Analysis of divergent series by Euler–Maclaurin summation

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Loughborough?

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$\approx 15,000$ students,
≈ 600 academic staff.

Nørlund's formula

- Consider the series $S_n = \sum_{s=0}^{\infty} (s+x)^n \mathrm{e}^{-(s+x)\delta}$ $(n = 0, 1, \dots)$.
- What happens as $\delta \to 0$? We want a bit more than 'it diverges'.
- Nørlund (1924) gives the formula

$$S_n = \int_0^\infty s^n \mathrm{e}^{-s\delta}\, \mathrm{d}s + \frac{B_{n+1}(x)}{n+1} + O(\delta),$$

  where $B_n(\cdot)$ is a Bernoulli polynomial.
- If we evaluate the integral, we obtain

$$S_n = \frac{n!}{\delta^{n+1}} + \frac{B_{n+1}(x)}{n+1} + O(\delta).$$

- How does this formula arise? What if the exponential was replaced by

$$\mathrm{e}^{-(s\delta)^2}, \quad \mathrm{erfc}\left[\left(s^2\delta^2 - 1\right)^{1/2}\right], \quad \dots$$

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Relating an integral over $[r_0, r_1]$ to a series

- Start with the integral from s to $s+1$ ($s \in \mathbb{Z}$):

$$\int_s^{s+1} f(x)\,\mathrm{d}x = \left[(x+c)f(x)\right]_s^{s+1} - \int_s^{s+1} (x+c)f'(x)\,\mathrm{d}x$$

$$= (s+1+c)f(s+1) - (s+c)f(s) - \int_s^{s+1} (x+c)f'(x)\,\mathrm{d}x.$$

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- Choose $c = -s - \frac{1}{2}$, and note that $s = [x]$ for $x \in [s, s+1)$:

$$\int_s^{s+1} f(x)\,\mathrm{d}x = \frac{1}{2}\left[f(s+1) + f(s)\right] - \int_s^{s+1} \left(x - [x] - \tfrac{1}{2}\right) f'(x)\,\mathrm{d}x.$$

- Now sum from $s = r_0$ to $s = r_1 - 1$ to obtain

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- the first derivative Euler–Maclaurin formula.

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The Euler–Maclaurin summation formula

- For $m \in \mathbb{N}$, we have

$$\sum_{s=r_0}^{r_1} f(s) = \int_{r_0}^{r_1} f(s)\,\mathrm{d}s + \mathcal{B}_m[f] + \Delta_m[f],$$

- where

$$\mathcal{B}_m[f] = \frac{f(r_1) + f(r_0)}{2} + \sum_{j=2}^{2m} \frac{B_j}{j!} \left[f^{(j-1)}(r_1) - f^{(j-1)}(r_0) \right],$$

$$\Delta_m[f] = \int_{r_0}^{r_1} \frac{\hat{B}_{2m+1}(x)}{(2m+1)!} f^{(2m+1)}(x)\,\mathrm{d}x.$$

- B_j is a Bernoulli number. N.B. $B_{2j+1} = 0$, $j \in \mathbb{N}$.
- $\hat{B}_j(x) = B_j(x - [x])$ is the periodic extension of $B_j(x)$ from $[0,1)$ to $\mathbb{R}$.

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The error term

- Lehmer (1940) proved $\left|\hat{B}_{2m+1}(x)\right| \le 2(2m+1)!/(2\pi)^{2m+1}$; hence

$$|\Delta_m[f]| \le \frac{2}{(2\pi)^{2m+1}} \int_{r_0}^{r_1} |f^{(2m+1)}(x)| \, \mathrm{d}x.$$

- We will need to apply EMS on $[\lambda, \infty)$ and then take a limit.
- If $f^{(2m+1)}(s)$ has fixed sign for $s > A$, we can write

$$\begin{aligned} |\Delta_m[f]| &\le \frac{2}{(2\pi)^{2m+1}} \left(\int_\lambda^A \left| f^{(2m+1)}(s) \right| \mathrm{d}s + \left| \int_A^\infty f^{(2m+1)}(s) \, \mathrm{d}s \right| \right) \\ &\le \frac{2}{(2\pi)^{2m+1}} \left(\left| \int_\lambda^A f^{(2m+1)}(s) \, \mathrm{d}s \right| + \left| f^{(2m)}(A) \right| \right). \end{aligned}$$

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- N.B. if $f(s;\delta) = (s+x)^n \mathrm{e}^{-s\delta}$ and $2m > n$, then $\Delta_m \to 0$ as $\delta \to 0$.

Generalised Nørlund formula

- Set $f(s;\delta) = (s+x)^n E(s;\delta)$, $E(s,0) = E_0$ (constant).
- Set $2m > n$, apply EMS over $[\lambda, \infty)$ (assume convergence as $s \to \infty$):

$$\sum_{s=\lambda}^{\infty} f(s;\delta) = \int_{\lambda}^{\infty} f(s;\delta)\,\mathrm{d}s + \mathcal{B}_m[f;\lambda] + o(1),$$

$$\mathcal{B}_m[f;\lambda] = \frac{f(\lambda;\delta)}{2} - \sum_{j=2}^{2m} \frac{B_j}{j!} f^{(j-1)}(\lambda;\delta).$$

- Provided that derivatives of E exist, they vanish as $\delta \to 0$, so

$$f^{(j-1)}(s;\delta) = \frac{n!}{(n+1-j)!}(s+x)^{n-j+1} E(s;\delta) + o(1),$$

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Generalised Nørlund formula (ctd.)

- Finally, we arrive at the result

$$\sum_{s=\lambda}^{\infty}(s+x)^n E(s;\delta) = \mathrm{I}_n(x;\delta,\lambda) - \frac{E_0}{n+1}B_{n+1}(\lambda+x) + o(1);$$

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- As before, $f(s;\delta) = (s+x)^n E(s;\delta)$; suppose that a sufficient number of derivatives exist on $[\Lambda, \infty)$, but not on $[\lambda, \Lambda)$. We can write

$$\sum_{s=\lambda}^{\infty} f(s;\delta) = \sum_{s=\lambda}^{\Lambda-1} f(s;\delta) + \sum_{s=\Lambda}^{\infty} f(s;\delta).$$

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- Suppose that $f(s) = g(s) + g(-s)$.
- Assuming convergence as $s \to \infty$, we have

$$\sum_{s=0}^{\infty} f(s) = \int_0^{\infty} f(s)\,\mathrm{d}s + \frac{f(0)}{2} - \sum_{j=2}^{2m} \frac{B_j}{j!} f^{(j-1)}(0) + \Delta_m[f;0]$$

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and we are free to choose m.

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- We can obtain explicit results in certain simple cases.
- Jacobi's imaginary transformation (Whittaker & Watson 1927) can be used to show that

$$\sum_{s=-\infty}^{\infty} \mathrm{e}^{-(s^2\delta^2)} = \frac{\sqrt{\pi}}{\delta} \sum_{s=-\infty}^{\infty} \mathrm{e}^{-(s\pi/\delta)^2}.$$

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