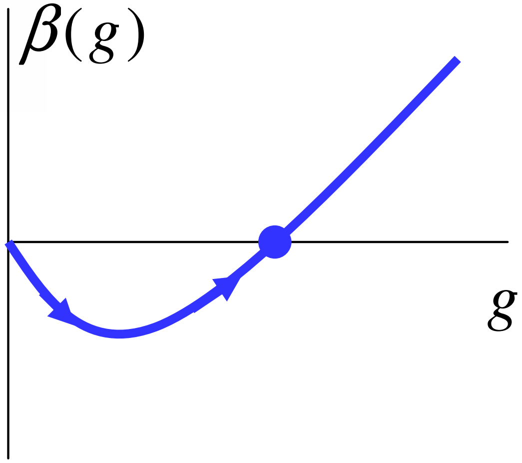
J. Frolich, Nucl. Phys. B **200** [FS4], 281 (1982).

M. Aizenman, Commun. Math. Soc. **86**, 1 (1982).

Can be proved rigorously:

(a) Triviality for d>4

(b) non-triviality for d<4

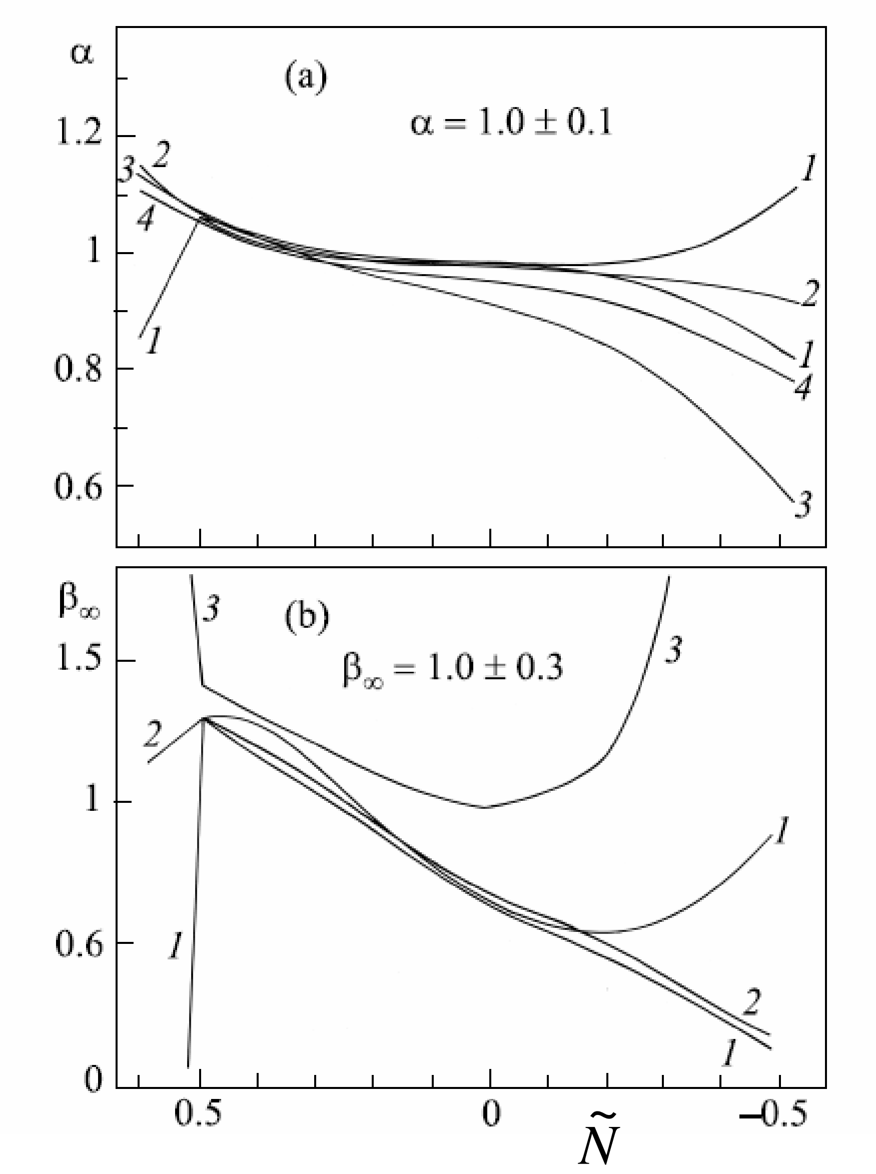


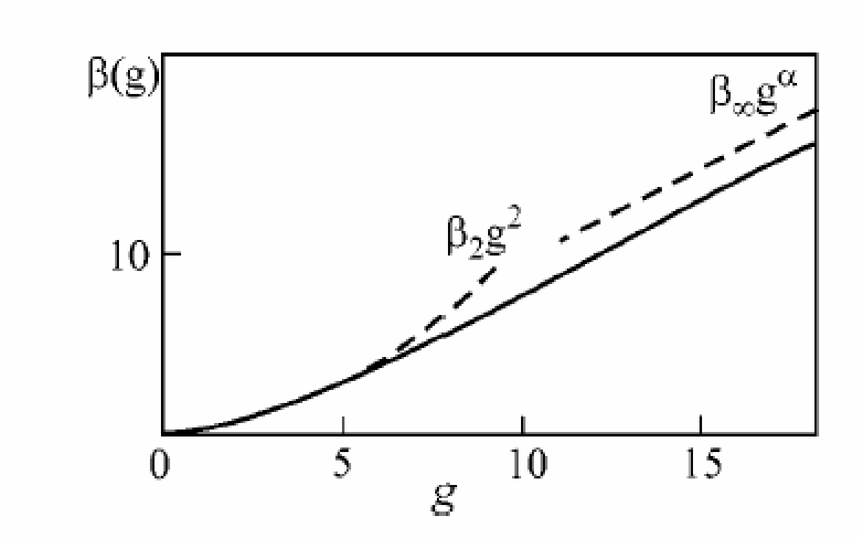
Exact asymptotics of the β-function in QED

Reconstruction of the β-function by summation of the perturbation series

I. M. Suslov, JETP Lett. 74, 191 (2001)

$$\beta(g) = \beta_\infty \; g^\alpha \qquad (g \to \infty)$$





From spectral representations:

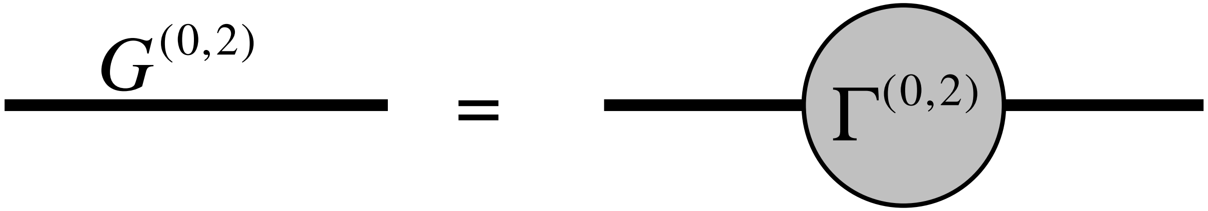
$$0 \leq \beta(g) < g$$

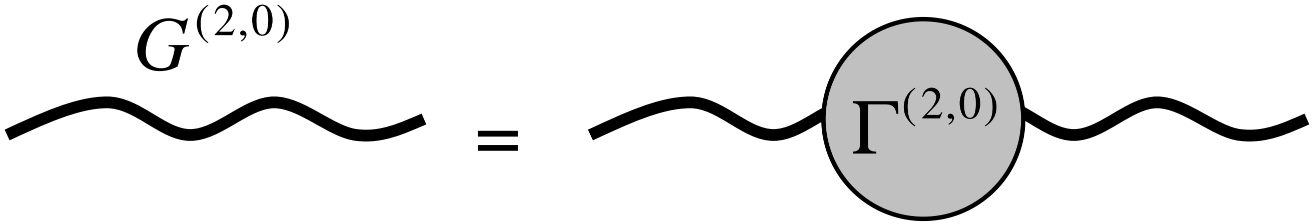
N. V. Krasnikov, Nucl. Phys. **B192**, 497 (1981);
H. Yamagishi, Phys. Rev. **D25**, 464 (1982).

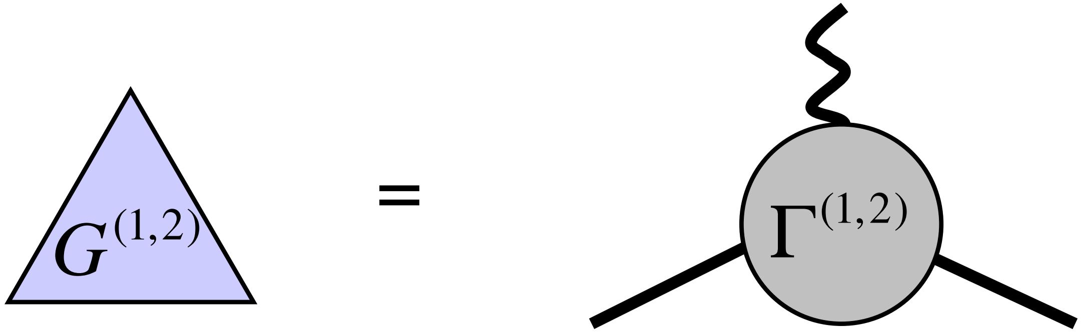
$$K_{M,2N} = \int DAD\bar{\psi}D\psi\, A_{\mu_1}(x_1)\ldots A_{\mu_M}(x_M)\,\psi(y_1)\bar{\psi}(z_1)\ldots\psi(y_N)\bar{\psi}(z_N)\exp\left(-S\{A,\psi,\bar{\psi}\}\right)$$

$$G^{(M,N)} = \frac{K_{MN}}{K_{00}}$$

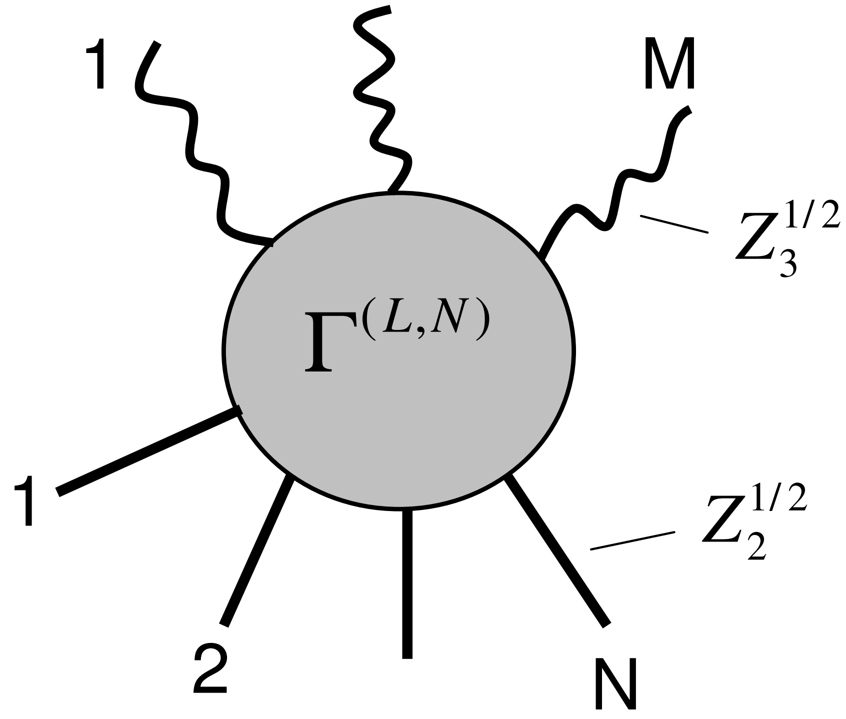
$$S\{A,\psi,\bar{\psi}\} = \int d^4x\left[\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \bar{\psi}(i\not{\partial} - m_0 + e_0\not{A})\psi\right]$$







Multiplicative renormalizability



$$\Gamma^{(M,N)}(q_i, p_i; e_0, m_0, \Lambda) = Z_3^{-M/2} Z_2^{-N/2} \Gamma_R^{(M,N)}(q_i, p_i; e, m)$$

Renormalization conditions:

$$\Gamma_R^{(0,2)}(p)\Big|_{p\to 0} = \not{p} - m\,,$$

$$\Gamma_R^{(2,0)}(q)\Big|_{q\to 0} = q^2\,,$$

$$\Gamma_R^{(1,2)}(q,p,p')\Big|_{q,p,p'\to 0} = e$$

Relation with the bare quantities:

$$Z_2 = \left(\frac{\partial}{\partial \not{p}}\,\Gamma^{(0,2)}(p;e_0,m_0,\Lambda)\Big|_{p=0}\right)^{-1}$$

$$Z_3 = \left(\frac{\partial}{\partial q^2}\,\Gamma^{(2,0)}(q;e_0,m_0,\Lambda)\Big|_{q=0}\right)^{-1}$$

$$m = -Z_2\,\Gamma^{(0,2)}(p;g_0,m_0,\Lambda)\Big|_{p=0}$$

$$e = Z_2 Z_3^{1/2}\,\Gamma^{(1,2)}(q,p,p';e_0,m_0,\Lambda)\Big|_{q,p,p'=0}$$

Definition of the β-function:

$$\beta(g) = \left.\frac{dg}{d\ln m^2}\right|_{e_0,\Lambda=const}$$

$$\Gamma^{(0,2)}(p) = \frac{K_{00}}{K_{02}(p)}\,, \qquad \Gamma^{(2,0)}(q) = \frac{K_{00}}{K_{20}(q)}\,, \qquad \Gamma^{(1,2)} = \frac{K_{12}K_{00}^2}{K_{02}^2 K_{20}}$$

$$K_{02}(p) = K_{02} + \tilde{K}_{02}\not{p}\,, \qquad K_{20}(q) = K_{20} + \tilde{K}_{20}q^2$$

$$Z_2 = -\frac{K_{02}^2}{K_{00}\tilde{K}_{02}}\,, \qquad Z_3 = -\frac{K_{20}^2}{K_{00}\tilde{K}_{20}}\,, \qquad m = \frac{K_{02}}{\tilde{K}_{02}}\,, \qquad g = -\frac{K_{12}^2 K_{00}}{\tilde{K}_{02}^2 \tilde{K}_{20}}$$

$$\beta(g) = \frac{m}{2}\left(-\frac{K_{12}^2 K_{00}}{\tilde{K}_{02}^2 \tilde{K}_{20}}\right)'_{m_0} \frac{dm_0}{dm}$$

$$\frac{dm}{dm_0} = \left(\frac{K_{02}}{\tilde{K}_{02}}\right)' = \frac{K'_{02}\tilde{K}_{02} - K_{02}\tilde{K}'_{02}}{\tilde{K}_{02}^2}$$

Parametric representation for the β-function:

$$g = -\frac{K_{12}^2 K_{00}}{\tilde{K}_{02}^2 \tilde{K}_{20}} \,,$$

(m_0, e_0, Λ)

$$\beta(g) = \frac{1}{2}\frac{K_{02}\tilde{K}_{02}}{K_{02}\tilde{K}'_{02} - K'_{02}\tilde{K}_{02}} \frac{K_{12}^2 K_{00}}{\tilde{K}_{02}^2 \tilde{K}_{20}} \left\{ \frac{2\,K'_{12}}{K_{12}} + \frac{K'_{00}}{K_{00}} - \frac{2\,\tilde{K}'_{02}}{\tilde{K}_{02}} - \frac{\tilde{K}'_{20}}{\tilde{K}_{20}} \right\}$$

For $\tilde{K}_{02} \to 0$ we have

$$g = -\frac{K_{12}^2 K_{00}}{\tilde{K}_{02}^2 \tilde{K}_{20}} \,, \qquad \beta(g) = -\frac{K_{12}^2 K_{00}}{\tilde{K}_{02}^2 \tilde{K}_{20}}$$

and the parametric representation is resolved

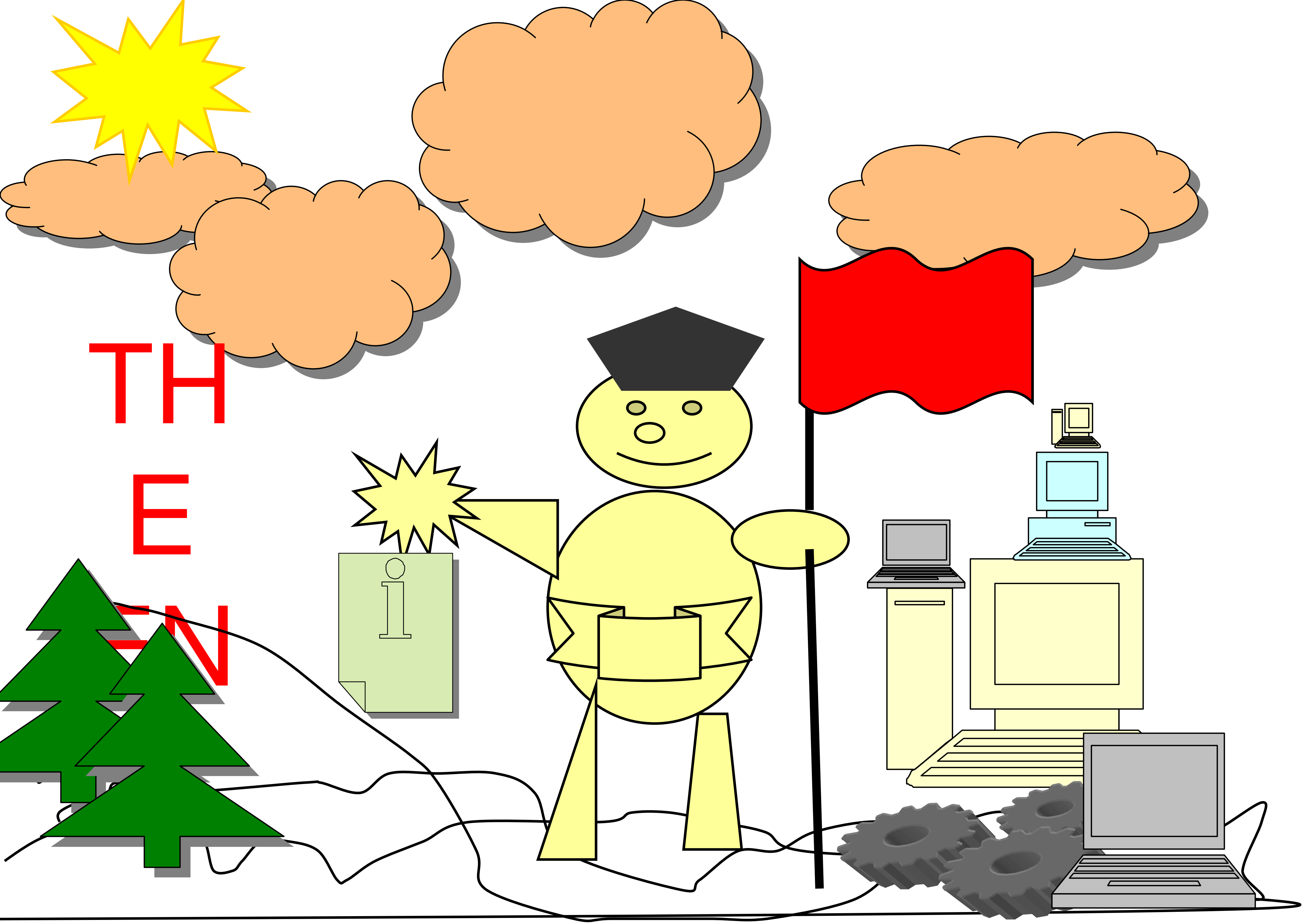
$$\boxed{\beta(g) = g \,, \qquad g \to \infty}$$

For $\tilde{K}_{20} \to 0$, one has

$$g \propto \frac{1}{\tilde{K}_{20}} \,, \qquad \beta(g) \propto \frac{1}{\tilde{K}_{20}^2} \,,$$

and hence

$$\beta(g) \propto g^2 \,, \qquad g \to \infty \,.$$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“Zero charge” problem

L. D. Landau, A. A. Abrikosov, and I. M. Khalatnikov,

Dokl. Akad. Nauk SSSR **95**, 497, 773, 1177 (1954).

$$g = \frac{g_0}{1 + \beta_2 g_0 \ln \Lambda^2/m^2}$$

For $\Lambda \to \infty$ we have $g \to 0$ (”zero charge”)

$$g_0 = \frac{g}{1 - \beta_2 g \ln \Lambda^2/m^2}$$